College Mathematics

COLLEGE MATHEMATICS

A GENERAL INTRODUCTION

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Preface

THIS TEXT presents the customary first-year course in college algebra, trigonometry, and analytic geometry, together with the notation and elementary processes and applications of the differential and integral calculus.

No attempt has been made either to "unify" or to keep separate the component subjects. What the author has tried to do is to present the entire subject matter in a single natural and orderly sequence. It is believed that the essential interdependence of the separate subjects can best be exhibited in this way.

Although this book is not intended for students with no previous training in algebra, the text begins with the elementary topics of that subject. The emphasis that will need to be placed on the early chapters will vary with the previous training and the ability of the class.

The exercises have been graded so that the instructor may select from them assignments suited to the needs of his class. Answers have been given to the odd-numbered exercises; those to the even-numbered exercises will be printed separately.

The author acknowledges with pleasure his indebtedness to Dr. Margaret Hansman for valuable suggestions and for assistance in seeing the book through the press.

C. H. S.

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College Mathematics

Chapter 1

Fundamental Operations

1. Literal Numbers. In algebra, it is customary to represent numbers by letters. We speak, for example, of the numbers a, b, x, y, and so on; meaning thereby certain numbers, the values of which may or may not be known to us, that appear in the problem under consideration. When a number is represented by a letter, it is spoken of as a *literal* number to distinguish it from *explicit* numbers, such as 7, 2, and -6, which are written in the Arabic symbols.

Thus, the numbers a, k, xy, $x^2 - y^2$, are literal numbers. The numbers 10, -15, 27, are explicit numbers.

2. Real Numbers. The numbers usually met with in elementary mathematics are called **real** numbers. Every real number may be classified as either positive, negative, or zero. Later, we shall introduce another type of numbers which we shall call **imaginary** numbers (Art. 26). Unless the contrary is indicated, the literal numbers we shall use will be real numbers.

3. Graphical Representation of Real Numbers. The Linear Scale. A clearer understanding of some of the properties of real numbers can be obtained by representing these numbers by the points on a straight line. The line on which this representation is made is a **linear scale.** We shall use linear scales very frequently as we proceed with this course.

On an unlimited straight line $X'X$ (Fig. 1), choose any fixed point O to represent the number zero. This point we shall call the **origin.**

To find the point P that represents a given number x, choose a unit of length and measure off from O a distance $OP = x$, to the right from O if x is a positive number and to the left if x is negative. The point P, at the end of this segment, is said to represent the number x and the number is said to be the **abscissa** of the point P. We have indicated on Fig. 1 the points representing the positive numbers 1, 2, 3, and 4 and the negative numbers -1, -2, -3, and -4. The point P_1 (read "P sub one"), half way between the points representing -2 and -3, represents the number $-\frac{5}{2}$ and the point P_2 represents the number* $\sqrt{7} = 2.646$, approximately.

Conversely, if we have given a point P on the line $X'X$ and wish to find its abscissa x, we measure the distance OP and prefix a plus or minus sign according as P lies to the right or to the left of O. We find in this

* A table of the squares and square roots of certain numbers will be found in Table IV at the back of the book.

way, in Fig. 1, that the abscissa of P_3 is 1.8, and of P_4 is -3.4, approximately.

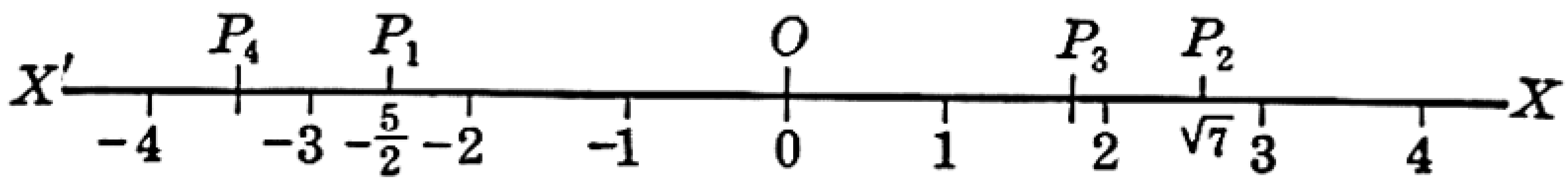

FIG. 1

Of two given numbers a and b, we say that *a is greater than b* if the point whose abscissa is a lies to the *right* of the point whose abscissa is b. In the contrary case, we say that *a is less than b*. For positive numbers, this statement is obvious; for negative numbers, it means, for example, that -1 is greater than -3 because the point whose abscissa is -1 lies to the right of the point whose abscissa is -3. We shall customarily write the statement, "a is greater than b," in the symbolic form $a > b$ and the statement, "a is less than b," in the form $a < b$.

Let P be the point representing a number a. The length of the segment OP, taken always as a *positive* number, is called the **absolute,** or **numerical,** value of a. We denote this absolute value by the symbol $|a|$. It follows that, if a is positive, then $|a| = a$ but, if a is negative, then $|a|$ is equal to a with its sign changed.

Thus, $|13| = 13,\quad |-13| = 13,\quad |7 - 12| = |12 - 7| = 5.$

Exercises

1. Represent on a graphical scale the numbers $4, -7, \frac{10}{3}, -\frac{5}{4}, \pi$.

2. Arrange the numbers in Ex. 1 in increasing order.

3. Choose two points at random on the graphical scale, one to the right and one to the left of O. Find, by measurement, the abscissas of these points to one decimal place.

4. Choose two negative numbers a and b such that $a > b$. Show that $a - b$ is a positive number.

5. If the abscissas of two points A and B on the graphical scale are 5 and -1, respectively, show that the abscissa of the midpoint of the segment AB is 2.

6. In Ex. 5, show that the length of the segment AB is 6.

7. Read each of the following expressions and state its value: $|-7|$, $|16|$, $|-\frac{5}{8}|$, $|-2|$, $|7 - 15|$, $|2 - 6|$, $|4 - 23 + 9|$.

4. Computations Involving Zero. In computations in which at least one of the numbers involved is zero, the following laws hold.

I. *If zero is added to, or subtracted from, a given number, the result equals the given number.*

Thus, $9 + 0 = 9$, $9 - 0 = 9$, $-3 + 0 = -3$, $-3 - 0 = -3$, $0 + 0 = 0$, $0 - 0 = 0$, $a + 0 = a$, $a - 0 = a$.

II. *The product of two numbers equals zero if, and only if, at least one of the factors is zero;* that is,

$$ab = 0 \text{ if, and only if, either } a = 0 \text{ or } b = 0.$$

Thus, $5 \times 0 = 0$, $0 \times \frac{4}{3} = 0$, $0 \times 0 = 0$, $0 \times (a + 3) = 0$.

EXAMPLE 1. Find the values of x that make $x^2 + x - 12 = 0$.
Factor the first member: $(x - 3)(x + 4) = 0$.
This product equals zero if, and only if, either

$$x - 3 = 0 \quad \text{or} \quad x + 4 = 0,$$

that is, either $x = 3$ or $x = -4$.
The student should show by substitution that each of these values satisfies the given equation.

III. *The quotient indicated by dividing a number by zero does not exist;* that is,

$$\text{the expressions } a \div 0 \text{ and } \frac{a}{0} \text{ are meaningless.}$$

Thus, none of the following expressions represents a number:

$$4 \div 0, \ \frac{27}{0}, \ \frac{0}{0}, \ \frac{5}{4 \times 0}, \ \frac{51}{2 - 2}, \ \frac{b}{a - a}.$$

The expression $\dfrac{7}{x - 3}$ represents a number for all values of x *except* $x = 3$. If $x = 3$, the denominator is zero and the expression is meaningless. Similarly, the expression $\dfrac{x - 7}{(x + 2)(x - 3)}$ has no value when $x = -2$ or $x = 3$.

EXAMPLE 2. Solve the equation $(x - 1)(x - 5) = 2(x - 1)$.
If we divide both sides of this equation by $x - 1$, we obtain $x - 5 = 2$, giving the solution $x = 7$.
The process of division by $x - 1$ is legitimate for all values of x except $x = 1$. It will be found by substitution that $x = 1$ satisfies the given equation but this solution was *lost* in the process of division. The correct solutions are $x = 1$ and $x = 7$.

We shall frequently perform the operation of dividing one literal expression by another but it must be remembered that this operation is meaningless whenever values are assigned to the literal numbers involved that make the divisor zero.

Thus, if any value other than 2 is assigned to x, the value of the quotient $(x^2 - 4) \div (x - 2)$ is $x + 2$ but, if $x = 2$, the division is impossible and no number results from the operation.

5. Computations Involving Signed Numbers. In computations involving positive and negative numbers, the following laws hold.

I. *To add two numbers having like signs, add their absolute values and prefix the common sign.*

Thus, $3 + 11 = 14$, $(-5) + (-7) = -(5 + 7) = -12$.

II. *To add two numbers having unlike signs, take the difference of their absolute values and prefix the sign of the number having the larger absolute value.*

Thus, $26 + (-17) = 26 - 17 = 9$, $14 + (-19) = -(19 - 14) = -5$, $(-32) + 15 = -(32 - 15) = -17$, $(-17) + 35 = 35 - 17 = 18$.

III. *To subtract one number from another, change the sign of the number to be subtracted and proceed as in addition.*

Thus, $38 - 11 = 38 + (-11) = 27$, $43 - (-29) = 43 + 29 = 72$, $(-15) - 4 = (-15) + (-4) = -19$, $(-31) - (-47) = (-31) + 47 = 16$.

IV. *To multiply (or divide) one number by another, first multiply (or divide) their absolute values. Then prefix a plus sign if the given numbers have the same sign or a minus sign if they have unlike signs.*

Thus, $(-9) \times (-7) = +(9 \times 7) = 63$, $(-17) \times 4 = -(17 \times 4) = -68$, $(-18) \div (-6) = +(18 \div 6) = 3$, $42 \div (-7) = -(42 \div 7) = -6$.

Exercises

In the following exercises, (*a*) add the two numbers, (*b*) subtract the second number from the first.

1. $\begin{array}{r} 57 \\ \underline{41} \end{array}$ **2.** $\begin{array}{r} -92 \\ \underline{41} \end{array}$ **3.** $\begin{array}{r} -57 \\ \underline{-39} \end{array}$ **4.** $\begin{array}{r} 93 \\ \underline{-37} \end{array}$ **5.** $\begin{array}{r} -76 \\ \underline{0} \end{array}$

6. $\begin{array}{r} 5a \\ \underline{2a} \end{array}$ **7.** $\begin{array}{r} -3x \\ \underline{-2x} \end{array}$ **8.** $\begin{array}{r} -7m \\ \underline{3m} \end{array}$ **9.** $\begin{array}{r} 3y \\ \underline{-5y} \end{array}$ **10.** $\begin{array}{r} 0 \\ \underline{3z} \end{array}$

Perform the indicated multiplications.

11. $21 \times (-8)$. **12.** $(-15) \times (-7)$. **13.** $(-4) \times (-3) \times (-2)$.
14. $6a \times 0$. **15.** $(-4x) \times 3$. **16.** $5z \times (-6z)$.

Perform the indicated divisions.

17. $28 \div (-4)$. **18.** $(-91) \div 13$. **19.** $(-136) \div (-17)$.
20. $(-6u) \div 3$. **21.** $8t \div (-4t)$. **22.** $(-12x) \div (-3x)$.

Perform the indicated operations.

23. $(-2a)(-3b)(-7c)$. **24.** $(ab + bc) \div (a + b - a)$. **25.** $(a + b + c)d$.

26. Find the value of each of the following expressions that has a value. Show that each of the others is meaningless.

$$\frac{14-8}{9-6},\ \frac{3-7}{13-12},\ \frac{5-5}{4+3},\ \frac{(-6)-8}{9+(-2)},\ \frac{-5+1}{4-2^2},\ \frac{3^2-9}{2^2-4},\ \frac{a-a}{b-b},\ \frac{a^2-a^2}{b^2-b^2}.$$

Find the value of the expression $\frac{x^2 + 2y - 3z}{xy + yz + zx}$, given:

27. $x = 3, \quad y = -2, \quad z = 5.$ **28.** $x = -2, \quad y = -5, \quad z = -9.$

Solve the following equations.

29. $5x - 8 = 27.$ **30.** $2x + 9 = 21.$ **31.** $8x + 1 = 3x - 14.$

32. $9(x + 3) = 5(2x + 4).$ **33.** $\frac{4x + 11}{7} = \frac{3x - 6}{5}.$

34. $\frac{5x + 3}{4} = \frac{10x - 7}{9}.$ **35.** $(x - 5)(x - 11) = 0.$

36. $(2x + 9)(3x - 7) = 0.$ **37.** $x^2 - 5x + 6 = 0.$

38. Johannesburg, South Africa, is 67° south of Istanbul, Turkey, which is in latitude 41° north. Find the latitude of Johannesburg.

39. The top of Mt. Whitney is 14,502 feet above sea level. Find the altitude of a point in the Imperial Valley 14,670 feet lower than the top of Mt. Whitney.

40. A is 61 and B is 67 miles south of a certain crossing. Both A and B are traveling north at the uniform rates of 43 and 47 miles an hour, respectively. How far, and in what direction, from the crossing will B overtake A?

6. Symbols of Grouping. The symbols most frequently used to indicate grouping are **parentheses** ( ), **brackets** [ ], and **braces** { }. When used in mathematical expressions, these symbols mean that the quantity included between them should be treated as a single number.

Thus, the expression $6 \times (27 - 14)$ means that we are first to subtract 14 from 27, then multiply the difference by 6; that is,

$$6 \times (27 - 14) = 6 \times 13 = 78.$$

Similarly, the expression $5x^2 + 9x - 7 - (3x^2 - 2x - 8)$ means that the number $3x^2 - 2x - 8$ is to be subtracted from the number $5x^2 + 9x - 7$.

A pair of symbols of grouping preceded by a plus sign may be removed (or may be inserted) without changing the sign of any term between the symbols.

A pair of symbols of grouping preceded by a minus sign may be removed (or may be inserted) provided that the sign of **every** *term between the symbols is changed.*

Thus,

$$x^2 - 5x + 7 + (2x^2 - 8x + 9) = x^2 - 5x + 7 + 2x^2 - 8x + 9 = 3x^2 - 13x + 16.$$
$$x^2 - 5x + 7 - (2x^2 - 8x + 9) = x^2 - 5x + 7 - 2x^2 + 8x - 9 = -x^2 + 3x - 2.$$
$$3x^2 + 4x + 8 + 5x^2 - 4x + 2 = 3x^2 + 4x + 8 + (5x^2 - 4x + 2).$$
$$3x^2 + 4x + 8 + 5x^2 - 4x + 2 = 3x^2 + 4x + 8 - (-5x^2 + 4x - 2).$$

To remove the symbols of grouping when one pair is enclosed within another pair, first remove the innermost pair, then the next innermost pair, and so on until all are removed.

Thus,

$$\begin{aligned}
&4a - 3 - (8a + 2 - [\{6a - 11\} - \{2a - 3\} - 5a] - 7)\\
&= 4a - 3 - (8a + 2 - [6a - 11 - 2a + 3 - 5a] - 7)\\
&= 4a - 3 - (8a + 2 - 6a + 11 + 2a - 3 + 5a - 7)\\
&= 4a - 3 - 8a - 2 + 6a - 11 - 2a + 3 - 5a + 7 = -5a - 6.
\end{aligned}$$

Exercises

Remove all symbols of grouping and combine like terms.

1. $3x + 5 - (4x + 9) + (7x - 3) - (2x - 1)$.

2. $-(4x + 3y - 2) - (9x + 6y - 11)$.

3. $(6u + 2v + 9) - (-5u + 11v + 6) - (4u - v + 6)$.

4. $ab - 2a + 6b - 13 - (5ab - 11a - 2b + 6) + (7ab + 4)$.

5. $(4r - 9 - [6r + 5 - \{11r + 2\} - 8r - 3] + 3r - 11) - 7$.

6. $-(6x + 5y - 3 - [2x - 8y + 7] + [3x - 11y + 2]) - (5x - 7y + 4)$.

7. $-(2s + 5t - 6) - ([-8s + 11t - 4] - [3s - 2t - 9]) - (6s - 2)$.

8. $4x - \{(x + 3) - [(2x + 7) - (3x + 1) + 8] - (7x - 9) - 2\}$.

9. Find the value of $a + bc + d$ and $(a + b)(c + d)$ when $a = 2$, $b = 5$, $c = -3$, and $d = 7$. Explain why the results are not equal.

10. Find the value of $(a \div b) \div c$ and $a \div (b \div c)$ when $a = 12$, $b = 6$, and $c = 2$. Explain why the results are not equal.

Write each of the following expressions with the last three terms in parentheses preceded by (*a*) a plus sign, (*b*) a minus sign.

11. $6x - 4y + 2z - 9x + 4y - 5z$. **12.** $5r^3 + 7r^2 - 9r - 6$.

13. Find (*a*) the sum and (*b*) the difference of $7x + 4y - 9z$ and $2x - 3y - 5z$.

Indicate the following operations, using parentheses, and find value of the result.

14. From $11a^2 + 3ab - 7b^2$ subtract the sum of $2a^2 - 5ab + 3b^2$ and $5a^2 + 9ab - b^2$.

15. From $5z^3 - 8z^2 + 2z + 12$ subtract $4z^3 + 2z^2 - 6z - 3$ and add $2z^3 + 9z^2 - 3z - 5$ to the result.

16. What must be added to $3t^2 + 7t - 4$ to give $5t^2 - 3t + 8$?

HINT. Denote the required expression by x. Then $3t^2 + 7t - 4 + x = 5t^2 - 3t + 8$. Solve this equation for x.

17. What must be subtracted from $9r^3 + 4r^2 - 6r + 8$ to give $5r^3 - 2r^2 + 4r + 3$?

7. Positive Integral Exponents. The symbol a^2 is used to denote the product $a \cdot a$ and the symbol a^3 to denote $a \cdot a \cdot a$. Similarly, if n is any positive integer, the meaning of the symbol a^n is defined by the equation

$$a^n = a \cdot a \cdot a \text{ and so on to } n \text{ factors.}$$

If we wish to multiply a^4 by a^3, we have, from the definition of the symbols,

$$\begin{aligned} a^4 \cdot a^3 &= (a \cdot a \cdot a \cdot a)(a \cdot a \cdot a) \\ &= a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a = a^{4+3} = a^7. \end{aligned}$$

Similarly, for the product of a^2 by a^6, we have

$$\begin{aligned} a^2 \cdot a^6 &= (a \cdot a)(a \cdot a \cdot a \cdot a \cdot a \cdot a) \\ &= a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a = a^{2+6} = a^8. \end{aligned}$$

If m and n are any two positive integers, we find, in precisely the same way, that

$$a^m \cdot a^n = a^{m+n}.$$

If we wish to divide a^5 by a^2, we have from the definition,

$$\frac{a^5}{a^2} = \frac{a \cdot a \cdot a \cdot a \cdot a}{a \cdot a} = a \cdot a \cdot a = a^{5-2} = a^3,$$

and if we wish to divide a^2 by a^5, we have

$$\frac{a^2}{a^5} = \frac{a \cdot a}{a \cdot a \cdot a \cdot a \cdot a} = \frac{1}{a \cdot a \cdot a} = \frac{1}{a^{5-2}} = \frac{1}{a^3}.$$

If m and n are any two positive integers, the same reasoning shows us that, if $m > n$,

$$\frac{a^m}{a^n} = a^{m-n}, \qquad m > n$$

and, if $m < n$,

$$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}. \qquad m < n$$

8. Multiplication of Monomials. An expression of the form $15a^2b^3xy^5$ is called a **monomial.** The number 15 is called the **numerical coefficient** (or simply the **coefficient**) and the factors a^2, b^3, x, and y^5 are the **literal factors.**

To multiply two monomials, *first find the product of the numerical coefficients, then multiply this result by the product of the literal factors of both monomials.*

Thus, $(3a^2x^3)(4a^4y) = (3 \times 4)(a^2 \cdot a^4)x^3y = 12a^6x^3y.$

$(5x^2y^3z)(4xy^2w^5) = (5 \times 4)(x^2 \cdot x)(y^3 \cdot y^2)zw^5 = 20x^3y^5zw^5.$

9. Multiplication of Polynomials. The sum of two or more monomials is a **polynomial.** Each of the monomials contained in a polynomial is called a **term** of the polynomial. In particular, a polynomial of two terms is called a **binomial** and one of three terms is a **trinomial.**

To multiply a polynomial by a monomial, *multiply each term of the polynomial by the monomial and combine the results.*

Thus, $(-2a^2xy^3)(4ax^3y - 7a^2xy^4 - 3x^2y^2) = -8a^3x^4y^4 + 14a^4x^2y^7 + 6a^2x^3y^5.$

To multiply two polynomials, *multiply each term of one polynomial* (the **multiplicand**) *by each term of the other polynomial* (the **multiplier**) *and combine the results to form the* **product.**

Both the multiplicand and the multiplier should be arranged, if possible, in descending or in ascending powers of a letter common to both.

A *complete check* on the result of the multiplication of two polynomials can be obtained by dividing the product by the multiplier (Art. 10). The quotient should be the multiplicand and the remainder should be zero. As the computations involved in checking in this way are usually long and difficult, it is customary, instead, to check (*partially*) by assigning to the letters numerical values, other than 0 and 1, that do not make either the multiplicand or the multiplier equal to zero. The product of the numerical values of the multiplicand and the multiplier should then equal the numerical value of the product.

EXAMPLE. Multiply $2m^2 - 5mn + 3n^2$ by $3m^2 + 4mn - 7n^2$.

$$\begin{array}{lllll}
2m^2 - 5mn + 3n^2 & & & & \textbf{(Multiplicand)} \\
\underline{3m^2 + 4mn - 7n^2} & & & & \textbf{(Multiplier)} \\
6m^4 - 15m^3n + 9m^2n^2 & & & & \\
\quad\quad 8m^3n - 20m^2n^2 + 12mn^3 & & & & \\
\quad\quad\quad\quad - 14m^2n^2 + 35mn^3 - 21n^4 & & & & \\
\hline
6m^4 - 7m^3n - 25m^2n^2 + 47mn^3 - 21n^4 & & & & \textbf{(Product)}
\end{array}$$

CHECK. Put $m = 2$, $n = 3$. Multiplicand $= 8 - 30 + 27 = 5$; multiplier $= 12 + 24 - 63 = -27$; product $= 96 - 168 - 900 + 2538 - 1701 = -135$. $5 \times (-27) = -135$.

Exercises

Perform the indicated multiplications.

1. $(3m^4)(13m^5)$.

2. $(4x^3y)(-9x^3y^5)$.

3. $(7a^2bd^4)(8a^3b^2c^5)$.

4. $(6xy^2z^3)(-\frac{7}{3}x^2z^6)$.

5. $(-3ar^4z^3)(-7a^3r^2z^5)$.

6. $(-\frac{15}{2}a^4t^2x^3)(\frac{4}{5}ab^2st^3)$.

7. $(\frac{4}{3}u^5vw^2x^3)(\frac{9}{2}uv^3x^2y)$.

8. $(\frac{8}{5}ab^2c^5d^3)(\frac{5}{4}a^2b^3c^2e^4)$.

9. $3(2a - 4b + c)$.

10. $7x(3x + 2y - 5z)$.

11. $4abc(a^2 + 7b^2 - 9c^2)$.

12. $-3p^2q(4p - 9q^2 + 7)$.

13. $7rs^2t(2st^2 - 5rt^3 + 3r^2s^3)$.

14. $5x^2y^2z^2(2x^3 - 3y^2 + 9z^4)$.

15. $-11a^2bc^3(3ab - 5a^2c - 2bc)$.

16. $3ux^2yz^2(5u^3 - 7x^3 + z^3)$.

17. $(4x + 7)(2x - 5)$.

18. $(3a - 4)(2a - 5)$.

19. $(5m - 2n)(3m + 8n)$.

20. $(7x - 3y)(2x - 5y)$.

21. $(x^2 - 4x - 3)(3x - 1)$.

22. $(a^2 - 3a - 2)(a^2 - 4a + 3)$.

23. $(x^2 - 2xy + 4y^2)(x^2 + 2xy - y^2)$.

24. $(y^2 + 3yz + z^2)(y^2 - yz - 7z^2)$.

25. $(x^2 + y^2 + z^2)(x + y + z)$.

26. $(a + x)(a^3 + 3a^2x + 3ax^2 + x^3)$.

27. $(4x + 3x^3 - 5x^2)(2x - 7 + x^2)$.

28. $(x - 2x^3 + 4x^4 - 7 - 4x^2)(2 - x^3)$.

10. Division of Polynomials.* If one quantity is divided by another, the quantity that is divided is the **dividend,** the quantity by which it is divided is the **divisor,** the result of the division is the **quotient,** and the quantity left over after the division is the **remainder.**

Thus, if 46 (the dividend) is divided by 7 (the divisor), the quotient is 6 and the remainder is 4.

To divide a monomial by a monomial, *multiply the quotient of the numerical coefficients by the quotients of the literal factors.*

Thus,

$$\frac{18x^3y^2}{6xy^5} = \frac{18}{6}\cdot\frac{x^3}{x}\cdot\frac{y^2}{y^5} = 3\cdot x^2\cdot\frac{1}{y^3} = \frac{3x^2}{y^3}.$$

$$\frac{15ab^2c^4}{9a^3bc^2d} = \frac{15}{9}\cdot\frac{a}{a^3}\cdot\frac{b^2}{b}\cdot\frac{c^4}{c^2}\cdot\frac{1}{d} = \frac{5}{3}\cdot\frac{1}{a^2}\cdot b\cdot c^2\cdot\frac{1}{d} = \frac{5bc^2}{3a^2d}.$$

To divide a polynomial by a monomial, *divide each term of the polynomial by the monomial and combine the results.*

Thus,

$$\frac{21x^3y^4 - 7x^4yz - 28xy^4z^3}{14x^2y^2z} = \frac{21x^3y^4}{14x^2y^2z} - \frac{7x^4yz}{14x^2y^2z} - \frac{28xy^4z^3}{14x^2y^2z}$$

$$= \frac{3xy^2}{2z} - \frac{x^2}{2y} - \frac{2y^2z^2}{x}.$$

To divide one polynomial by another polynomial, *we proceed in the following way:*

1. *Arrange both the dividend and the divisor in descending powers of a letter common to both.*
2. *To obtain the first term of the quotient, divide the first term of the dividend by the first term of the divisor.*
3. *Multiply the entire divisor by the first term of the quotient and subtract the result from the dividend.*
4. *Consider the result obtained from Step* 3 *as a new dividend and repeat the operation.*
5. *Continue in this way until a remainder is obtained that is either zero or of lower degree than the divisor.*

A *complete check* for division may be obtained from the relation:

Dividend = Divisor × Quotient + Remainder,

that is, if we multiply the quotient by the divisor and add the remainder we should obtain the dividend.

For most purposes, it is adequate to obtain a *partial check* by assuming, for the letters involved, numerical values, other than 0 and 1, that do not make either the dividend or the divisor equal to zero.

* **Division by factoring will be discussed in Chap. II.**

EXAMPLE. Divide $22x^3 + 6x^5 - 13x^4 - 21x - 15x^2 + 58$ by $2x^2 + 7 - x$.

$$\begin{array}{rl}
 & 3x^3 - 5x^2 - 2x + 9 \quad \textbf{(Quotient)} \\
\textbf{(Divisor)}\ 2x^2 - x + 7 & \overline{)\,6x^5 - 13x^4 + 22x^3 - 15x^2 - 21x + 58} \quad \textbf{(Dividend)} \\
 & 6x^5 - 3x^4 + 21x^3 \\
 & \overline{\quad - 10x^4 + x^3 - 15x^2 - 21x + 58} \\
 & \quad - 10x^4 + 5x^3 - 35x^2 \\
 & \overline{\qquad - 4x^3 + 20x^2 - 21x + 58} \\
 & \qquad - 4x^3 + 2x^2 - 14x \\
 & \overline{\qquad\qquad 18x^2 - 7x + 58} \\
 & \qquad\qquad 18x^2 - 9x + 63 \\
 & \overline{\qquad\qquad\qquad 2x - 5} \quad \textbf{(Remainder)}
\end{array}$$

The quotient is $3x^3 - 5x^2 - 2x + 9$ and the remainder is $2x - 5$.

CHECK. Put $x = 2$. Divisor $= 8 - 2 + 7 = 13$; dividend $= 192 - 208 + 176 - 60 - 42 + 58 = 116$; quotient $= 24 - 20 - 4 + 9 = 9$; remainder $= 4 - 5 = -1$. $116 = 13 \times 9 - 1$.

Exercises

Perform the indicated divisions and check by multiplication.

1. $52x^8 \div 13x^3$. **2.** $35a^3b^5 \div 14ab^2$. **3.** $57x^4yz^2 \div 76x^3y^3$.

4. $\dfrac{72t^4x^6y^2z^3}{96t^2x^7y^5z}$. **5.** $\dfrac{154a^3b^8c^2d^4}{66a^6b^2c^7d^3}$. **6.** $\dfrac{54x^9y^5z^3u^2}{30x^8y^3v^3w}$.

7. $(21x^4 - 33x^3 - 18x^2) \div 3x^2$. **8.** $(45t^7 - 20t^5 + 9t^2) \div 15t^3$.

9. $\dfrac{21x^2y^3z + 16x^4y^3 + 24yz^3}{6xyz^2}$. **10.** $\dfrac{48a^5c^3 + 60ab^5c^4 + 18a^4b}{24a^2bc^3}$.

11. $(2x^2 + x - 15) \div (x + 3)$. **12.** $(6m^2 + 3m - 23) \div (2m + 5)$.

13. $\dfrac{4x^3 - 16x^2 - 3x + 17}{2x - 7}$. **14.** $\dfrac{6x^2 + 13xy - 9y^2}{3x - 4y}$.

Perform the indicated divisions and check by substituting numerical values for the letters.

15. $(15x^2 - 14x + 4) \div (3x + 2)$. **16.** $(6a^2 + 21a + 17) \div (2a + 3)$.

17. $(8a^2 - 26ab + 7b^2) \div (2a - 5b)$. **18.** $(4z^6 - 14z^3 + 3) \div (2z^3 - 3)$.

19. $(3a^4b^2 + 8a^2bt^3 + t^6) \div (a^2b + 4t^3)$. **20.** $(6h^3 - 13h^2 + 1) \div (2h - 7)$.

21. $\dfrac{10v^3 + 17v^2 - 15v - 56}{2v^2 + 7v + 9}$. **22.** $\dfrac{x^4 + x^2y^2 + y^4}{x^2 + xy + y^2}$.

23. $\dfrac{21r^3s^3 + 5r^2s^2t + 20rst^2}{3r^2s^2 - rst + 4t^2}$. **24.** $\dfrac{x^6 - y^6}{x^2 - y^2}$.

25. $(m^4n^4 + 2m^3n^3z - 3m^2n^2z^2 + 2mnz^3 + 9z^4) \div (mn + 3z)$.

26. $(k^3 + 9k + k^4 + 21 - 5k^2) \div (4k + 6 + k^2)$.

27. $(9y^4 + 10y^5 + 34 - 5y^3 - 10y^2 - 10y) \div (2y + 3)$.

28. $(20x^3 + 41x^2 + 3x^6 - 9x^5 - 2x - 3 - 7x^4) \div (x^3 - 3x - 4)$.

Chapter 2

Factoring and Fractions

11. Special Products. The following products occur so frequently in mathematical computations that they should be memorized. The correctness of each equation should also be verified by multiplying together the factors in the second member and showing that the result can be reduced to the expression given in the first member.

$$\begin{aligned}
\text{I.}\quad & ab + ac = a(b + c).\\
\text{II.}\quad & a^2 - b^2 = (a - b)(a + b).\\
\text{III.}\quad & a^2 + 2ab + b^2 = (a + b)^2.\\
\text{IV.}\quad & a^2 - 2ab + b^2 = (a - b)^2.\\
\text{V.}\quad & x^2 + (a + b)x + ab = (x + a)(x + b).\\
\text{VI.}\quad & acx^2 + (ad + bc)x + bd = (ax + b)(cx + d).\\
\text{VII.}\quad & a^3 + b^3 = (a + b)(a^2 - ab + b^2).\\
\text{VIII.}\quad & a^3 - b^3 = (a - b)(a^2 + ab + b^2).
\end{aligned}$$

Exercises

Write out the following products.

1. $2a(3x + 9y)$.
2. $5st(3s - 4t + 2)$.
3. $(x - 3)(x + 3)$.
4. $2x^2y^3(3x^2 + 7xy - 4y^2)$.
5. $(100 - 3)(100 + 3)$.
6. 49×51.
7. $(3x + 4y)^2$.
8. $(5a - 3b)^2$.
9. $(x + 7)(x + 4)$.
10. $(3a - 2b)(2a + 3b)$.
11. $(3x - 5)(3x + 1)$.
12. $(4y - 5)(3y + 7)$.
13. $(3x + y)(9x^2 - 3xy + y^2)$.
14. $(2x - 5y)(4x^2 + 10xy + 25y^2)$.
15. $(z^2 + 3)(z^2 - 7)$.
16. $(xy - 5z)(xy + 3z)$.
17. $xy(x - y)(x + y)$.
18. $ab(a - 2b)(a - 3b)$.
19. $(2a^2 + 5b^2)(3a^2 + 2b^2)$.
20. $(u^3 - 2v^3)(3u^3 + 5v^3)$.
21. $(x - 2)(x + 2)(x^2 + 4)$.
22. $(a - b)(a + b)(a^2 + b^2)$.
23. $(x - y)(x^2 + xy + y^2)(x + y)(x^2 - xy + y^2)$.
24. $(a + b + c)(a + b - c)(a - b + c)(a - b - c)$.

12. Factoring by Inspection. If a polynomial can be recognized as belonging to one of the types given in the first members of the formulas of the preceding article, it can be factored by inspection.

In the following discussion of factoring, we shall factor each polynomial into as many factors as possible such that, for each factor, all the coefficients and all the exponents are integers. Such a factor is called a **prime factor** of the given polynomial.

EXAMPLE 1. Factor $18zx^2 - 32zy^2$.

By formula I of Art. 11, we have

$$18zx^2 - 32zy^2 = 2z(9x^2 - 16y^2).$$

By formula II, $9x^2 - 16y^2 = (3x)^2 - (4y)^2 = (3x - 4y)(3x + 4y)$.
Hence, $18zx^2 - 32zy^2 = 2z(3x - 4y)(3x + 4y)$.

EXAMPLE 2. Factor $x^2 + 8x - 84$.

We can apply formula V of Art. 11 if we can find two numbers, a and b, such that $a + b = 8$ and $ab = -84$. By trial, we find that $a = 14$ and $b = -6$ satisfy these conditions. Hence,

$$x^2 + 8x - 84 = (x + 14)(x - 6).$$

EXAMPLE 3. Factor $6y^2 - 11y - 35$.

We first seek four numbers a, b, c, and d, such that

$$ac = 6, \quad ad + bc = -11, \quad \text{and} \quad bd = -35.$$

By trial, we find that $a = 3$, $b = 5$, $c = 2$, and $d = -7$ satisfy these conditions.
Hence, $6y^2 - 11y - 35 = (3y + 5)(2y - 7)$.

EXAMPLE 4. Factor $m^6 - n^6$.

By formulas II, VII, and VIII, we have

$$\begin{aligned} m^6 - n^6 &= (m^3)^2 - (n^3)^2 = (m^3 - n^3)(m^3 + n^3) \\ &= (m - n)(m^2 + mn + n^2)(m + n)(m^2 - mn + n^2). \end{aligned}$$

Exercises

Factor the following polynomials into their prime factors. Monomial factors, if there are any, should be factored out first.

1. $15xy + 21y^2$.
2. $14uxy + 6uyz$.
3. $36x^2 - 25y^2$.
4. $49r^2s^2 - 25t^2$.
5. $4p^2 + 36pq + 81q^2$.
6. $121a^2 - 132ab + 36b^2$.
7. $18ax^2 - 12ax + 2a$.
8. $4abc^2 - 36abd^2$.
9. $a^2b^2 - 9c^2$.
10. $a^2b^2 - 4abcd + 4c^2d^2$.
11. $8u^4 + 27uv^3$.
12. $32rs^3 - 108rt^3$.
13. $5z^2 - 18z - 8$.
14. $3a^2 - a - 10$.
15. $20a^2 + 47ab + 21b^2$.
16. $7ax^2 - 56axy + 105ay^2$.
17. $x^4 - 2x^2y^2 + y^4$.
18. $x^4 - 13x^2 + 36$.
19. $x^3y^3 - 8c^6$.
20. $125z^3 + 8x^3y^3$.
21. $(x + y)^2 - z^2$.
22. $(a + b)^2 - 2(a + b) - 15$.
23. $(u + v)^2 - 4(u - v)^2$.
24. $x^2 - (y - z)^2$.
25. $(a + 1)^3 - b^3$.
26. $(a + b)^3 + 1$.

13. Factoring by Grouping. Many expressions that do not come directly under one of the types given in Art. 11 can be reduced to one of these types by a suitable grouping of the terms.

EXAMPLE 1. Factor $6bc - ad - 2bd + 3ac$.

After rearranging the terms, we have

$$6bc - 2bd + 3ac - ad = 2b(3c - d) + a(3c - d) = (2b + a)(3c - d).$$

EXAMPLE 2. Factor $x^3 + 7x^2 - 9x - 63$.

$$\begin{aligned} x^3 + 7x^2 - 9x - 63 &= x^2(x + 7) - 9(x + 7) \\ &= (x^2 - 9)(x + 7) = (x - 3)(x + 3)(x + 7). \end{aligned}$$

EXAMPLE 3. Factor $4a^4 - 25x^6 + 4a^2b^2 - 9 - 30x^3 + b^4$.

$$\begin{aligned} 4a^4 - 25x^6 + 4a^2b^2 - 9 - 30x^3 + b^4 &= (4a^4 + 4a^2b^2 + b^4) - (25x^6 + 30x^3 + 9) \\ = (2a^2 + b^2)^2 - (5x^3 + 3)^2 &= (2a^2 + b^2 - 5x^3 - 3)(2a^2 + b^2 + 5x^3 + 3). \end{aligned}$$

Sometimes it is necessary to add and subtract one or more terms, as in the following example.

EXAMPLE 4. Factor $x^4 + 3x^2y^4 + 4y^8$.

$$\begin{aligned} x^4 + 3x^2y^4 + 4y^8 &= x^4 + 4x^2y^4 + 4y^8 - x^2y^4 = (x^2 + 2y^4)^2 - (xy^2)^2 \\ &= (x^2 + 2y^4 - xy^2)(x^2 + 2y^4 + xy^2). \end{aligned}$$

Exercises

Factor the following polynomials into their prime factors.

1. $6ax + 9bx + 10ay + 15by$.
2. $2ux - 3vy + uy - 6vx$.
3. $x^2 + 4xy - 3xz - 12yz$.
4. $10a^2 + 4ac - 15ab - 6bc$.
5. $2x^3 + 5x^2 - 8x - 20$.
6. $4y^3 - y^2 - 36y + 9$.
7. $x^2 + 4xy + 4y^2 - 7x - 14y$.
8. $x^2 + 4xy + 4y^2 - 9$.
9. $9a^2 - 15ac + 5bc - b^2$.
10. $a^2 - 4b^2 + 3a - 6b$.
11. $u^2 + 6uv + 9v^2 + u + 3v - 6$.
12. $x^2 + 4xy + 4y^2 - 4z^2 + 12z - 9$.
13. $x^4 - 4x^2y^2 - x^2z^2 + 4y^2z^2$.
14. $(a - b)^2 - 7(a - b) + 10$.
15. $x^4 + x^2y^2 + y^4$.
16. $x^2 - ax - bx - 3x + ab + 3a$.
17. $x^2 - 4xy + 4y^2 - 5x + 10y + 6$.
18. $x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$.
19. $y^3 + 27z^3 - 15xz - 5xy$.
20. $x^4 - 3x^2y^2 + y^4$.

14. Lowest Common Multiple and Highest Common Factor. One polynomial is a **multiple** of another polynomial if it has that polynomial as a factor. It is a **common multiple** of two or more polynomials if it is a multiple of each of those polynomials. It is their **lowest common multiple** (L.C.M.) if it is their common multiple that has the smallest possible number of prime factors.

To find the L.C.M. of two or more polynomials, first factor each polynomial into its prime factors. Then the product of every factor occurring in any of the given polynomials, to the highest power that it occurs in any one of them, is the required L.C.M.

The L.C.M., when it has been found, should be left in the factored form.

Similarly, the **highest common factor** (H.C.F.) of two or more polynomials is the polynomial containing the greatest number of prime factors which is a factor of each of the given polynomials. *It is the product of every factor occurring in all of the given polynomials to the lowest power to which it occurs in any one of them.*

EXAMPLE 1. Find the L.C.M. and the H.C.F. of $30a^2b^3c^3e$, $12ab^4cd^2$, and $54a^3b^2f$.

We have
$$30a^2b^3c^3e = 2 \cdot 3 \cdot 5 \cdot a^2 \cdot b^3 \cdot c^3 \cdot e,$$
$$12ab^4cd^2 = 2^2 \cdot 3 \cdot a \cdot b^4 \cdot c \cdot d^2,$$
$$54a^3b^2f = 2 \cdot 3^3 \cdot a^3 \cdot b^2 \cdot f.$$

Hence, the L.C.M. is $2^2 3^3 5a^3b^4c^3d^2ef$ and the H.C.F. is $2 \cdot 3ab^2$.

EXAMPLE 2. Find the L.C.M. and the H.C.F. of $x^2 - 4$, $x^2 + 4x + 4$, and $3x^2 + x - 10$.

We have
$$x^2 - 4 = (x - 2)(x + 2),$$
$$x^2 + 4x + 4 = (x + 2)^2,$$
$$3x^2 + x - 10 = (x + 2)(3x - 5).$$

Hence, the L.C.M. is $(x - 2)(x + 2)^2(3x - 5)$ and the H.C.F. is $x + 2$.

Exercises

Find the L.C.M. of each of the following sets of polynomials.

1. $9a^2bcd^3$, $36b^6c^2d$, $24abc^4$.
2. $24w^2xy^3z$, $60v^2wx^3$, $18xyz^4$, $20x^2y^6z^2$.
3. $x^2 - y^2$, $x^2 + 2xy + y^2$, $x^2 - 2xy + y^2$.
4. $ax^2 + axy$, $a^2xy - a^2y^2$, $ax^2 - ay^2$.
5. $x^2 - x - 6$, $x^2 + 7x + 10$, $x^2 + 2x - 15$.
6. $x^2 - 3xy + 2y^2$, $a^2 + 2ab + b^2$, $ax - ay - by + bx$.

Find the L.C.M. and the H.C.F. of each of the following sets of polynomials.

7. $x^3 - 5x^2 + 6x$, $x^4 - 4x^2$, $x^3 - 3x^2 - 4x + 12$.
8. $(x + y)^2 - z^2$, $(x + z)^2 - y^2$, $x^2 - (y + z)^2$.
9. $9x^2 - 6xy + y^2 - 4$, $(3x + 2)^2 - y^2$.
10. $x^3 - x^2y$, $x^2y^2 + xy^3 + y^4$, $x^3 - y^3$.

15. Simplification of Fractions. The value of the fraction a/b is the number obtained by dividing the number a by the number b. We call a the **numerator** and b the **denominator** of the fraction. The denominator must be different from zero since division by zero is excluded from mathematical operations (Art. 4).

The value of a fraction is not changed if its numerator and denominator are both multiplied, or both divided, by the same number different from zero.

Thus, $\dfrac{4}{11} = \dfrac{4 \times 5}{11 \times 5} = \dfrac{20}{55}$; $\dfrac{12}{-5} = \dfrac{-12}{5} = -\left(\dfrac{12}{5}\right)$;

$$\frac{65}{39} = \frac{5 \times 13}{3 \times 13} = \frac{5}{3}; \quad \frac{12x^2yz^3}{8xy^3w} = \frac{3xz^3 \cdot 4xy}{2y^2w \cdot 4xy} = \frac{3xz^3}{2y^2w}.$$

A fraction is said to be **simplified,** or to be **reduced to its lowest terms,** if all of the factors common to both its entire numerator and its entire denominator have been removed by division according to the preceding theorem.

A fraction must *not* be simplified by cancelling merely a single term that is common to both the numerator and the denominator.

Thus, the fraction $\dfrac{4x + 3y}{4x - z}$ is already in its lowest terms. It cannot be simplified further by cancelling the term $4x$ in the numerator and the denominator.

EXAMPLE 1. Simplify: $\dfrac{(3x + 2)(x - 4)}{(x^2 + 5)(x - 4)}$.

By dividing both the numerator and the denominator by $x - 4$, we obtain

$$\frac{(3x + 2)(x - 4)}{(x^2 + 5)(x - 4)} = \frac{3x + 2}{x^2 + 5}.$$

EXAMPLE 2. Simplify: $\dfrac{2x^2 + 20x + 42}{6x^2 + 8x - 30}$.

If we factor both the numerator and the denominator and divide both of them by the common factor $2(x + 3)$, we have

$$\frac{2x^2 + 20x + 42}{6x^2 + 8x - 30} = \frac{2(x + 3)(x + 7)}{2(x + 3)(3x - 5)} = \frac{x + 7}{3x - 5}.$$

EXAMPLE 3. Simplify: $\dfrac{2a^2b + 3ab^2c + ac}{5b^2c + 3ab^2c + a^2c}$.

There are no factors common to both the numerator and the denominator. The fraction, as given, is in its lowest terms.

From the definition of a fraction, it follows that *the quotient of two polynomials may be written as a fraction having the dividend as its numerator and the divisor as its denominator.* That is

$$a \div b = \frac{a}{b}.$$

The resulting fraction should be simplified, as in the preceding examples.

EXAMPLE 4. Divide $4a^2bx^3y$ by $6ab^4xy^5$.

The required quotient may be written in the form

$$\frac{4a^2bx^3y}{6ab^4xy^5} = \frac{2ax^2 \cdot 2abxy}{3b^3y^4 \cdot 2abxy} = \frac{2ax^2}{3b^3y^4}.$$

EXAMPLE 5. Divide $3x^2 - 2x - 1$ by $3x^2 + 13x + 4$.

The quotient equals: $\dfrac{3x^2 - 2x - 1}{3x^2 + 13x + 4} = \dfrac{(3x + 1)(x - 1)}{(3x + 1)(x + 4)} = \dfrac{x - 1}{x + 4}$.

Exercises

Reduce each fraction to its lowest terms.

1. $\frac{91}{39}$.
2. $\frac{108a}{24a}$.
3. $\frac{65xy^2z^4}{30x^2yz^2}$.
4. $\frac{42a^3b^5c^2d}{28a^2bc^3e^3}$.
5. $\frac{15x^2(x-2y)}{9y(x-2y)}$.
6. $\frac{(3x+2y)^2}{9x^2-4y^2}$.
7. $\frac{x^2+2xy}{3xy^2+6y^3}$.
8. $\frac{x^2-3x-10}{3x^2+7x+2}$.
9. $\frac{2x^2+7x+5}{4x^2+4x-15}$.
10. $\frac{12x^2-5x-2}{6x^2-19x+10}$.
11. $\frac{xy+ax+by+ab}{xy+ax-by-ab}$.
12. $\frac{(x+2y)^2-3x-6y}{x^2y+2xy^2-3xy}$.
13. $\frac{(a+b)^2-(c+d)^2}{(a+c)^2-(b+d)^2}$.
14. $\frac{x^3+y^3}{4x^2+xy-3y^2}$.
15. $\frac{2x^3+2x^2y+2xy^2}{x^3-y^3}$.
16. $\frac{r^2s^2-7rst^2+12t^4}{r^2s^2-rst^2-6t^4}$.

Write each of the following quotients as a fraction in its lowest terms.

17. $84 \div 132$.
18. $42x^3y^5z^2 \div 63xy^2w$.
19. $(n^3-3n^2+9n) \div (n^3+27)$.
20. $(x^2-8x+12) \div (x^2+8x-20)$.

16. Multiplication and Division of Fractions. *To multiply two fractions, form a new fraction whose numerator is the product of the numerators of the given fractions and whose denominator is the product of their denominators;* that is

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}.$$

The resulting fraction should be simplified, as in the preceding article. For this purpose, it is usually best to factor the numerators and denominators of the given fractions before performing the multiplications so that the common factors can be found and removed.

If one of the numbers to be multiplied is integral, it may be thought of as a fraction with denominator unity, thus

$$\frac{a}{b}\cdot c=\frac{a}{b}\cdot\frac{c}{1}=\frac{ac}{b}.$$

EXAMPLE 1. Multiply $\frac{21}{22}$ by $\frac{55}{14}$.

We have $$\frac{21}{22}\times\frac{55}{14}=\frac{3\times 7}{2\times 11}\times\frac{5\times 11}{2\times 7}=\frac{3\times 7\times 5\times 11}{2\times 11\times 2\times 7}=\frac{15}{4}.$$

EXAMPLE 2. Multiply $\frac{x^3+8y^3}{x^2-9y^2}$ by $\frac{x^2-4xy+3y^2}{x^2-2xy-8y^2}$.

$$\frac{x^3+8y^3}{x^2-9y^2}\cdot\frac{x^2-4xy+3y^2}{x^2-2xy-8y^2}=\frac{(x+2y)(x^2-2xy+4y^2)}{(x-3y)(x+3y)}\cdot\frac{(x-3y)(x-y)}{(x+2y)(x-4y)}$$
$$=\frac{(x^2-2xy+4y^2)(x-y)}{(x+3y)(x-4y)}.$$

To divide one fraction by another, invert the divisor fraction and proceed as in multiplication; that is,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}.$$

The **reciprocal** of a number is *unity divided by that number.*

Thus, the reciprocal of 3 is $\frac{1}{3}$; of a is $\frac{1}{a}$; of $\frac{c}{d}$ is $1 \div \frac{c}{d}$ or $\frac{d}{c}$.

With this definition of the reciprocal of a number, we may state the law of division of fractions as follows: *to divide one fraction by another, multiply the dividend fraction by the reciprocal of the divisor.*

EXAMPLE 1. Divide $\frac{40}{91}$ by $\frac{15}{14}$.

$$\frac{40}{91} \div \frac{15}{14} = \frac{40}{91} \times \frac{14}{15} = \frac{8 \times 5 \times 2 \times 7}{13 \times 7 \times 3 \times 5} = \frac{16}{39}.$$

EXAMPLE 2. Divide $\frac{6a^2b}{5c^2d}$ by $\frac{9ab^3}{10c^4d^5}$.

$$\frac{6a^2b}{5c^2d} \div \frac{9ab^3}{10c^4d^5} = \frac{6a^2b}{5c^2d} \cdot \frac{10c^4d^5}{9ab^3} = \frac{6a^2b \cdot 10c^4d^5}{5c^2d \cdot 9ab^3} = \frac{4ac^2d^4}{3b^2}.$$

EXAMPLE 3. Divide $\frac{x^2 + x - 2}{x^2 - 7x}$ by $\frac{x^2 + 2x}{x^2 - 13x + 42}$.

$$\frac{x^2 + x - 2}{x^2 - 7x} \div \frac{x^2 + 2x}{x^2 - 13x + 42} = \frac{x^2 + x - 2}{x^2 - 7x} \cdot \frac{x^2 - 13x + 42}{x^2 + 2x}$$

$$= \frac{(x - 1)(x + 2)(x - 6)(x - 7)}{x(x - 7)x(x + 2)} = \frac{(x - 1)(x - 6)}{x^2}.$$

Exercises

Find the reciprocals of the following expressions.

1. -1. **2.** $\frac{5}{9}$. **3.** $5x + 1$. **4.** $\frac{2a - b}{3a + 5b}$.

Perform the indicated operations and simplify the results.

5. $\frac{22}{63} \times \frac{14}{121}$. **6.** $\frac{5}{51} \div \frac{1}{17}$. **7.** $\frac{5a}{2b^2} \cdot \frac{8b^3}{15a^4}$.

8. $\frac{5x^3}{7yz^2} \div \frac{10x^3}{63y^3z}$. **9.** $4x^4yz \cdot \frac{19ab^2}{72xy^5z^7}$.

10. $\frac{4r^4s^2t^7}{21r^7v^2w^5} \cdot \frac{35r^2v^5w^8}{6r^3s^6t^4}$. **11.** $\frac{a^3 - 5a^2}{b^2} \cdot \frac{2b}{ac - 5c}$.

12. $\frac{x + 3y}{u - 2v} \cdot \frac{3u - 6v}{5x + 15y}$. **13.** $\frac{xy + 4y^2}{6x^2} \cdot \frac{y^2}{x^2 + 4xy}$.

14. $\frac{2y - xy}{5w - yw} \cdot \frac{3y^2 - 15y}{2xz - 4z}$. **15.** $\frac{3x - 3y}{(2x + y)^2} \cdot \frac{4x^2 - y^2}{2x^2 - 2y^2}$.

16. $\dfrac{x^2-2x-8}{x^2+3x-28}\cdot\dfrac{2x^2+13x-7}{2x^2+7x+6}.$ **17.** $\dfrac{4x^2-x-14}{8x^2+18x+7}\cdot(2x+1).$

18. $\dfrac{z^2+3z-10}{3z^2-4z-4}\cdot(3z+2).$ **19.** $\dfrac{2y^2-y-6}{6y^2+17y+12}\cdot\dfrac{3y^2+y-4}{2y^2+7y-4}.$

20. $\dfrac{3x^2+11xy-4y^2}{3x^2+8xy+4y^2}\cdot\dfrac{x^2+3xy+2y^2}{3x^2+5xy+2y^2}.$

21. $\dfrac{x^2+4xy+4y^2}{4x^2+12xy+9y^2}\cdot\dfrac{2x^2+xy-3y^2}{x^2-xy-6y^2}.$

22. $\dfrac{(x-2)^2}{3x+1}\div\dfrac{2x-4}{9x+3}.$ **23.** $\dfrac{2x+1}{3x-2}\div\dfrac{4x^2-1}{9x^2-4}.$

24. $\dfrac{16h^2-k^2}{h^2-4k^2}\div\dfrac{8h+2k}{3h-6k}.$ **25.** $\dfrac{x^2-6x+9}{2x-3}\div\dfrac{5x-15}{4x^2-12x+9}.$

26. $(3u^2-uv-4v^2)\div\dfrac{u^2-v^2}{2uv}.$ **27.** $\dfrac{2r^2-rs}{3rs-s^2}\div\dfrac{rs^2-2r^2s}{2rs-6r^2}.$

28. $\dfrac{a^2-5ab+4b^2}{a^2-9ab+18b^2}\div\dfrac{2a^2+11ab+5b^2}{a^2+7ab+10b^2}.$

17. Addition and Subtraction of Fractions. *The sum of two or more fractions having the same denominator is equal to the sum of the numerators divided by the common denominator;* that is,

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c},\qquad \frac{a}{d}+\frac{b}{d}+\frac{c}{d}=\frac{a+b+c}{d},$$

and so on for any number of such fractions.

If some of the fractions are to be subtracted, instead of being added, subtract, instead of adding, the corresponding numerators; thus,

$$\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c},\qquad \frac{a}{d}+\frac{b}{d}-\frac{c}{d}=\frac{a+b-c}{d},$$

and so on.

Example 1. Perform the addition: $\dfrac{2x-1}{3x+2}+\dfrac{4-x}{3x+2}.$

$$\frac{2x-1}{3x+2}+\frac{4-x}{3x+2}=\frac{(2x-1)+(4-x)}{3x+2}=\frac{x+3}{3x+2}.$$

Example 2. Perform the indicated operations: $\dfrac{3x-5}{x^2+1}-\dfrac{4x-9}{x^2+1}+\dfrac{5x+2}{x^2+1}.$

$$\frac{3x-5}{x^2+1}-\frac{4x-9}{x^2+1}+\frac{5x+2}{x^2+1}=\frac{(3x-5)-(4x-9)+(5x+2)}{x^2+1}=\frac{4x+6}{x^2+1}.$$

Example 3. Perform the indicated operations and simplify the result:

$$\frac{3x^2-2}{x-2}-\frac{x^2+3x+4}{x-2}-\frac{8-x^2}{2-x}.$$

To make the last denominator the same as the other two, we shall multiply the numerator and denominator of the last fraction by -1.

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$$\frac{3x^2-2}{x-2}-\frac{x^2+3x+4}{x-2}-\frac{x^2-8}{x-2}=\frac{(3x^2-2)-(x^2+3x+4)-(x^2-8)}{x-2}$$
$$=\frac{x^2-3x+2}{x-2}=x-1.$$

If the fractions to be combined do not have the same denominators, first transform all of them into fractions having the L.C.M. (Art. 14) *of the denominators as their common denominator, then combine the resulting fractions having the same denominator.*

EXAMPLE 4 Perform the indicated operations:

$$\frac{3x-4}{x+2}+\frac{5x-3}{x+1}-\frac{2x+7}{x}.$$

The L.C.M. of the denominators is $(x+2)(x+1)x$.

To transform the first fraction into an equivalent fraction having this L.C.M. as its denominator, we must multiply its numerator and denominator by

$$(x+2)(x+1)x \div (x+2) = (x+1)x;$$

that is, by the quotient obtained by dividing the L.C.M. of the denominators by the denominator of the given fraction.

Similarly, we must multiply the numerator and denominator of the second and third fractions, respectively, by

$$(x+2)(x+1)x \div (x+1) = (x+2)x \quad \text{and} \quad (x+2)(x+1)x \div x = (x+2)(x+1).$$

These multiplications give us

$$\frac{3x-4}{x+2}+\frac{5x-3}{x+1}-\frac{2x+7}{x}=\frac{(3x-4)(x+1)x}{(x+2)(x+1)x}+\frac{(5x-3)(x+2)x}{(x+2)(x+1)x}-\frac{(2x+7)(x+2)(x+1)}{(x+2)(x+1)x}$$
$$=\frac{(3x-4)(x+1)x+(5x-3)(x+2)x-(2x+7)(x+2)(x+1)}{(x+2)(x+1)x}$$
$$=\frac{(3x^3-x^2-4x)+(5x^3+7x^2-6x)-(2x^3+13x^2+25x+14)}{(x+2)(x+1)x}$$
$$=\frac{6x^3-7x^2-35x-14}{x^3+3x^2+2x}.$$

A **mixed expression**, such as $a+\frac{b}{c}$, where a is integral, may be combined by first writing the quantity a as a fraction, in the form $\frac{a}{1}$, and adding the fractions according to the preceding rule.

EXAMPLE 6. Perform the indicated operations: $x-y+\dfrac{3x^2+2xy}{x+y}$.

$$x-y+\frac{3x^2+2xy}{x+y}=\frac{x-y}{1}+\frac{3x^2+2xy}{x+y}=\frac{(x-y)(x+y)}{x+y}+\frac{3x^2+2xy}{x+y}$$
$$=\frac{(x^2-y^2)+(3x^2+2xy)}{x+y}=\frac{4x^2+2xy-y^2}{x+y}.$$

Exercises

Perform the indicated operations.

1. $\frac{5}{7} - \frac{1}{7} + \frac{2}{7} - \frac{3}{7}$.

2. $2 - \frac{4}{3} - \frac{6}{5} + \frac{13}{15}$.

3. $\dfrac{6}{x} + \dfrac{3x - 7}{x} - \dfrac{4 - x}{x}$.

4. $\dfrac{4y + 7}{3y + 4} - \dfrac{2y - 8}{3y + 4} - \dfrac{3 - 5y}{3y + 4}$.

5. $\dfrac{7x + 2}{10} - \dfrac{2x - 9}{10}$.

6. $\dfrac{3a - 2b}{6} - \dfrac{5a + 3b}{14}$.

7. $\dfrac{4}{x^3} - \dfrac{3}{2x^2} + \dfrac{5}{x} - 2$.

8. $\dfrac{2a}{bc} - \dfrac{b}{ca} + \dfrac{3c}{ab}$.

9. $\dfrac{s}{x} - \dfrac{t}{y} - \dfrac{2s + 3t}{xy}$.

10. $\dfrac{a + x}{b + y} - \dfrac{a}{b}$.

11. $\dfrac{5}{x + 3} - \dfrac{2}{x - 1}$.

12. $\dfrac{2}{x - 5} + \dfrac{3}{2x + 1}$.

13. $\dfrac{8x - 5}{(x - 1)(x + 4)} - \dfrac{2}{x + 4} + \dfrac{5}{x - 1}$.

14. $\dfrac{3x - 7}{(x + 1)^2} - \dfrac{5}{x + 1} - \dfrac{3}{x - 2}$.

15. $3 - 4x - \dfrac{5 - 3x - 7x^2}{x + 2}$.

16. $3z^2 + 7z + 2 - \dfrac{z^2 - 10}{2z - 5}$.

17. $\dfrac{5x - 1}{x^2 + 2x - 3} - \dfrac{4x + 1}{x^2 - x - 12}$.

18. $\dfrac{3t + 5}{t^2 - 4} + \dfrac{6 - 4t}{t^2 - 4t + 4}$.

19. $\dfrac{x}{y - z} + \dfrac{y}{z - x} + \dfrac{z}{x - y}$.

20. $\dfrac{1}{x + y} + \dfrac{1}{y + z} + \dfrac{1}{z + x}$.

21. $\dfrac{5}{x + 1} - \dfrac{3}{x + 3} - \dfrac{2x + 7}{(x + 1)^2} + \dfrac{3x - 5}{(x + 3)^2}$.

18. Complex Fractions. A **complex fraction** is one in which at least one term of the numerator or of the denominator is, itself, a fraction.

To simplify a complex fraction, reduce the numerator and denominator each, separately, to a simple fraction. Then divide the numerator by the denominator.

EXAMPLE 1. Simplify: $\dfrac{\dfrac{a + b}{b} - \dfrac{b}{a + b}}{\dfrac{1}{b} + \dfrac{2}{a}}$.

$$\text{Numerator} = \frac{a + b}{b} - \frac{b}{a + b} = \frac{(a + b)^2 - b^2}{b(a + b)} = \frac{a^2 + 2ab}{b(a + b)}.$$

$$\text{Denominator} = \frac{1}{b} + \frac{2}{a} = \frac{a + 2b}{ab}.$$

$$\text{Hence, } \frac{\dfrac{a + b}{b} - \dfrac{b}{a + b}}{\dfrac{1}{b} + \dfrac{2}{a}} = \frac{\dfrac{a^2 + 2ab}{b(a + b)}}{\dfrac{a + 2b}{ab}} = \frac{a(a + 2b)}{b(a + b)} \cdot \frac{ab}{a + 2b} = \frac{a^2}{a + b}.$$

Frequently the computation can be shortened considerably by multiplying both the numerator and the denominator by the L.C.M. of

the denominators of all the fractions that appear in either the numerator or the denominator of the given expression, as in the following example.

EXAMPLE 2. Simplify: $\dfrac{\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}}{\dfrac{1}{yz}+\dfrac{1}{zx}+\dfrac{1}{xy}}$.

If we multiply every term in the numerator and the denominator by xyz, the given fraction will be transformed at once into a simple fraction.

$$\frac{\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}}{\dfrac{1}{yz}+\dfrac{1}{zx}+\dfrac{1}{xy}}=\frac{\dfrac{xyz}{x}+\dfrac{xyz}{y}+\dfrac{xyz}{z}}{\dfrac{xyz}{yz}+\dfrac{xyz}{xz}+\dfrac{xyz}{xy}}=\frac{yz+xz+xy}{x+y+z}.$$

Exercises

Simplify the following complex fractions.

1. $\dfrac{\dfrac{6}{5}-\dfrac{3}{4}}{\dfrac{7}{4}-\dfrac{2}{5}}$.

2. $\dfrac{\dfrac{7}{5}-\dfrac{13}{12}}{\dfrac{1}{3}+\dfrac{4}{5}}$.

3. $\dfrac{\dfrac{2}{a}-3}{\dfrac{1}{a}+5}$.

4. $\dfrac{\dfrac{a}{c}-\dfrac{b}{c}}{\dfrac{x}{c}+\dfrac{y}{c}}$.

5. $\dfrac{\dfrac{x}{y}-1}{\dfrac{y}{x}-1}$.

6. $\dfrac{\dfrac{x}{y}+\dfrac{y}{x}}{\dfrac{x}{y}-\dfrac{y}{x}}$.

7. $\dfrac{2-\dfrac{b}{a+b}}{1+\dfrac{3a}{b-a}}$.

8. $\dfrac{\dfrac{2}{y}+\dfrac{3}{x}}{\dfrac{9}{x^2}-\dfrac{4}{y^2}}$.

9. $\dfrac{\dfrac{a}{b}-\dfrac{b}{a}}{\dfrac{1}{b}-\dfrac{1}{a}}$.

10. $\dfrac{a+b-\dfrac{a^2}{a+b}}{\dfrac{1}{a}-\dfrac{1}{a+b}}$.

11. $\dfrac{\dfrac{(x-y)^2}{xy}+4}{1+\dfrac{2y}{x-y}}$.

12. $\dfrac{\dfrac{1}{x+h}-\dfrac{1}{x}}{h}$.

13. $\dfrac{\dfrac{x+h}{x+h+1}-\dfrac{x}{x+1}}{h}$.

14. $\dfrac{1+\dfrac{1}{x+3}}{x-\dfrac{12}{x+1}}$.

15. $\dfrac{\dfrac{u}{u+v}-\dfrac{v}{v-u}}{\dfrac{v}{u+v}-\dfrac{u}{v-u}}$.

Chapter 3

Linear Equations in One Unknown

19. Equations. An **equation** is a statement that two expressions are equal. The expression to the left of the sign of equality is the **first member** (or **first side**) and the one to the right is the **second member** (or **second side**) of the equation.

Thus, the statement

$$2x + y = x^2 + y^2,$$

is an equation in which $2x + y$ is the first member and $x^2 + y^2$ is the second member.

We must distinguish between two kinds of equations: identical equations and equations of condition.

An **identical equation,** or **identity,** *is one that is true for all values of the letters involved in it for which both sides of the equation have a meaning.*

Thus, the equations

$$a^2 - b^2 = (a - b)(a + b),$$

$$\frac{x^3 + 1}{x + 1} = x^2 - x + 1,$$

and

$$\frac{m}{n} + \frac{n}{m} = \frac{m^2 + n^2}{mn},$$

are identities. The first one is true for all values of a and b; the second, for all values of x except $x = -1$; and the third for all values of m and n except those for which either $m = 0$ or $n = 0$.

An identity may be **proved** by reducing one member to the form given in the other member. For example, the first identity in the preceding illustration may be proved by multiplying together the factors in the second member and simplifying the result.

An **equation of condition,** or **conditional equation,** *is one that is true only for certain values, or sets of values, of the letters contained in it.*

Thus, the equation $x^2 + 4x - 21 = 0$ is true only if $x = -7$ or if $x = 3$ but not for any other value of x.

The equation $x + y = 5$ is true for certain sets of values of x and y, such as $x = 2$, $y = 3$, or $x = 7$, $y = -2$, and so on, but it is not true for many other sets, such as $x = 8$, $y = 3$, or $x = 1$, $y = 7$, and so on.

The symbol $=$ is used to denote equality, both in identities and in equations of condition. Occasionally, in the case of identities, the symbol $\equiv$ (read, "is identically equal to") is used when one wishes to emphasize the fact that the given equation is an identity.

20. Linear Equations in One Unknown. An equation of the form

$$ax + b = 0, \qquad a \neq {}^* 0$$

where a and b are known numbers and a is different from zero, is called a **linear equation** in x. It has one, and only one, root, namely

$$x = -\frac{b}{a}.$$

Equations that are not given in the form stated above can frequently be reduced to that form, as is shown in the following examples.

EXAMPLE 1. Solve:

$$(2x - 5)(x + 3) - (x - 2)(3x + 1) + (4 - x)(5 - x) = 0.$$

Multiply out each product:

$$(2x^2 + x - 15) - (3x^2 - 5x - 2) + (20 - 9x + x^2) = 0.$$

Remove the parentheses and simplify. The result is

$$-3x + 7 = 0.$$

The root of this equation is $\frac{7}{3}$. The student should check this result.

EXAMPLE 2. Solve: $5 + \dfrac{5x}{1 - x} = \dfrac{1}{1 - x} + \dfrac{2}{x}.$

Multiply through by $x(1 - x)$, which is the L.C.M. of the denominators

$$5x(1 - x) + 5x^2 = x + 2(1 - x),$$

or

$$5x = 2 - x.$$

The root of this equation is $x = \frac{1}{3}$, which checks in the given equation.

Exercises

State which of the following equations are identities and which are equations of condition. Prove the identities and solve the equations of condition.

1. $3x + 2 - (5x - 3 - [2x + 1] - 7x + 9) = 7x - 3.$
2. $4x - 1 - (5 - 2x - [3x - 2] + 1) = 5x - 1.$
3. $x^2 + 6x - 9 = (x + 7)(x - 3).$
4. $2x^2 + 11x + 12 = (2x + 3)(x + 4).$
5. $3a^2 + 8ab + 4b^2 = (3a + 2b)(a + 2b).$
6. $\dfrac{x}{3} - \dfrac{3}{x} + 2 = \dfrac{x^2 + 2x + 3}{3x}.$

Solve the following equations of condition.

7. $4x + 8 = x - 7.$
8. $3x - 5 = 2(2x - 4).$
9. $5.1x + 3.7 = 2.4x - 7.1.$
10. $0.3(4.3x - 1.8) = 0.54x + 0.27.$

* The symbol $\neq$ is read, "is not equal to."

11. $7(3x+2)-5(2x+3)=5x+7.$

12. $2(3x+1)+4(7-x)=7x+8$

13. $\dfrac{2x-5}{7}+\dfrac{3x+8}{2}+1=0.$

14. $\dfrac{3}{x}-\dfrac{9}{4}=\dfrac{7}{2x}-\dfrac{5}{3}.$

15. $4-\dfrac{15x+13}{3x-2}=0.$

16. $\dfrac{x+10}{3x+4}+\dfrac{3}{4}=0.$

17. $\dfrac{1}{2x-1}-\dfrac{2}{x+1}=\dfrac{15}{4x-2}.$

18. $\dfrac{15}{2x+1}-\dfrac{5}{4x^2-1}=\dfrac{6}{1-2x}.$

19. $\dfrac{3x+4}{3x+1}-\dfrac{3x+1}{3x-1}=\dfrac{1}{1-9x^2}.$

20. $\dfrac{6x+1}{x+3}+\dfrac{2x-7}{1-x}=\dfrac{4x^2+8}{x^2+2x-3}.$

21. $\dfrac{4x-1.7}{2x-1.3}=\dfrac{2x+4.7}{x+1.2}.$

22. $\dfrac{6x+2.2}{2x+2.9}+\dfrac{1-3x}{x+2.5}=0.$

23. $x=\dfrac{1}{a}-a-\dfrac{x}{a}.$

24. $\dfrac{a}{x-2a}+\dfrac{x-a}{a}=\dfrac{x}{a}.$

25. $\left(\dfrac{a}{b}+\dfrac{b}{a}\right)x-2x=\dfrac{b}{a}-\dfrac{a}{b}.$

26. $\dfrac{x-a}{x+a}+\dfrac{2(a+b)}{x}=\dfrac{x+b}{x-b}.$

Solve the following equations for x.

27. $y=mx+b.$

28. $axy+bx+cy+d=0.$

29. $A=P(1+xn).$

30. $pv=p_0v_0\left(1-\dfrac{x}{273}\right).$

21. Problems Leading to Linear Equations. In the applications of mathematics, it frequently happens that the value of the unknown must be found, not by solving an equation that is given to us, but from information that is stated in words and out of which we must ourselves set up the equation to be solved. Frequently it is more difficult to write down this equation from the verbal statement of the problem than it is to solve the equation after it has been written.

Whenever a verbal problem is given to us, we should think of it as a succession of problems, each with its own special difficulties and methods of solution. Each of these component problems should be solved, by itself, before the next one is undertaken.

The following is the usual sequence of component problems which must be solved in order to obtain the final solution of a verbal problem in algebra.

1. Read the problem, several times if necessary, until you understand exactly what is given and what is to be found. Look up the meaning of any words that you do not understand.

2. Choose one of the numbers whose values are to be found and write the statement that x equals that number.

3. If there are other unknown numbers in the problem, write out the value of each of these other unknowns in terms of x. To do this, you must either know or look up the formula expressing the value of each unknown in terms of x.

4. The problem will state that two expressions containing these unknowns are equal. Write this statement as an equation in x.

5. Solve this equation.

6. Check your results by testing whether they satisfy the conditions of the problem as stated verbally.

EXAMPLE. A man has \$20,000 invested in 4% and 5% bonds. His annual income from the 4% bonds exceeds that from the 5% bonds by \$125. Find the amount of the bonds of each kind that he owns.

Let $x =$ the number of dollars invested in 4% bonds.

Then $20000 - x =$ the number of dollars invested in 5% bonds.

Hence $\frac{4x}{100} =$ the number of dollars of annual income from the 4% bonds and $\frac{5(20000 - x)}{100} =$ the number of dollars of annual income from 5% bonds.

From the statement of the problem,

$$\frac{4x}{100} = \frac{5(20000 - x)}{100} + 125.$$

Clear of fractions: $4x = 100000 - 5x + 12500.$

Simplify: $$9x = 112500,$$

or $$x = 12500,$$

and $$20000 - x = 7500.$$

The man has \$12,500 invested in 4% bonds and \$7,500 invested in 5% bonds.

CHECK. \$12,500 + \$7,500 = \$20,000; further, the annual interest on \$12,500 at 4% is \$500 and that on \$7,500 at 5% is \$375. \$500 − \$375 = \$125.

Problems

1. A stone weighing 685 pounds is broken into two parts such that the smaller weighs two-thirds as much as the larger. Find the weight of each part.

2. The sum of $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{6}$ of a number exceeds $\frac{1}{2}$ the number by 105. Find the number.

3. The sum of the ages of three men is 70 years. In how many years will the sum of their ages be four times as great as it was ten years ago?

4. The circumference of each of the larger wheels of a wagon exceeds that of each of the smaller wheels by 13 inches. The smaller wheels make as many revolutions in going 184 feet as the larger ones do in going 207 feet. Find the circumference of each wheel.

5. A merchant bought some articles for \$3.20 each. He marked them for sale at a price such that, by selling them for 10% less than the marked price, he still made 35% over cost. Find the marked price.

6. The speed of a passenger plane exceeds that of a cargo plane by 52 miles an hour. They make the trip between two cities in five hours and seven hours, respectively. Find the speed of each.

7. A boy unpacked 12 dozen fragile articles. He was to receive three cents for each article he unpacked safely and to pay twenty-five cents for every one he broke. If the amount due him for his work was \$2.36, how many did he break?

8. A man bought some shares of stock for \$40 a share. Later, he bought twice as many shares for \$30 a share. He sold all these shares for \$32 a share, thereby losing \$1200. How many shares did he buy at his first purchase?

9. A farmer sold some corn for \$728. The following year, with the price 50% higher, he sold 400 bushels less for \$756. How many bushels did he sell the first year?

10. One alloy of tin and copper is composed of 3 parts tin and 5 parts copper; another is composed of 4 parts tin and 9 parts copper. How many tons of each must be taken to make 105 tons of an alloy composed of 1 part tin and 2 parts copper?

11. An automobile radiator contains 15 quarts of a mixture which is 25% alcohol and 75% water. How much of this mixture must be drained off and replaced by pure alcohol to produce a mixture which is 40% alcohol?

12. If a new set of spark plugs, costing \$5.50, will increase the mileage of a car from 16 to 18 miles per gallon of gasoline, and if gasoline costs 22 cents a gallon, how many miles must the car be driven in order that the saving in gasoline will pay the cost of the spark plugs?

13. A marksman heard his bullet strike the target two seconds after it was fired. If the bullet traveled 1400 feet per second and if the sound traveled 1100 feet per second, find the distance of the target.

14. A can paint a house in 12 hours and B can paint it in 18 hours. A and B work together on it for a certain time, then B finishes it in twice as many hours more. Find the number of hours each man worked.

Chapter 4

Ratio, Proportion, and Variation

22. Ratio. The value of the ratio of a number a to a second number b is the quotient, a divided by b. It may be expressed in any one of the following three forms

$$a : b = a \div b = \frac{a}{b}.$$

For purposes of mathematical computation, the last of these three forms is usually the most convenient.

When the value of the ratio is stated in the first of the above mentioned forms, the number a is called the **antecedent** and b, the **consequent**; in the second form, a is the **dividend** and b is the **divisor**; and in the last form, a is the **numerator** and b, the **denominator.** The two numbers a and b are the **terms** of the ratio.

When one is dealing with the ratio of two concrete numbers of the same kind, both should be expressed in the same units.

Thus, the ratio of 37 minutes to 3 hours is 37:180; the ratio of 13 quarts to 4 gallons is 13:16, and the ratio of 5 feet to 2 yards is 5:6.

23. Proportion. The statement that two ratios are equal is called a **proportion.** The proportion, a is to b as c is to d, may be written in any one of the following forms:

$$a : b = c : d, \quad a \div b = c \div d, \quad \text{or} \quad \frac{a}{b} = \frac{c}{d}.$$

In this proportion, a and d are the **extremes** and b and c are the **means.** The number d is called the **fourth proportional** to a, b, and c.

In the proportion

$$a : b = b : d,$$

in which the means are equal, b is called a **mean proportional** between a and d and d is called a **third proportional** to a and b.

Exercises

Write each ratio as a fraction and simplify.

1. $15 : 42$.
2. $\frac{51}{32} : \frac{85}{44}$.
3. $12x^5y^3z^7 : 30x^2y^4z^2$.
4. $a(x^2 + xy) : ab(xy + y^2)$.
5. $a^2(x^2 - y^2) : abc(x^2 + 2xy - 3y^2)$.
6. $(x^2 - 3x + 2) : (x^2 + 3x - 10)$.
7. 90 seconds to 2 minutes.
8. 7 feet to 66 inches.

If $a : b = c : d$, show that:

9. $ad = bc$. **10.** $a : c = b : d$. **11.** $d : c = b : a$.

12. $\dfrac{a+b}{b} = \dfrac{c+d}{d}$. **13.** $\dfrac{a-b}{b} = \dfrac{c-d}{d}$. **14.** $\dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}$.

15. If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f}$, show that $\dfrac{a}{b} = \dfrac{a+c+e}{b+d+f}$.

HINT. Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$. Show that $a = kb$, $c = kd$, and $e = kf$. Add these equations and solve for k.

Solve for x.

16. $x + 7 : x - 9 = 9 : 5$. **17.** $2x - 7 : 7x + 3 = 3 : 13$.
18. $9x - 8 : 4 = 5x + 8 : 3$. **19.** $8 : 5x + 7 = 5 : 3x - 2$.
20. $x - 3 : x - 4 = x - 7 : x - 12$. **21.** $x^2 - 7 : x^2 - 3 = 1 : 3$.

22. Find x, given that if x is subtracted from each of the numbers 12, 8, 19, and 11, the resulting numbers form a proportion.

23. The dimensions of a given rectangle are 4 feet and 7 feet. Find the dimensions of a rectangle similar to the given one and having an area of 448 square feet.

24. The sides of a given triangle are 6, 11, and 13 inches. The perimeter (that is, the sum of the sides) of a triangle similar to the given one is ten feet. Find the lengths of the sides of the second triangle.

Find the fourth proportional to the following three numbers.

25. 6, 4, 15. **26.** 4, 5, 6. **27.** 3, $\frac{1}{2}$, 5.
28. x, y, z. **29.** xy, yz, zx. **30.** $a + 1$, $a^2 - 1$, $a - 1$.

Find the positive number which is a mean proportional between the given numbers.

31. 18, 8. **32.** 75, 27. **33.** 5, 7. **34.** a^2, b^2.

Find the third proportional to the given numbers.

35. 4, 6. **36.** 5, 15. **37.** 7, 2. **38.** x, y.

24. The Language of Variation. In this article, we shall consider several forms of statement that occur frequently in the applications of mathematics. We shall show how to replace each of these statements by an equation so that we can deal mathematically with the quantities involved more conveniently than would otherwise be possible.

(a) *Direct variation.* Suppose it is stated, of two variables y and x, that

y varies as x, or **y varies directly as x,** or

y is proportional to x, or **y is directly proportional to x.**

These four forms of statement are equivalent. They mean that y and x *vary in such a way that,*

$$y = kx,$$

where k is a constant. Whenever any one of these four statements occurs, therefore, the student should replace it, mentally, by the statement: *y equals some constant times x.*

The constant k, that appears in the statement $y = kx$, is called the **constant of proportionality.** We can determine the value of this constant k *provided we know the value of y that corresponds to some one value of x other than $x = 0$.*

EXAMPLE 1. If an automobile is traveling at a uniform speed, then the distance s that it has traveled varies as the time t during which it has been traveling; that is,

$$s = kt.$$

The constant of proportionality, in this case, is the speed. We can determine its value if we know how far the automobile has gone at the end of some definite time t. Suppose, for example, that we are given the further information that the automobile has gone 85 miles at the end of 2.5 hours. Then we have

$$85 = k2.5,$$

from which we find that $k = 34$.

The equation expressing the distance that the automobile has traveled, in terms of the time, now becomes

$$s = 34t.$$

EXAMPLE 2. If the base of a variable triangle is constant, then the area is proportional to the altitude, that is,

$$A = kh.$$

Suppose, further, that we know that the area is 20 square inches when the altitude is 5 inches. Then

$$20 = k5,$$

so that $k = 4$. The expression for the area in terms of the altitude now becomes

$$A = 4h.$$

(*b*) *Inverse variation.* Either one of the statements

y varies inversely as x, or

y is inversely proportional to x,

means that y and x vary in such a way that

$$y = \frac{k}{x},$$

where k is a constant.

EXAMPLE. Boyle's law, in physics, states that, for a fixed quantity of gas at a constant temperature, the pressure p is inversely proportional to the volume v, that is

$$p = \frac{k}{v}.$$

Suppose, further, that $p = 76$ when $v = 3$. Then

$$76 = \frac{k}{3}.$$

Hence $k = 228$ and the equation connecting p and v becomes

$$p = \frac{228}{v}.$$

(c) *Joint variation. Combined variation.* The statement

z varies jointly as x and y,

means that there exists a constant k such that

$$z = kxy.$$

Thus, the area of a variable triangle varies jointly as the base and the altitude, since

$$A = \tfrac{1}{2}bh.$$

The preceding forms of statement may be combined. For example, the statement: y varies jointly as x and the square of z, and inversely as w and the cube of v, may be expressed as an equation of the form

$$y = \frac{kxz^2}{wv^3}.$$

Conversely, the equation

$$z = \frac{kt^2x^3}{yw^2},$$

may be stated in words as follows: z varies jointly as the square of t and the cube of x, and inversely as y and the square of w.

Exercises

Write each of the following statements as an equation, using a constant k. Then find the value of k and rewrite the equation replacing k by its value.

1. S is proportional to e^2 and $S = 150$ when $e = 5$.

2. F varies directly as m and inversely as r^2. Also $F = 15$ when $m = 6000$ and $r = 4$.

3. P varies jointly as A and the square of v. Also, $P = 30$ when $A = 84$ and $v = 15$.

4. L varies jointly as b and the square of d and inversely as l. When $b = 4$, $d = 3$, and $l = 24$, the value of L is 372.

State each of the following formulas in words, using the language of variation.

5. $t = 73pv.$ **6.** $v = 3.14r^2h.$ **7.** $F = 0.61\frac{m}{r^2}.$

8. $Q = 4.3\frac{d\sqrt{h}}{p}.$ **9.** $C = \frac{4\pi^2R}{t^2}.$ **10.** $F = 2\pi\sqrt{\frac{I}{mgh}}.$

11. The pressure of still air against a moving automobile varies as the square of the speed. If the pressure is 32 pounds at 20 miles an hour, what is it at 45 miles an hour?

12. Assuming that the value of a used car is inversely proportional to its age, if the value of a car was $420 when it was three years old, what was its value when it was seven years old?

13. The time of vibration of a simple pendulum varies as the square root of its length. If a pendulum 39 inches long vibrates in one second, what is the length of a pendulum that vibrates in one-half of a second?

14. If a vessel with a horizontal bottom and vertical sides contains water, the pressure on the bottom varies jointly as the area of the bottom and the depth of the water. If the area of the bottom is 36 square inches and the depth is two feet, the pressure is 31.2 pounds. Find the pressure when the area of the bottom is three square feet and the depth is 18 inches.

15. The density of a solid is directly proportional to its mass and inversely proportional to its volume. If the density of a body occupying 6 cubic feet and weighing 1875 pounds is 5, what is the density of a body weighing 4050 pounds and occupying a cubic yard?

16. The load that can safely be supported by a circular column of a given material varies as the fourth power of its radius and inversely as the square of its length. If a column three inches in radius and ten feet high can support four tons, how many tons can a column one foot in diameter and 16 feet high support?

17. The loss of pressure of water flowing through a pipe varies as the length and inversely as the diameter of the pipe. If the loss is 75 pounds in a pipe 250 yards long and two inches in diameter, what is the loss in one 1000 yards long and three inches in diameter?

18. If two unequal weights are connected by a cord passing over a pulley, the distance the lighter weight is drawn up in a given time varies jointly as the difference of the weights and the square of the time and inversely as the sum of the weights. If 5 pounds raises 3 pounds 16 feet in 2 seconds, how far will 13 pounds raise 12 pounds in 10 seconds?

Chapter 5

Exponents and Radicals

25. Laws of Exponents. If n is any positive integer, the symbol a^n was defined in Art. 7 as the product

$$a^n = a \cdot a \cdot a \text{ and so on to } n \text{ factors.}$$

From this definition, one readily derives the following five laws of exponents for positive, integral values of m and n.

$$\text{I.} \qquad a^m \cdot a^n = a^{m+n}.$$

$$\text{II.} \qquad (a^m)^n = a^{mn}.$$

$$\text{III. (1)} \qquad \frac{a^m}{a^n} = a^{m-n}, \qquad \text{if } m > n.$$

$$\text{(2)} \qquad \frac{a^m}{a^n} = \frac{1}{a^{n-m}}, \qquad \text{if } m < n.$$

$$\text{IV.} \qquad (ab)^n = a^n b^n.$$

$$\text{V.} \qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

ILLUSTRATIONS.

$$a^3a^5 = (a \cdot a \cdot a)(a \cdot a \cdot a \cdot a \cdot a) = a^{3+5} = a^8.$$
$$(a^2)^3 = a^2 \cdot a^2 \cdot a^2 = (a \cdot a)(a \cdot a)(a \cdot a) = a^{2 \cdot 3} = a^6.$$
$$\frac{a^6}{a^4} = \frac{a \cdot a \cdot a \cdot a \cdot a \cdot a}{a \cdot a \cdot a \cdot a} = a \cdot a = a^{6-4} = a^2.$$
$$\frac{a^2}{a^5} = \frac{a \cdot a}{a \cdot a \cdot a \cdot a \cdot a} = \frac{1}{a \cdot a \cdot a} = \frac{1}{a^{5-2}} = \frac{1}{a^3}.$$
$$(ab)^3 = (ab)(ab)(ab) = (a \cdot a \cdot a)(b \cdot b \cdot b) = a^3b^3.$$
$$\left(\frac{a}{b}\right)^4 = \frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} = \frac{a^4}{b^4}.$$

Exercises

Find the value of each of the following expressions.

1. 3^4. **2.** $(-2)^4$. **3.** $-(2)^4$. **4.** $2^4 \cdot 3^3$.

5. $(\frac{3}{2})^3$. **6.** $(0.2)^5$. **7.** $(-0.03)^3$. **8.** $(-0.03)^4$.

9. $\frac{2^7}{2^3}$. **10.** $\frac{4^3}{2^5}$. **11.** $(2^2)^5$. **12.** $\frac{3^3}{3^6}$.

Perform the indicated operations.

13. $x^7 \cdot x^5$. **14.** $(a^3)^2$. **15.** $z^7 \div z^6$. **16.** $z^4 \div z^7$.

17. $(3y)^4$. **18.** $3(y)^4$. **19.** $h^{4n} \div h^{2n}$. **20.** $a^{2n-1} \div a^{n-1}$.

21. $\left(\dfrac{t^a}{t^b}\right)^{a+b}$. **22.** $\dfrac{(a^{n+1})^{n-1}}{a^{n^2}}$. **23.** $(a^l b^m c^n)^p$. **24.** $\left(\dfrac{a^r}{b^s}\right)^t$.

25. $\dfrac{4^{n+2} + 4^3 4^{n-2}}{2^{2n}}$. **26.** $\dfrac{(x^{m-1})^{m+1} \div x^m}{(x^{m+1})^m \div x^{2m}}$.

26. Square Roots. Imaginary Numbers. Any number x that satisfies the equation $x^2 = a$ is a *square root* of the number a.

Every positive number has two real square roots. These roots are equal in numerical * value but one is positive and the other is negative. The positive square root of a is called the **principal square root** of a and is denoted by the symbol $\sqrt{a}$. The negative square root is denoted by $-\sqrt{a}$.

The square root of a negative number cannot be either positive, negative, or zero. For, the square of either a positive or negative number is positive and the square of zero is zero. We shall, accordingly, assume the existence of another kind of numbers, which have the property that their squares are negative. The numbers are called **imaginary numbers.**

In particular, we assume that -1 has two imaginary square roots. We denote one of these by $\sqrt{-1}$ and the other by $-\sqrt{-1}$. For brevity, we shall usually denote the first of these roots by i and the second by $-i$, so that $i = \sqrt{-1}$ and $-i = -\sqrt{-1}$.

If a is any negative number, we may put $a = -|a|$, where $|a|$ is the numerical value of a. Then,

$$\sqrt{a} = \sqrt{-|a|} = \sqrt{-1}\sqrt{|a|} = i\sqrt{|a|},$$

and

$$-\sqrt{a} = -\sqrt{-|a|} = -\sqrt{-1}\sqrt{|a|} = -i\sqrt{|a|}.$$

These two imaginary numbers, $i\sqrt{|a|}$, and $-i\sqrt{|a|}$, are the two imaginary square roots of the negative number a.

The positive and negative numbers, and zero, are called **real numbers** to distinguish them from the imaginary ones. *Unless otherwise indicated, the numbers considered in this book are assumed to be real numbers.*

27. Roots of Any Positive, Integral Order. Principal Roots. If n is any positive integer, and x is any number, real or imaginary, such that $x^n = a$, then x is said to be an **nth root, or root of order n,** of the number a.

Thus, -2 is a cube root of -8, since $(-2)^3 = -8$.

It will be shown, in Chapter XXXIII, that *any number, except zero, has n distinct nth roots*. If the given number a is real, the following statements are true concerning the reality of its nth roots.

* The numerical, or absolute, value of a real number a was defined in Art. 3.

(1) *If n is an odd integer, the real number a has one, and only one, real nth root.* This real root is called the **principal nth root** of a and is denoted by $\sqrt[n]{a}$. This principal nth root of a is positive, negative, or zero according as a is positive, negative, or zero.

Thus, $\sqrt[5]{32} = 2,\ \sqrt[3]{-64} = -4,\ \sqrt[7]{0} = 0.$

(2) *If n is an even integer, and if a is positive, then a has two, and only two, real nth roots. These roots are numerically equal but one is positive and the other is negative.* The positive root is the **principal nth root** of a and is denoted by $\sqrt[n]{a}$. The negative real root is denoted by $-\sqrt[n]{a}$.

Thus, $\sqrt[4]{81} = 3$ and $-\sqrt[4]{81} = -3$ are the real fourth roots of 81. Of these, $\sqrt[4]{81} = 3$ is the principal fourth root of 81.

(3) *If n is an even integer and a is negative, then a has no real root.* In this case, we may choose any one of the imaginary nth roots to be denoted by the symbol $\sqrt[n]{a}$ but we shall not call the root so chosen a principal root. Moreover, since, in this case, the value of $\sqrt[n]{a}$ is not definitely fixed by its symbol, we shall exclude such roots from the following discussion of the laws obeyed by radicals. It can be shown, in fact, that these roots obey only in part the laws that hold for principal roots.

Exercises

Assuming that the numbers indicated by letters are positive, find the indicated principal root. If there is a real root that is not a principal root, indicate it by a suitable symbol and find its value.

1. $\sqrt{49}$. **2.** $\sqrt[3]{-64}$. **3.** $\sqrt{0.25}$. **4.** $\sqrt[5]{-32}$.

5. $\sqrt[9]{-1}$. **6.** $\sqrt[4]{\dfrac{x^4}{y^{20}}}$. **7.** $\sqrt[3]{\dfrac{-27x^6}{y^9}}$. **8.** $\sqrt[7]{\dfrac{128a^{14}}{b^{35}}}$.

Find the value of each of the following expressions.

9. $(\sqrt{19})^2$. **10.** $(-\sqrt[4]{81})^3$. **11.** $\left(\sqrt[5]{\dfrac{a^{15}b^{25}}{c^{35}d^5}}\right)^2$.

28. Rational Numbers. *Any real number that can be expressed as a fraction whose numerator and denominator are both* **integers** *is a* **rational number.** Zero, also, is classed as a rational number. All other real numbers are **irrational.**

Thus, $\frac{4}{5}$, 0.041, 7, and -5.3 are rational numbers since they can be written, respectively, as $\dfrac{4}{5}, \dfrac{41}{1000}, \dfrac{7}{1}$, and $\dfrac{-53}{10}$.

The numbers $\sqrt{2}$, $\sqrt[3]{45}$, $\sqrt{1+\sqrt{5}}$, and π are irrational numbers since none of them can be written as a fraction whose numerator and denominator are both *integers.*

Because rational numbers are, in some ways, simpler to deal with than irrational ones, it will sometimes be necessary for us, in this and subsequent chapters, to make use of this distinction between rational and irrational numbers. In the following article, for example, we shall consider the definition of a number to a power when the exponent is any rational number.

29. Rational Exponents. The definition of a^n given in Arts. 7 and 25 holds only if n is a positive integer. It is, indeed, quite meaningless for all other values of n. We shall now extend this definition of the symbol a^n in such a way that the exponent may be any rational number whatever. *We shall choose these extended definitions in such a way that the five laws of exponents stated in Art.* 25 *shall continue to be true.*

(1) *Fractional exponents.* Let m and n be any two positive integers. We wish to define the symbol $a^{\frac{m}{n}}$.

Assuming that Law II of Art. 25 holds, we shall have

$$\left(a^{\frac{m}{n}}\right)^n = a^{\frac{mn}{n}} = a^m.$$

It follows from the definition of an nth root of a number (Art. 27), that $a^{\frac{m}{n}}$ must be an nth root of a^m. If a^m has a principal nth root * we shall define $a^{\frac{m}{n}}$ as this principal nth root. Hence, by definition,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

In particular, if $m = 1$, we have

$$a^{\frac{1}{n}} = \sqrt[n]{a},$$

that is, *the symbols* $a^{\frac{1}{n}}$ *and* $\sqrt[n]{a}$ *are equivalent. Either may be used in place of the other.*

ILLUSTRATIONS. $36^{\frac{1}{2}} = \sqrt{36} = 6$, $(-125)^{\frac{1}{3}} = \sqrt[3]{-125} = -5$,

$$8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4, \quad (a^6)^{\frac{3}{2}} = \sqrt{a^{18}} = a^9.$$

(2) *Zero exponents.* Let $a \neq 0$. If a^0 is to obey Law I of Art. 25, we must have

$$a^n \cdot a^0 = a^{n+0} = a^n, \quad \text{or} \quad a^0 = a^n \div a^n = 1.$$

We accordingly make the following definition

$$a^0 = 1. \qquad a \neq 0$$

ILLUSTRATIONS. $5^0 = 1$, $(10{,}000)^0 = 1$, $(54xy^5z^3)^0 = 1$.

* If a^m is negative and n is an even integer, a^m does not have a principal nth root. In this case, we still make the definition

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

where $\sqrt[n]{a^m}$ is some one of the nth roots of a^m. As the value of $a^{\frac{m}{n}}$ is not definitely fixed, we shall exclude this case from the following discussion. It can, in fact, be shown that certain modifications of Laws I to V are necessary if $a^{\frac{m}{n}}$ is not a principal root of a^m.

(3) *Negative exponents.* Let $a \neq 0$ and let n be any positive rational number. If a^{-n} is to obey Law I of Art. 25, we must have

$$a^n \cdot a^{-n} = a^{n-n} = a^0 = 1, \quad \text{or} \quad a^{-n} = \frac{1}{a^n}.$$

Hence, we make the following definition

$$\boldsymbol{a^{-n} = \frac{1}{a^n}}. \qquad \boldsymbol{a \neq 0}$$

ILLUSTRATIONS. $5^{-3} = \dfrac{1}{5^3} = \dfrac{1}{125}$, $27^{-\frac{2}{3}} = \dfrac{1}{27^{\frac{2}{3}}} = \dfrac{1}{9}$, $x^{-7} = \dfrac{1}{x^7}$.

It will be observed that, with this definition of a negative exponent, *any* **factor** *of the entire numerator of a fraction may be written as a* **factor** *of the entire denominator, and conversely, provided the sign of its exponent is changed at the same time.*

EXAMPLE 1. $\dfrac{ab^{-2}}{c} = \dfrac{a}{c} \cdot \dfrac{1}{b^2} = \dfrac{a}{cb^2}$.

EXAMPLE 2. $\dfrac{3xy^{-1}}{z^{-3}} = \dfrac{3x\frac{1}{y}}{\frac{1}{z^3}} = \dfrac{3x}{y} \cdot \dfrac{z^3}{1} = \dfrac{3xz^3}{y}$.

EXAMPLE 3. $$\frac{x^{-2} + y^{-2}}{x^{-3} + y^{-3}} = \frac{\frac{1}{x^2} + \frac{1}{y^2}}{\frac{1}{x^3} + \frac{1}{y^3}} = \frac{\frac{y^2 + x^2}{x^2y^2}}{\frac{y^3 + x^3}{x^3y^3}} = \frac{y^2 + x^2}{x^2y^2} \cdot \frac{x^3y^3}{y^3 + x^3}$$
$$= \frac{xy^3 + yx^3}{y^3 + x^3}.$$

It is proved in advanced mathematics that the numbers $a^{\frac{m}{n}}$, a^0, and $a^{-\frac{m}{n}}$, as defined in this article, obey *all* of the laws of exponents stated in Art. 25, provided that the symbols $a^{\frac{m}{n}}$ and $a^{-\frac{m}{n}}$ are interpreted to mean the *principal* nth roots of a^m and a^{-m}, respectively. *In all future computations with exponents, the operations are to be performed according to these laws.*

Exercises

Find the values of the following expressions.

1. $169^{\frac{1}{2}}$. **2.** $(-125)^{-\frac{1}{3}}$. **3.** $(0.0016)^{-\frac{1}{4}}$. **4.** $81^{\frac{3}{4}}$.

5. $7^0 \cdot 2^{-5}$. **6.** $\left(\dfrac{4^0}{2^{-3}} - 5^0\right)^{-2}$. **7.** $\left(\dfrac{5^{\frac{3}{4}}}{5^{-\frac{1}{4}}}\right)^{-2}\left(\dfrac{5}{5^{\frac{1}{2}}}\right)^4$. **8.** $(3^{-2} + 4^{-2})^{-\frac{1}{2}}$.

Change to an equivalent form having only positive exponents.

9. $3x^2y^{-4}$. **10.** $-5x^{-5}$. **11.** $\dfrac{(-8)^{-5}\,b^0}{4^{-6}\,a^{-2}}$. **12.** $\dfrac{3^0u^{-2}v^{-3}}{w^{-1}x^{-2}}$.

13. $2a^{-2} - (2a)^{-2}$. 14. $\dfrac{(x-y)^0}{(x+y)^{-2}}$. 15. $\dfrac{(x^2-y^2)^{-1}}{(x-y)^{-2}}$. 16. $\dfrac{x^{-1}+y^{-1}}{x^{-2}-y^{-2}}$.

17. $\dfrac{x^{-3}y^{-3}}{x^{-3}+y^{-3}}$. 18. $\dfrac{a^{-1}-b^0}{a^{-2}-b^0}$. 19. $(x^{-1}+y^{-2})^{-1}$. 20. $(x^{-1}+y^{-1}+z^{-1})^{-1}$.

Write without a denominator, using negative exponents.

21. $\dfrac{3a}{bc^2}$. 22. $\dfrac{2x^{\frac{1}{2}}}{\sqrt{y^3z^3}}$. 23. $\dfrac{5^0x}{\sqrt{yz}}$. 24. $\dfrac{a^2\sqrt{x^3}}{\sqrt[3]{x^2\sqrt{x}}}$.

Replace the fractional exponents by radicals and simplify, if possible.

25. $9a^{\frac{3}{2}}$. 26. $(9a)^{\frac{3}{2}}$. 27. $(4x^6y^2z^8)^{\frac{1}{4}}$. 28. $(a^2+b^2)^{\frac{1}{2}}$.

Replace the radicals by fractional exponents and simplify, if possible.

29. $\sqrt[3]{8a^6b^{12}}$. 30. $\sqrt[n]{a^{2n}a^{n^2}}$. 31. $\sqrt{a\sqrt[3]{a\sqrt[4]{a}}}$. 32. $\sqrt{x^0 - 2x^{-1} + x^{-2}}$.

Perform the indicated multiplications and divisions.

33. $x^{\frac{4}{3}}(x^{\frac{2}{3}} + 3x^{\frac{1}{3}} - 5x^{-\frac{1}{3}})$.

34. $(a^{\frac{1}{2}} + b^{\frac{1}{2}})(a^{\frac{1}{2}} - b^{\frac{1}{2}})$.

35. $(y^{\frac{1}{2}} + 2)(y^{\frac{1}{2}} + 5)$.

36. $(3x^{\frac{4}{3}} - 2x^{\frac{1}{3}})(2x^{\frac{5}{3}} - 5x^{\frac{2}{3}})$.

37. $(x^{\frac{8}{3}}y^{\frac{4}{5}} - 3x^{\frac{5}{3}}y^{\frac{7}{5}} + x^{\frac{4}{3}}y^{\frac{9}{5}}) \div x^{\frac{2}{3}}y^{\frac{3}{5}}$.

38. $(y^{-2} + 2y^{-1} - 3) \div (y^{-1} + 3)$.

30. Radicals. The expression $\sqrt[n]{a}$, which denotes the principal nth root of a (Art. 27), is called a **radical.** The number n is the **index,** or **order,** of the radical and a is the **radicand.** The index is customarily omitted if $n = 2$, that is, $\sqrt{a} = \sqrt[2]{a}$.

Thus, $\sqrt[5]{3x+2y}$ is a radical for which the index is 5 and the radicand is $3x + 2y$.

Since (by Art. 29) $\sqrt[n]{a} = a^{1/n}$, the process of operating with radicals must be made according to the laws of exponents (Art. 25). The laws we shall use most frequently, written in the radical form, are the following ones.

$$\text{I.}\quad (\sqrt[n]{a})^n = a,$$

$$\text{II.}\quad \sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a},$$

$$\text{III.}\quad \sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab},$$

$$\text{IV.}\quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}},$$

where the roots involved are the principal roots, in each case.

The student should verify the correctness of each of these formulas by replacing each radical by the corresponding expression involving fractional exponents and using the laws of exponents.

31. Simplification of Radicals. (*a*) *Removal of factors from the radicand.* If the radicand contains one or more factors that are perfect nth powers, we may write these factors separately and remove them from under the radical sign.

EXAMPLE 1. Remove all fourth-degree factors from the radicand: $\sqrt[4]{\dfrac{112a^6b^2}{81x^5y^{11}}}$.

$$\sqrt[4]{\frac{112a^6b^2}{81x^5y^{11}}} = \sqrt[4]{\frac{2^4a^4\cdot 7a^2b^2}{3^4x^4y^8\cdot xy^3}} = \sqrt[4]{\frac{2^4a^4}{3^4x^4y^8}}\cdot\sqrt[4]{\frac{7a^2b^2}{xy^3}} = \frac{2a}{3xy^2}\sqrt[4]{\frac{7a^2b^2}{xy^3}}.$$

Sometimes we must use the converse process of introducing a coefficient under the radical sign. This process is illustrated by the following example.

EXAMPLE 2. Introduce the coefficient of $\dfrac{3xy^2}{z}\sqrt[3]{\dfrac{4axz^2}{189y^8}}$ under the radical sign.

$$\frac{3xy^2}{z}\sqrt[3]{\frac{4axz^2}{189y^8}} = \sqrt[3]{\frac{3^3x^3y^6}{z^3}}\sqrt[3]{\frac{4axz^2}{3^3\cdot 7y^8}} = \sqrt[3]{\frac{3^3x^3y^6\cdot 4axz^2}{z^3\cdot 3^3\cdot 7y^8}} = \sqrt[3]{\frac{4ax^4}{7y^2z}}.$$

(*b*) *Rationalization of the denominator.* If the radicand is a simple fraction, the radical can be transformed into an expression in which the radicand is integral by multiplying both the numerator and the denominator of the fraction by an expression that will make the denominator a perfect nth power and then removing the denominator from under the radical sign.

EXAMPLE 3. $\sqrt[4]{\dfrac{5x^2y}{27a^3b^6}} = \sqrt[4]{\dfrac{5x^2y\cdot 3ab^2}{3^4a^4b^8}} = \dfrac{\sqrt[4]{15x^2yab^2}}{\sqrt[4]{3^4a^4b^8}} = \dfrac{\sqrt[4]{15x^2yab^2}}{3ab^2}.$

EXAMPLE 4. Write the radicand of $\sqrt{2x+\dfrac{y^2}{2x^2}}$ as a simple fraction and rationalize the denominator.

$$\sqrt{2x+\frac{y^2}{2x^2}} = \sqrt{\frac{4x^3+y^2}{2x^2}} = \frac{\sqrt{8x^3+2y^2}}{\sqrt{4x^2}} = \frac{\sqrt{8x^3+2y^2}}{2x}.$$

(*c*) *Reduction of order.* Whenever the radicand can be written as a number raised to a power k where k is a factor of n, the order of the radical can be reduced. This process is illustrated by the following examples.

EXAMPLE 5. $\sqrt[12]{8a^3b^6} = \sqrt[12]{(2ab^2)^3} = (2ab^2)^{\frac{3}{12}} = (2ab^2)^{\frac{1}{4}} = \sqrt[4]{2ab^2}.$

EXAMPLE 6. $\sqrt[6]{4a^2+20ab+25b^2} = \sqrt[6]{(2a+5b)^2} = \sqrt[3]{2a+5b}.$

A radical is defined to be in its **simplest form** if the following conditions are satisfied.

(1) *The radicand contains no factor to a power as high as the order of the radical.*

(2) *The radicand contains no fractions.*

(3) *The index of the radical is as small as possible.*

The radical should be reduced to the simplest form, as defined above, before any computations are made which involve it.

Exercises

Remove as many factors as possible from under the radical sign.

1. $\sqrt[3]{162x^5y^7}$. 2. $\sqrt[5]{\dfrac{128x^9}{y^{11}}}$. 3. $\sqrt[4]{\dfrac{80a^5b^7c^2}{81xy^6z^{11}}}$.

4. $\sqrt[3]{x^6y^3+x^9}$. 5. $\sqrt[n]{\dfrac{x^{2n+4}y^{n+1}}{z^{3n+5}}}$. 6. $\sqrt[n]{x^{n^2}y^{mn}(x-y)^{3n+2}}$.

Introduce the coefficient under the radical. Then simplify the radicand.

7. $5\sqrt[3]{2}$. 8. $\dfrac{2x}{3y^2}\sqrt{\dfrac{15uy^3}{14vx}}$. 9. $\dfrac{u+v}{u-v}\sqrt{\dfrac{u-v}{u+v}}$.

Rationalize the denominator and remove, when possible, perfect powers from the numerator of the radicand.

10. $\sqrt{\dfrac{121}{2x^3}}$. 11. $\sqrt{\dfrac{a}{b}}$. 12. $\sqrt[3]{\dfrac{16}{a^2b}}$.

13. $\sqrt[3]{\dfrac{24x^6y^4}{5u^5v^2}}$. 14. $\sqrt[t]{\dfrac{a^{4t+3}}{2b^{t-2}c^{3t-5}}}$. 15. $\sqrt[3]{\dfrac{x-y}{(x+y)^2}}$.

Reduce to a radical of lower order.

16. $\sqrt[4]{25a^6b^{10}}$. 17. $\sqrt[6]{\dfrac{27x^9y^{15}}{125z^{21}}}$. 18. $\sqrt[4]{\dfrac{(x-y)^2}{x^2y^2}}$.

19. $\sqrt[10]{32(u-v)^{15}}$. 20. $\sqrt[6]{\dfrac{9(u-v)^2}{(u+v)^2}}$. 21. $\sqrt[3n]{x^ny^{n^2}(x+y)^{2n}}$.

Reduce the radicals to their simplest forms.

22. $\sqrt{90}$. 23. $\sqrt[3]{40x^4y^{10}}$. 24. $\sqrt[14]{\dfrac{a^{21}b^7}{c^{35}}}$.

25. $\sqrt{\dfrac{1}{a^2}+\dfrac{1}{b^2}}$. 26. $\sqrt[3]{\dfrac{8}{x}-\dfrac{3}{y^2}}$. 27. $\sqrt{\dfrac{x-y}{y}+\dfrac{y-x}{x}}$.

32. Addition and Subtraction of Radicals. Two radicals are said to be **the same** *only* if they have the *same radicand* and the *same index*. They are said to be **similar** if, after they have both been reduced to the simplest form, the resulting radicals are the same. In all other cases, the radicals are **dissimilar**.

Thus, $\sqrt{\frac{1}{2}}$ and $\sqrt{8}$ are similar, because $\sqrt{\frac{1}{2}} = \frac{1}{2}\sqrt{2}$ and $\sqrt{8} = 2\sqrt{2}$. The radicals $\sqrt{5}$ and $\sqrt[3]{5}$ are dissimilar, because they have different indices; $\sqrt{3}$ and $\sqrt{7}$ are dissimilar, because they have different radicands.

To add two or more terms having the same radical factor: combine the coefficients of the radical and multiply the result by the common radical factor.

Illustration. $3a\sqrt{2b} - 4b\sqrt{2b} - c\sqrt{2b} = (3a - 4b - c)\sqrt{2b}.$

To add two or more terms having similar radical factors: reduce each radical to the simplest form, then add according to the rule for adding terms having the same radical factor.

Illustration.

$$\sqrt{\frac{4a^3}{b}} - 5x\sqrt[4]{81a^2b^2} - 6y\sqrt{a^3b^3} = \frac{2a}{b}\sqrt{ab} - 15x\sqrt{ab} - 6aby\sqrt{ab}$$
$$= \left(\frac{2a}{b} - 15x - 6aby\right)\sqrt{ab}.$$

The sum of two or more dissimilar radicals should only be indicated.

Illustration.

$$\sqrt{4xy^2} - \sqrt{x^5y^4} + \sqrt[3]{\frac{8x^4}{y^3}} + \sqrt[3]{\frac{xy^6}{27}} = 2y\sqrt{x} - x^2y^2\sqrt{x} + \frac{2x}{y}\sqrt[3]{x} + \frac{y^2}{3}\sqrt[3]{x}$$
$$= (2y - x^2y^2)\sqrt{x} + \left(\frac{2x}{y} + \frac{y^2}{3}\right)\sqrt[3]{x}.$$

Exercises

Simplify the following expressions and find the value of each to four significant figures, using Table IV.

1. $3\sqrt{7} + 5\sqrt{63}.$

2. $11\sqrt{24} - 5\sqrt{54}.$

3. $4\sqrt{175} + 2\sqrt{28} - 5\sqrt{63}.$

4. $\sqrt{245} - \sqrt{45} + 4\sqrt{99} - \sqrt{176}.$

Simplify and collect similar terms.

5. $\sqrt{9a^2x} - \sqrt{b^2x} - \sqrt{36a^2b^2x^3}.$

6. $\sqrt{4x^3} + \sqrt{9xy^2} - \sqrt{x(x+y)^2}.$

7. $\sqrt{(a+b)^3} - 3\sqrt{a^3 + a^2b}.$

8. $\sqrt[3]{24x^4y} - \sqrt[3]{375xy^4}.$

9. $\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}.$

10. $\sqrt{3x^2 + 6x + 3} + \sqrt{3x^2 - 6x + 3}.$

11. $\sqrt[3]{\frac{3x}{y^2}} + \sqrt[3]{\frac{24y}{x^2}} - \sqrt[3]{\frac{81a^3b^3}{x^2y^2}}.$

12. $\sqrt{x^{-2} + y^{-2}} - \sqrt{x^2y^2 + y^4}.$

13. $\sqrt[4]{\frac{x^5}{8}} + \sqrt[4]{32x^{13}}.$

14. $\sqrt[3]{t^{-1}} - \sqrt[3]{27t^5} + \sqrt[3]{8t^{-4}}.$

15. $\sqrt[n]{\dfrac{u^{4n^2+3n+2}}{v^{2n^2-3n-1}}} - \sqrt[2n]{\dfrac{u^{6n+4}}{v^{8n-2}}}$. 16. $\sqrt[3]{(2x-y)^5} + \sqrt{(2x-y)^3}$.

17. $\sqrt[3]{2x^3y} - \sqrt[3]{2xy^3} - \sqrt[3]{\dfrac{16y}{x^3}} + \sqrt[3]{\dfrac{250x}{y^3}}$.

18. $\sqrt{\dfrac{a-b}{a+b}} + \sqrt{\dfrac{a+b}{a-b}} + \sqrt{\dfrac{4b^2}{a^2-b^2}}$.

33. Multiplication and Division of Radicals. Two or more radicals of different orders may be transformed into radicals of the same order by taking, as the common order, the least common multiple of the orders of the given radicals.

EXAMPLE 1. Write $\sqrt{3}$, $\sqrt[3]{5}$, and $\sqrt[4]{7}$ as radicals of the same order and write these three numbers in order of increasing magnitude.

Since 12 is the least common multiple of the orders of these radicals, we write: $\sqrt{3} = 3^{\frac{1}{2}} = (3^6)^{\frac{1}{12}} = \sqrt[12]{729}$: $\sqrt[3]{5} = 5^{\frac{1}{3}} = (5^4)^{\frac{1}{12}} = \sqrt[12]{625}$; $\sqrt[4]{7} = 7^{\frac{1}{4}} = (7^3)^{\frac{1}{12}} = \sqrt[12]{343}$.

Since $\sqrt[12]{343} < \sqrt[12]{625} < \sqrt[12]{729}$, we have $\sqrt[4]{7} < \sqrt[3]{5} < \sqrt{3}$.

EXAMPLE 2. Express as radicals of the same order: $\sqrt{\dfrac{x}{y}}$, $\sqrt[3]{\dfrac{2y}{x^2}}$, and $\sqrt[9]{\dfrac{7x^4}{y^7}}$.

The least common multiple of the orders of the radicals is 18. We have

$$\sqrt{\frac{x}{y}} = \left(\frac{x}{y}\right)^{\frac{1}{2}} = \left[\left(\frac{x}{y}\right)^9\right]^{\frac{1}{18}} = \sqrt[18]{\frac{x^9}{y^9}};\quad \sqrt[3]{\frac{2y}{x^2}} = \left(\frac{2y}{x^2}\right)^{\frac{1}{3}} = \left[\left(\frac{2y}{x^2}\right)^6\right]^{\frac{1}{18}} = \sqrt[18]{\frac{64y^6}{x^{12}}};$$

$$\sqrt[9]{\frac{7x^4}{y^7}} = \left[\left(\frac{7x^4}{y^7}\right)^2\right]^{\frac{1}{18}} = \sqrt[18]{\frac{49x^8}{y^{14}}}.$$

To multiply two radicals, first transform them to the same order, then apply the formula $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$.

Similarly, *to divide one radical by another, transform them to the same order, then apply the formula* $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$.

EXAMPLE 3. Multiply $\sqrt{\dfrac{24x^5}{y}}$ by $\sqrt[3]{\dfrac{y^2}{2x}}$.

$$\sqrt{\frac{24x^5}{y}}\cdot\sqrt[3]{\frac{y^2}{2x}} = \sqrt[6]{\frac{2^93^3x^{15}}{y^3}}\sqrt[6]{\frac{y^4}{2^2x^2}} = \sqrt[6]{\frac{2^93^3x^{15}y^4}{y^32^2x^2}} = \sqrt[6]{2^73^3x^{13}y} = 2x^2\sqrt[6]{54xy}.$$

EXAMPLE 4. Divide: $\sqrt[4]{18a^3b^2c^2}$ by $\sqrt[3]{3abc^4}$.

$$\frac{\sqrt[4]{18a^3b^2c^2}}{\sqrt[3]{3abc^4}} = \frac{\sqrt[12]{2^33^6a^9b^6c^6}}{\sqrt[12]{3^4a^4b^4c^{16}}} = \sqrt[12]{\frac{2^33^2a^5b^2}{c^{10}}} = \frac{1}{c}\sqrt[12]{72a^5b^2c^2}.$$

EXAMPLE 5. Simplify: $\dfrac{\sqrt[3]{3a^2b}\,\sqrt[4]{8a^5b^3}}{\sqrt[6]{12a^3b^5}}$.

$$\frac{\sqrt[3]{3a^2b}\,\sqrt[4]{8a^5b^3}}{\sqrt[6]{12a^3b^5}} = \frac{\sqrt[12]{3^4a^8b^4}\,\sqrt[12]{2^9a^{15}b^9}}{\sqrt[12]{2^43^2a^6b^{10}}} = \sqrt[12]{\frac{3^4a^8b^42^9a^{15}b^9}{2^43^2a^6b^{10}}}$$

$$= \sqrt[12]{3^22^5a^{17}b^3} = a\sqrt[12]{288a^5b^3}.$$

Exercises

Write the expressions in each exercise as radicals of the same order. In exercises 1 to 3, arrange the numbers in order of increasing magnitude.

1. $\sqrt[3]{9}$, $\sqrt[4]{17}$. **2.** $\sqrt{11}$, $\sqrt[3]{31}$. **3.** $\sqrt{6}$, $\sqrt[3]{13}$, $\sqrt[6]{143}$.

4. $\sqrt[4]{2u^2v^3}$, $\sqrt[6]{3u^5v^7}$. **5.** $\sqrt[2n]{xy}$, $\sqrt[3n]{x^2y^5}$. **6.** $\sqrt[4]{ab^2c^3}$, $\sqrt[5]{a^3bc^7}$, $\sqrt[10]{2a^5b^3c^{11}}$.

Perform the indicated operations.

7. $\sqrt{\dfrac{15}{22}}\sqrt[3]{\dfrac{44}{75}}$. **8.** $\sqrt[3]{\dfrac{35}{18}}\sqrt[5]{\dfrac{108}{245}}$. **9.** $\sqrt{\dfrac{72}{55}} \div \sqrt[3]{\dfrac{36}{11}}$.

10. $\sqrt{5y^3z}\,\sqrt[4]{2y^5z^3}$. **11.** $\sqrt{2u^3vw^4}\,\sqrt[5]{uv^2w^2}$. **12.** $\sqrt{a}\,\sqrt[3]{2b^2}\,\sqrt[4]{3a^2b}$.

13. $\dfrac{\sqrt{6a^3b^2}}{\sqrt[4]{2ab^5}}$. **14.** $\dfrac{\sqrt{2a^3b^2c^5}}{\sqrt[5]{a^4b^3c^7}}$. **15.** $\dfrac{\sqrt{10x^2y^5}\,\sqrt[3]{3x^5y^4}}{\sqrt[6]{15x^5y^4}}$.

16. $\dfrac{\sqrt[4]{x^2z^2+3xyz^2}}{\sqrt[6]{xyz+3y^2z}}$. **17.** $\sqrt[n]{x^{m+n}y^{m-n}}\,\sqrt[m]{x^{m-n}\,y^{m+n}}$.

34. Binomials Involving Radicals. Rationalization of the Denominator. The following types of products and quotients involving radicals appear frequently in certain types of mathematical computations.

EXAMPLE 1. Multiply $2\sqrt{a}+\sqrt{b}$ by $\sqrt{a}+3\sqrt{b}$.

We perform this multiplication according to the customary process for multiplying two binomials, applying the rule for the multiplication of radicals.

$$\begin{array}{l} 2\sqrt{a} + \sqrt{b} \\ \underline{\sqrt{a} + 3\sqrt{b}} \\ 2a \quad + \sqrt{ab} \\ \underline{\qquad\quad 6\sqrt{ab} + 3b} \\ 2a \quad + 7\sqrt{ab} + 3b. \end{array}$$

Hence, $(2\sqrt{a}+\sqrt{b})(\sqrt{a}+3\sqrt{b}) = 2a+7\sqrt{ab}+3b.$

By actual multiplication, we find that

$$(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b}) = a-b.$$

It follows from this relation that, if we have given a fraction whose denominator is either of the expressions $\sqrt{a}+\sqrt{b}$ or $\sqrt{a}-\sqrt{b}$, the denominator will be freed of radicals if we *multiply both the numerator and the denominator by the denominator with the sign of the second term changed.* This process is called **rationalizing the denominator** and is illustrated by the following example.

EXAMPLE 2. Rationalize the denominator of $\dfrac{7\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$.

$$\frac{7\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}} = \frac{7\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}} \cdot \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}} = \frac{35+8\sqrt{15}+3}{5-3}$$
$$= \frac{38+8\sqrt{15}}{2} = 19+4\sqrt{15}.$$

Exercises

Perform the indicated operations and simplify the results.

1. $\sqrt{10}(3\sqrt{5}+7\sqrt{2})$.
2. $(3-\sqrt{7})(2+5\sqrt{7})$.
3. $(2\sqrt{6}+11\sqrt{5})(3\sqrt{6}-2\sqrt{5})$.
4. $(a\sqrt{b}+b\sqrt{a})(b\sqrt{b}-a\sqrt{a})$.
5. $\left(4\sqrt{\dfrac{2y}{z}}-3\sqrt{\dfrac{x}{y}}\right)\left(\sqrt{\dfrac{2y}{z}}+2\sqrt{\dfrac{x}{y}}\right)$.
6. $(a\sqrt{5v}+b\sqrt{3u})(c\sqrt{5v}+d\sqrt{3u})$.
7. $\dfrac{7\sqrt{3}+2\sqrt{5}}{2\sqrt{3}-\sqrt{5}}$.
8. $\dfrac{14}{3-\sqrt{7}}$.
9. $\dfrac{\sqrt{\frac{7}{3}}+\sqrt{\frac{3}{5}}}{\sqrt{\frac{7}{3}}-\sqrt{\frac{3}{5}}}$.
10. $\dfrac{4\sqrt{x}-7\sqrt{y}}{2\sqrt{x}-3\sqrt{y}}$.
11. $\dfrac{a^2\sqrt{a}}{\sqrt{a+4}-2}$.
12. $\dfrac{\sqrt{3x}+\sqrt{x+y}}{\sqrt{3x}+2\sqrt{x+y}}$.
13. $\dfrac{x+3}{\sqrt{x^2-5}+2}$.
14. $\dfrac{\sqrt{abc}}{\sqrt{ab}+\sqrt{bc}}$.
15. $(\sqrt{x-3}+\sqrt{x+5})^2$.
16. $(\sqrt{x+1}+5\sqrt{x-1}) \div (3\sqrt{x+1}-2\sqrt{x-1})$.
17. $(2\sqrt{z^2+7}+6\sqrt{z^2+3}) \div (\sqrt{z^2+7}-\sqrt{z^2+3})$.

Solve the following equations for x and simplify the result.

18. $\sqrt{5}x-14=0$.
19. $(2+\sqrt{3})x-4+\sqrt{2}=0$.
20. $\sqrt{7}x+\sqrt{2}+\sqrt{3}x-\sqrt{11}=0$.
21. $x\sqrt{2a}+x\sqrt{3b}+3\sqrt{2a}-5\sqrt{3b}=0$.
22. Is $\dfrac{4+\sqrt{10}}{3}$ a root of $3x^2-8x+3=0$?
23. Is $\dfrac{-3+\sqrt{14}}{5}$ a root of $5x^2+6x-1=0$?

Chapter 6

Functions, Coördinates, and Graphs

35. Constants and Variables. A number symbol that retains a fixed value throughout a given problem is a **constant.** One that may assume different values during the course of the problem is a **variable.**

Thus, when we are dealing with freely falling bodies near the surface of the earth, we have the physical law

$$s = \tfrac{1}{2}gt^2,$$

where $\frac{1}{2}$ and g are constants and s and t, the distance the body has fallen from rest and the time, are variables.

Similarly, if we are dealing with circles, we know that

$$C = 2\pi r,$$

where 2 and π are constants but C and r, the circumference and the radius, are variables since they may take different values for different circles.

36. Functions. The equation

$$s = \tfrac{1}{2}gt^2,$$

which expresses the law of falling bodies, enables us to compute the value of s when the value of t is given. For, taking $g = 32$, approximately, we find

$$\text{if } t = 1, \quad \text{then } s = 16; \qquad \text{if } t = 3, \quad \text{then } s = 144;$$

and so on. We shall express this fact that s can be computed when t is given by the statement: s is a function of t.

The fact that the value of one variable can sometimes be computed when the value of another one is given is not new to the student. What probably is unfamiliar is the mathematical way of saying that such a computation is possible. As we shall, from now on, frequently use this technical mathematical form of statement, we shall ask the student to learn the following definition.

The statement, y is a function of x, means that, when a value has been assigned to x, then the value (or a set of values) of y is definitely fixed.

The variable to which a value has been assigned is called the **independent variable.** The one whose value is then determined is the **dependent variable.**

EXAMPLE 1. The area of a circle is a function of the radius. For, there exists a formula

$$A = \pi r^2,$$

by means of which the value of A can be found when the value of r is given.

In this case, since we are assigning values to r and computing A, we call r the independent variable and A the dependent one.

EXAMPLE 2. The first-class postage on a letter is a function of its weight. For, this postage is fixed by law at three cents for each ounce or fraction of an ounce.

37. The Function Notation. We deal so frequently in mathematics with two variables, one of which is a function of the other, that a special mathematical symbolism has been devised to express this relationship. The statement, y is a function of x, is written symbolically in the form

$$y = f(x).$$

This symbol should be read either as, "y is a function of x" or as, "y equals the f-function of x."

It should be clearly understood that the symbol $f(x)$ does not represent a product, f times x; it is a symbol standing for whatever formula is needed to compute y when x is given.

Thus, the statement, the area of a circle is a function of the radius, may be written, briefly, in the symbolic form

$$A = f(r).$$

This statement, by itself, does not tell us what the formula is that expresses the value of A in terms of r. It merely tells us that there is such a formula. It is only from a theorem proved in geometry that we learn, in this case, that $f(r) = \pi r^2$, so that

$$A = f(r) = \pi r^2.$$

Similarly, if we are dealing with a problem in which the value of y may be computed by means of the equation

$$y = x^3 - 2x + 8,$$

then (1) y is a function of x because its value is fixed when the value of x is given and (2) in this problem, $f(x) = x^3 - 2x + 8$ because this is the particular expression we use to compute the value of y. We may therefore write

$$y = f(x) = x^3 - 2x + 8,$$

the first equality stating that y is a function of x and the second defining the particular formula we are using to find the value of y.

Suppose, in this problem, we wish to find the value of y when $x = 2$. We merely replace x by 2 everywhere in the function. We have

$$f(2) = 2^3 - 2 \cdot 2 + 8 = 12,$$

giving $y = 12$ when $x = 2$. Similarly, by putting $x = 5$, we find $f(5) = 123$ so that $y = 123$ when $x = 5$.

Sometimes two or more functions occur in one problem. In that case, we denote them by different letters. We may, in fact, use any one of the symbols

$$f(x), \qquad g(x), \qquad F(x), \qquad H(x), \qquad \phi(x),$$

and so on, to denote a function of x. If, then, two different functions occur in one problem, we may denote one of them by one of the above symbols and the other by another symbol.

Exercises

In each of the following exercises, first write the given statement in symbolic form, then give the formula by means of which the value of the dependent variable may be computed.

1. The volume of a cube is a function of the length of its edge.

2. The total surface of a cube is a function of the length of its edge.

3. The amount due on a loan of \$100 at 4%, simple interest, is a function of the time in years.

4. The Fahrenheit temperature of a body is a function of its Centigrade temperature.

5. The length of the hypotenuse of an isosceles right triangle is a function of the length of one of its legs.

6. The perimeter of an equilateral triangle is a function of the length of one of its sides.

7. If $f(x) = 2x + 5$, find $f(1), f(-3), f(0), f(\frac{1}{2})$.

8. If $f(n) = 3n^2 + 2n$, find $f(-2), f(-\frac{2}{3}), f(5), f(\frac{1}{5})$.

9. If $f(t) = \sqrt{t} + 1/t$, find $f(1), f(4), f(6), f(a)$.

10. If $F(x) = x^2 - 3x + 3$, find $F(0), F(-1), F(2), F(h-1)$.

11. If $g(x) = \dfrac{3x+1}{x^2+1}$, find $g(8), g(-2), g(1/y), g(z^2)$.

12. If $H(x) = 12x^2 + 10x - 4$, find $H(2x) - 2H(x)$, $H(x/2) - \frac{1}{2}H(x)$.

13. If $H(x) = 3x + 7$, find $H(x^3), H(1/y), H[H(x)]$.

14. If $f(x) = \dfrac{x+1}{x-1}$ and $F(x) = 2x^2 + 3$, find $f(3) + F(1)$ and $f(3) \cdot F(1)$.

Solve the following equations, given $f(x) = 3x - 7$ and $g(x) = 2x + 9$.

15. $f(x) = 5$. **16.** $f(x) = 2g(x)$. **17.** $f(2x) = g(x)$.

18. $f(x) \cdot g(x) = 0$. **19.** $f(x/3) = 3g(x)$. **20.** $f[g(x)] = 0$.

21. Two numbers differ by 11. Express (*a*) their product and (*b*) the sum of their squares as a function of the smaller number.

22. Two men start from the same place at the same time in the same direction. One travels 40, and the other, 55 miles an hour. Find the distance between them as a function of the time.

38. Rectangular Coördinates. Let $X'X$ and $Y'Y$ (Fig. 2) be two given, mutually perpendicular lines, intersecting at O. These two lines are

called the **coördinate axes**; $X'X$ is the **x-axis** and $Y'Y$ is the **y-axis.** Their intersection, O, is the **origin.** Distances on the x-axis are positive if they are measured from left to right and negative if they are measured in the opposite direction. Distances on the y-axis are positive if they are measured upward and negative if measured downward.

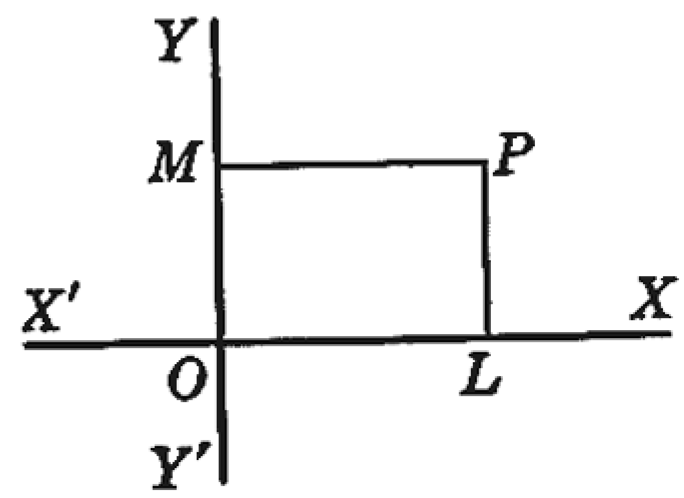


FIG. 2

Let P be any point in the plane of the coördinate axes. From P, drop perpendiculars to the x- and y-axes and denote the feet of these perpendiculars by L and M respectively. The length of the segment OL, measured from O to L and taken with its proper sign, is called the **x-coördinate,** or **abscissa,** of P. Similarly, the length of OM measured from O to M and taken with its proper sign, is the **y-coördinate,** or **ordinate,** of P. The two numbers, x and y, are the coördinates of P and are written thus: (x, y). Observe that the two coördinates are enclosed by parentheses and separated by a comma.

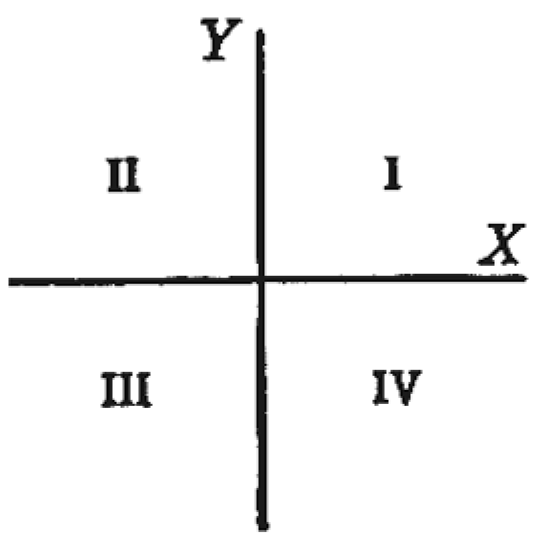


FIG. 3

It will be seen from the figure that, numerically, $MP = OL$ and $LP = OM$. Moreover, these lengths will agree in sign if MP is considered to be positive if it is measured to the right and LP is positive if it is measured upward. We shall, accordingly, frequently find it convenient to speak of the lengths of MP and LP, rather than those of OL and OM, as the coördinates of P.

The coördinate axes divide the plane into four parts, called **quadrants,** which are numbered as shown in the adjoining figure (Fig. 3).

39. Plotting Points. If we are given a pair of real numbers (x, y), we can always find a point P having x for its abscissa and y for its ordinate. Suppose, for example, the given coördinates are $(3, -2)$. We first determine the point L by laying off, on the x-axis, 3 units to the right from O. The point P is then located by measuring, on a parallel to the y-axis, 2 units downward from L (Fig. 4).

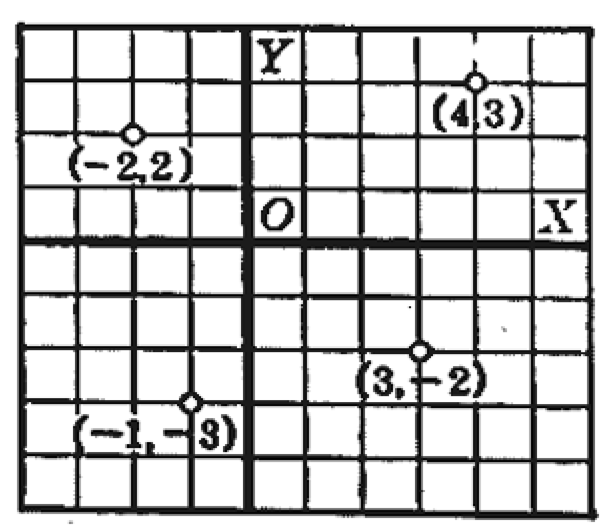


FIG. 4

When a point is located in this way, by means of its coördinates, it is said to be **plotted.** In the adjoining figure, we have plotted the points having the coördinates $(3, -2)$, $(4, 3)$, $(-2, 2)$, and $(-1, -3)$.

When it is necessary to plot points, time can be saved, and greater accuracy can be secured, by using coördinate paper, that is, paper which has been ruled with equally spaced lines parallel to the coördinate axes. The use of such paper is recommended in all work dealing with the plotting of points.

Exercises

In all exercises dealing with coördinates, throughout the entire book, drawing a suitable figure is an essential part of the exercise.

1. Plot the points whose coördinates are: $(3, 5)$, $(-2, 4)$, $(-4, -3)$, $(1, -5)$, $(5, 0)$, $(0, -3)$.

2. Plot the points whose coördinates are: $(2, -3)$, $(4, 1)$, $(-6, 2)$, $(-4, -3)$, $(0, 4)$, $(-2, 0)$.

3. Draw the triangle having the following points as vertices:

(*a*) $(4, 2)$, $(1, -1)$, $(-3, 5)$; (*b*) $(1, 4)$, $(5, -2)$, $(-2, -4)$.

4. Find the lengths of the sides and the hypotenuse of the right triangle whose vertices are:

(*a*) $(-1, 2)$, $(3, 2)$, $(3, 5)$; (*b*) $(-2, -5)$, $(-2, 7)$, $(3, 7)$.

5. Draw the rectangle and find its area, given that its vertices are:

(*a*) $(4, 3)$, $(-2, 3)$, $(-2, 1)$, $(4, 1)$; (*b*) $(4, 7)$, $(-3, 7)$, $(-3, 2)$, $(4, 2)$.

6. Three vertices of a rectangle are $(3, 5)$, $(7, 5)$, and $(7, -1)$. Find the coördinates of the fourth vertex and the length of a diagonal.

7. Find the coördinates of the midpoint of the segment joining:

(*a*) $(0, 0)$ to $(8, -6)$; (*b*) $(1, 3)$ to $(5, 9)$.

8. Find the coördinates of the point 2 units to the right, and 4 units above, the point $(2, -1)$.

9. The center of a square is at the origin; its sides are parallel to the coördinate axes and are 12 units long. Find the coördinates of the vertices of the square.

10. In what quadrant does a point lie (*a*) if both of its coördinates are positive; (*b*) if both are negative?

11. What is the ordinate of any point lying on the x-axis?

12. What is the abscissa of any point lying on the y-axis?

13. What is the locus of a point such that (*a*) $x = 4$; (*b*) $y = -2$?

40. The Graph of a Function. *The graph of the function $f(x)$ is the curve formed by the points whose coördinates (x, y) satisfy the equation*

$$y = f(x).$$

This curve is also called the *graph (or locus) of the equation $y = f(x)$.*

One outstanding advantage of the graph of a function is that it presents quickly to the eye the relationship of the values of the function to those of the independent variable. For this reason, graphs are widely used in non-mathematical, as well as in mathematical, studies of the behavior of functions.

Some methods of drawing the graph of a given function, at least approximately, are illustrated by the following examples.

EXAMPLE 1. Draw the graph of the function $f(x) = 2x + 2$.

We first equate the given function to y, giving

$$y = 2x + 2.$$

Next, we assign to x a set of values, chosen to suit our own convenience. We substitute each of these values of x in the given equation, find the corresponding values of y, and make a table of the resulting pairs of values, as follows:

x	-2	-1	0	1	2	3
y	-2	0	2	4	6	8

We next plot on coördinate paper the points whose coördinates are the pairs of values of x and y given in this table (Fig. 5a). A smooth curve drawn through these points is, at least approximately, the required graph (Fig. 5b).

It is shown in Figure 5b that the graph of the equation $y = 2x + 2$ is a straight line. This is an illustration of the fact, which we shall prove in Art. 162, that *the graph of an equation of the form*

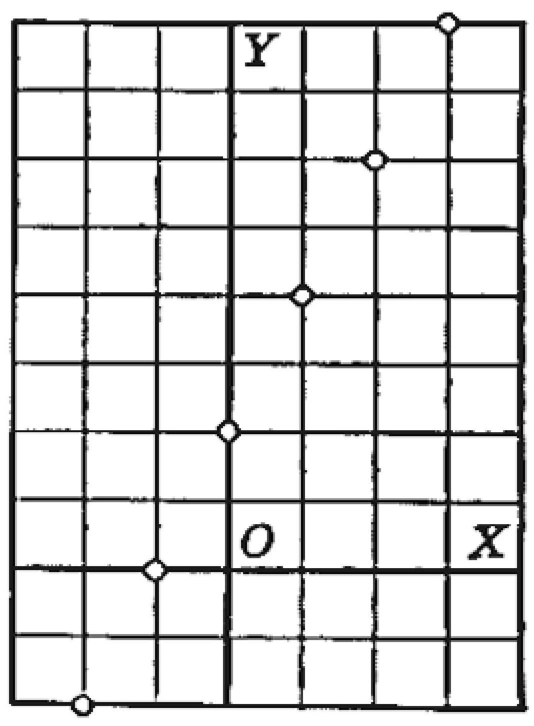


FIG. 5a

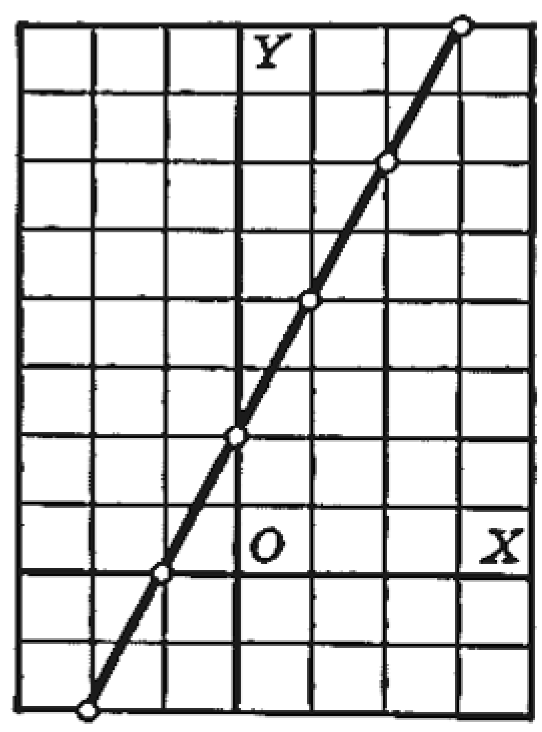


FIG. 5b

$$ax + by + c = 0,$$

where a, b, and c are constants such that a and b are not both zero, is a straight line. Because of this interesting property of its graph, *an equation of the form* $ax + by + c = 0$ *is called a* **linear equation.**

EXAMPLE 2. Draw the graph of the function $f(x) = x^2 + 2x - 3$.

Equate the given function to y: $y = x^2 + 2x - 3$.

Assign values to x, compute the corresponding values of y from the given equation, and tabulate the results.

x	-4	-3	-2	-1	0	1	2
y	5	0	-3	-4	-3	0	5

By plotting the points whose coördinates are given in this table and drawing a smooth curve through them, we obtain an approximate graph of the given function (Fig. 6). This curve is called a **parabola** and will be studied more fully in Chapter 23.

The graph of a given equation can be drawn more accurately by assuming also fractional values for x and thus plotting more points on the curve. Whenever the form of the curve is not clearly determined by the points already plotted, or when a more accurately drawn curve

is required, the number of values assumed for x should be increased in this way.

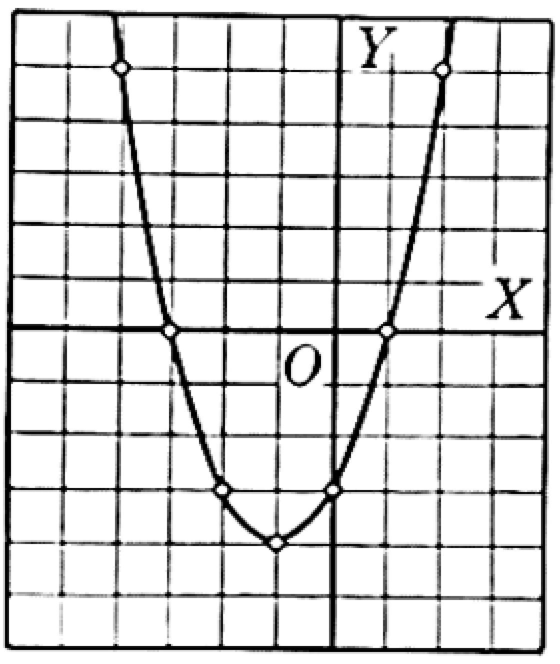


FIG. 6

Because the graph shows so clearly how the function changes with respect to the independent variable, it is also often used to exhibit these changes when the values of the function are obtained, not from an equation but from direct observation of the data.

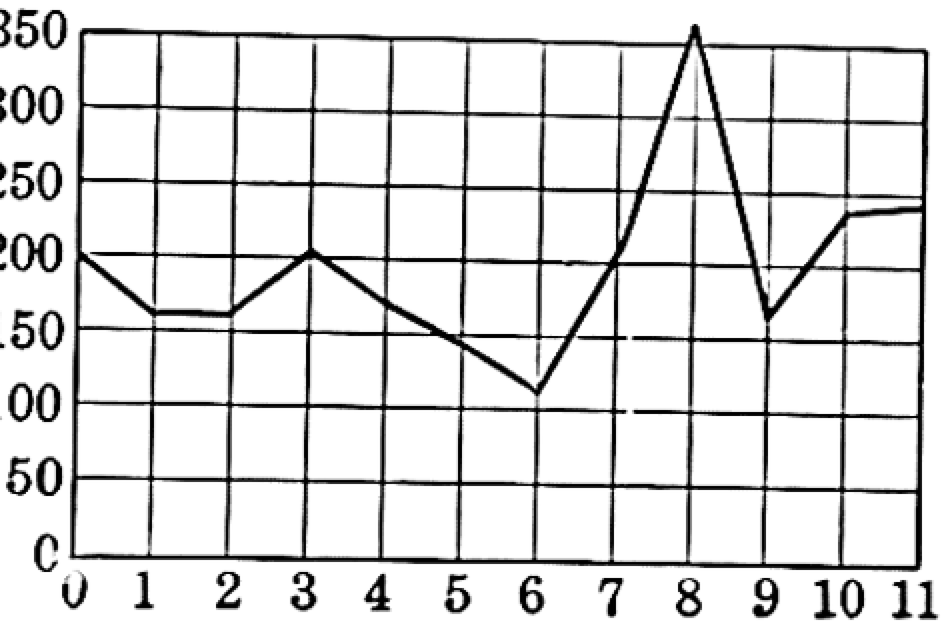


FIG. 7

In such cases, if nothing is known, or can reasonably be inferred, about the variation of the function between the values given in the table, the plotted points should be joined, not by a smooth curve, but by segments of straight lines, as in Figure 7.

EXAMPLE 3. Exhibit graphically the following table which gives the total (in millions of dollars) of new security issues in the United States, by months, during a twelve-months period, as reported to the Federal Reserve Board.

Month	0	1	2	3	4	5	6	7	8	9	10	11
Amount	200	158	157	203	169	145	109	201	357	163	240	246

In this figure, we have used units of different lengths on the two axes. It is often necessary to do this to secure a graph that is satisfactory in appearance.

41. Symmetry. The problem of drawing the graph of an equation can often be simplified considerably by noticing certain of its properties which are obvious from its equation. One of the most useful of these properties is **symmetry.**

Two points, (x, y) and $(-x, y)$, whose coördinates differ only in the signs of their abscissas, are said to be symmetric with respect to the y-axis. Similarly, the points (x, y) and $(x, -y)$ are symmetric with respect to the x-axis.

A curve is symmetric with respect to the y-axis if the symmetric point, with respect to the y-axis, of every point on it also lies on the curve. It is symmetric with respect to the x-axis if the symmetric point, with respect to the x-axis, of every point on it lies on the curve.

If the equation of a curve is formed by equating to zero a polynomial in x and y, and if this polynomial contains no odd powers of x, the curve is symmetric with respect to the y-axis. For, if the coördinates (x, y) of a point satisfy the equation, so, also, do those of the symmetric point $(-x, y)$. Similarly, *if the polynomial contains no odd powers of y, the curve is symmetric with respect to the x-axis.*

Thus, the graph of the equation $2y - x^2 = 0$ is symmetric with respect to the y-axis since its equation contains no odd powers of x (Fig. 8). The graph of $x^2 + y^2 - 25 = 0$ is symmetric with respect to both axes since its equation does not contain odd powers of either x or y (Fig. 9).

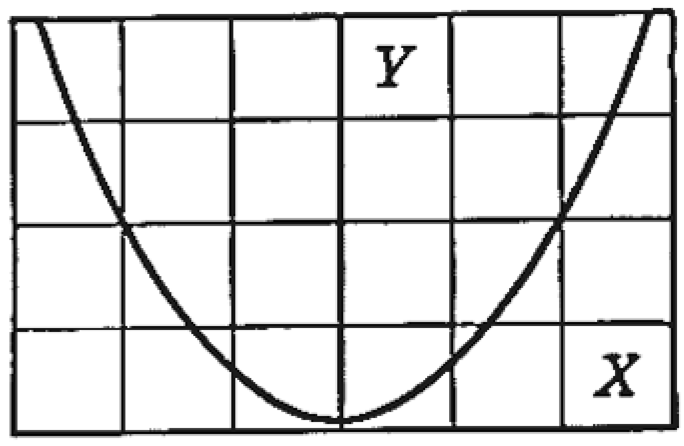


FIG. 8

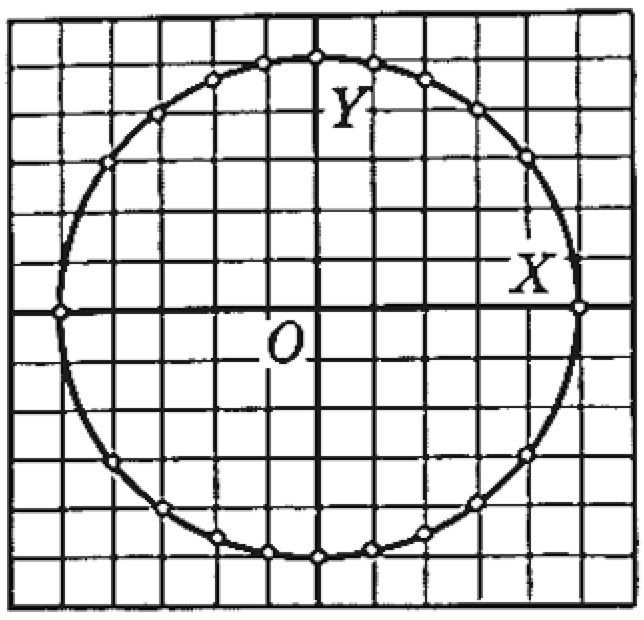


FIG. 9

If a curve is known to be symmetric with respect to the y-axis, for example, and if the part of it to the right of the y-axis has been drawn, we can plot as many points as we please on it, to the left of the y-axis, by choosing points on the graph to the right of the y-axis and plotting their symmetric points with respect to the y-axis.

Exercises

Draw the graphs of the following functions.

1. $-3x$. **2.** $4x - 5$. **3.** $7 - 2x$. **4.** $\frac{5}{2}x + \frac{2}{3}$.

Draw the graphs of the following equations.

5. $2x - y - 5 = 0$. **6.** $4x + y - 9 = 0$. **7.** $8x - 3y + 4 = 0$.

Draw the graphs of the following functions in the intervals indicated. State any symmetries with respect to either axis.

8. $x^2 - 4$, $(-4 \text{ to } 4)$. **9.** $4 - x^2$, $(-4 \text{ to } 4)$.
10. $x^2 - 2x$, $(-2 \text{ to } 4)$. **11.** $x^2 - 5x + 6$, $(0 \text{ to } 6)$.
12. $2x^2 - 5x - 2$, $(-2 \text{ to } 5)$. **13.** x^3, $(-3 \text{ to } 3)$.
14. $x^3 - 9x$, $(-4 \text{ to } 4)$. **15.** $x^4 - 9x^2$, $(-4 \text{ to } 4)$.

Draw the graphs of the following equations in the intervals for x indicated. State any symmetries with respect to either axis.

16. $y^2 = x$, $(0 \text{ to } 9)$. **17.** $y^2 = 3x + 3$, $(-1 \text{ to } 6)$.
18. $x^2 + y^2 = 36$, $(-6 \text{ to } 6)$. **19.** $y^2 - x^2 = 1$, $(-5 \text{ to } 5)$.
20. $y^2 = x^3$, $(0 \text{ to } 4)$. **21.** $y^3 = x^2$, $(-8 \text{ to } 8)$.

22. According to the Table of Mortality used by many life insurance companies, out of 100,000 people living at age 10, the number (in thousands) living at certain other ages is given by the following table.

Age	10	15	20	25	30	35	40	45	50	55	60	65	70	75
No.	100	96	93	89	85	82	78	74	70	65	58	49	39	26

Draw the graph of the number living as a function of the age. Estimate from your graph the number living at age 38. At approximately what age are there 62,000 living?

23. The amount due on \$1, at 5% compound interest, varies with the time according to the following table.

Years	0	2	4	6	8	10	12	14
Amount	1.00	1.10	1.22	1.34	1.48	1.63	1.80	1.98

Draw the graph of the amount due as a function of the time.

24. The number of customers who entered a certain store during the successive hours that the store was open on a certain day is given by the following table.

Time	1	2	3	4	5	6	7	8	9	10	11
Number	31	46	69	73	57	65	71	93	97	63	52

Show graphically the number of customers as a function of the time. Draw a graph formed by line segments joining the successive points.

Chapter 7

Simultaneous Linear Equations

42. Linear Equations in Two Unknowns. An equation of the form

$$ax + by = c,$$

where a, b, and c are constants and a and b are not both zero, is called an **equation of first degree** or, since its graph is a line * (Art. 40), a **linear equation,** in x and y.

Suppose we have two such equations

$$a_1x + b_1y = c_1,$$

and

$$a_2x + b_2y = c_2.$$

We shall seek a pair of values of x and y which, when substituted in these two equations, will make both equations true. Such a pair of values is a **simultaneous solution** of the two equations.

In the following articles, we shall discuss several of the most important ways of finding the simultaneous solution of two linear equations.

43. Solution by Addition or Subtraction. In this method, we multiply the two equations by suitable constants in such a way that the coefficients of one variable are made to be either equal, or equal but opposite in sign. By subtracting, or adding, the resulting equations, we obtain an equation in which this variable does not appear.

The variable that does not appear in the resulting equation is said to be **eliminated.** The method is, accordingly, sometimes called the method of *elimination by addition or subtraction.*

EXAMPLE. Solve for x and y: $6x - 4y = 17,$

$$10x + 3y = 9.$$

We can make the coefficients of x equal, in the two equations, by multiplying the first equation by 5 and the second by 3. This gives

$$30x - 20y = 85,$$
$$30x + \ \ 9y = 27.$$

If we subtract the first equation from the second, member by member, we obtain an equation in which x does not appear; that is, x will be eliminated between the two equations. The result is

$$29y = -58, \quad \text{or} \quad y = -2.$$

If we substitute this value of y in the first of the given equations, we have

$$6x + 8 = 17,$$

so that

$$6x = 9, \quad \text{or} \quad x = \tfrac{3}{2}.$$

* We shall use the word "line", throughout, to mean a straight line.

As a check, we substitute $x = \frac{3}{2}$, $y = -2$ in the second given equation. We obtain $15 - 6 = 9$, which is true. Hence, $x = \frac{3}{2}$, $y = -2$ is the required simultaneous solution.

As an exercise, the student should solve these two simultaneous equations by first eliminating y.

44. Solution by Substitution. In this method, we first solve one equation for one variable in terms of the other and substitute the resulting expression in the other equation.

EXAMPLE. Solve for x and y: $2x + 5y = 19$,

$$7x - 3y = 5.$$

Solve the first equation for x in terms of y:

$$x = \frac{19 - 5y}{2}.$$

Substitute this value for x in the second equation:

$$7\left(\frac{19 - 5y}{2}\right) - 3y = 5,$$

or

$$133 - 35y - 6y = 10.$$

Hence,

$$-41y = -123, \quad \text{or} \quad y = 3.$$

On substituting $y = 3$ in the first given equation, we have

$$2x + 15 = 19,$$

from which

$$2x = 4, \quad \text{or} \quad x = 2.$$

CHECK. Substitute $x = 2$, $y = 3$ in the second given equation. We have $14 - 9 = 5$. Hence, $x = 2$, $y = 3$ is the required solution.

45. Solution by Graphs. Let it be required to solve the simultaneous equations

$$4x + 3y = 2,$$

and

$$2x + 5y = 8.$$

The graph of the first equation is a line l_1 (Fig. 10) having the property that the coördinates of every point on this line satisfy the first equation. Similarly, the coördinates of every point on the line l_2 satisfy the second given equation. Consider the point of intersection, P, of l_1 and l_2. Since P lies on both lines, its coördinates satisfy both equations, that is, they constitute the simultaneous solutions of the two given equations.

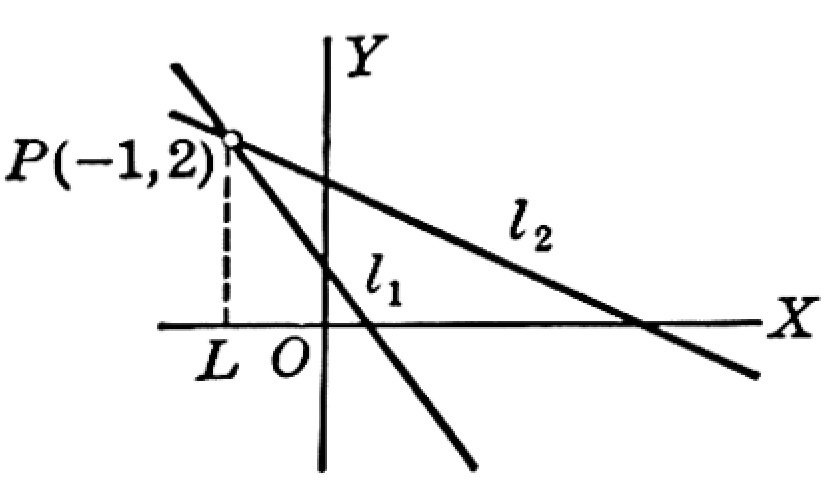


FIG. 10

From the figure, we find by measurement that the coördinates of P are $x = OL = -1$ and $y = LP = 2$. These values check in the given equations and constitute the required simultaneous solution of the given equations.

By applying precisely the same reasoning to any two linear equations, we find that, *to solve graphically two linear equations*

$$a_1x + b_1y = c_1,$$

and

$$a_2x + b_2y = c_2,$$

draw the graphs of the two equations and locate the point of intersection of the resulting lines. The coördinates of this point (if it exists) are the simultaneous solution of the two equations.

The graphical solution is usually only an approximate one since we cannot be sure either that the graphs have been drawn with perfect accuracy or that the coördinates of their intersection have been measured exactly. This method is, however, helpful in enabling us to visualize the work we have been doing algebraically and it also helps to clarify certain types of exercises, such as those given in the following two examples, in which the meaning of the results of the algebraic computations are rather obscure.

EXAMPLE 1. Solve:

$$3x - 4y = 5,$$
$$6x - 8y = -7.$$

The graphs of these two equations are parallel lines (Fig. 11). If the two equations had a simultaneous solution, this solution would be constituted by the coördinates of the point of intersection of the lines. Since parallel lines do not intersect, there can be no simultaneous solution.

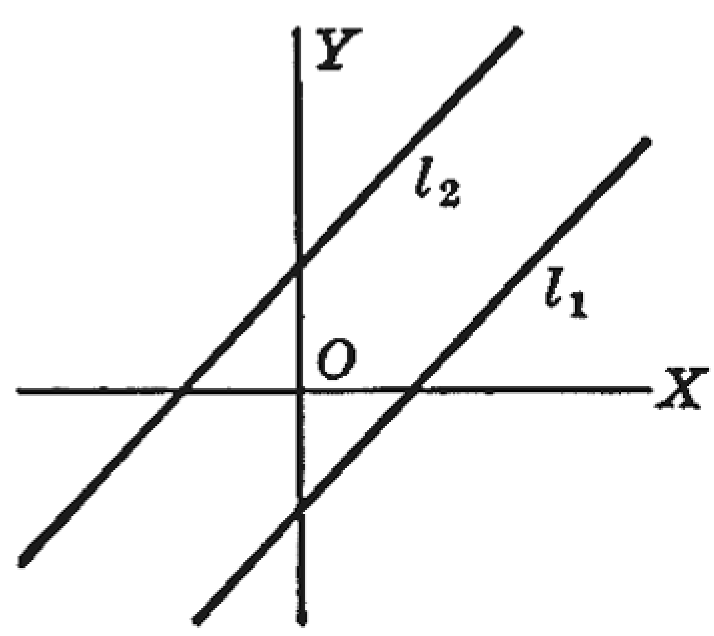


FIG. 11

If we attempt to solve the equations algebraically by the method of addition or subtraction, we would multiply the first equation by 2 and subtract the second one from the result. This gives

$$\begin{array}{r} 6x - 8y = 10, \\ \underline{6x - 8y = -7,} \\ 0x - 0y = 17. \end{array}$$

The final equation means that, if there were a solution, it would have to be a pair of values of x and y such that $0x - 0y = 17$. Since no such values of x and y exist, there is no simultaneous solution.

Two equations that do not have a simultaneous solution are said to be **inconsistent**. The equations of Ex. 1 constitute an illustration of a pair of inconsistent equations.

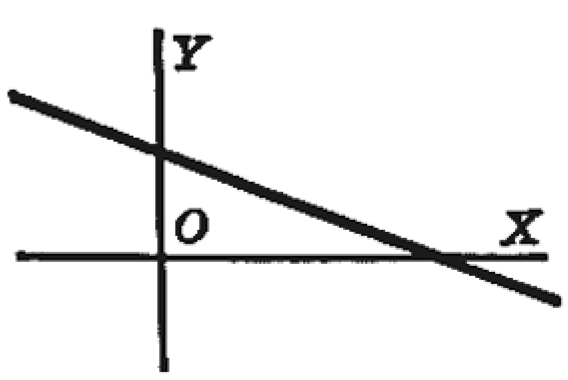


FIG. 12

EXAMPLE 2. Solve:

$$2x + 5y = 4,$$
$$6x + 15y = 12.$$

When we draw the graphs of these equations, we find that they coincide (Fig. 12). Hence, every point that lies on one of the lines lies on the other also. It follows that every solution of one of the equations must also satisfy the other one. The second equation is, in fact, obtained from the first by multiplying it by 3.

Two linear equations are said to be **dependent** if one of them can be obtained from the other by multiplying by a constant. Every solution of one of two dependent equations is a solution of the other also.

Exercises

Solve by the method of addition or subtraction.

1. $3x + y = 15,$
$3x - 2y = 6.$

2. $5x + 6y = 8,$
$x + 2y = 4.$

3. $4x + 7y = 5,$
$3x - 2y = 11.$

4. $4x - 7y = 11,$
$3x - 2y = 5.$

5. $7a - b = 3,$
$a + 2b = 4.$

6. $3r - 7s = 18,$
$5r + 3s = 8.$

Solve by the method of substitution.

7. $x = 5 - 3y,$
$4x + 7y = 15.$

8. $2y = 4x + 9,$
$3x - 8y + 10 = 0.$

9. $5x - 3y = 14,$
$2x + y = 9.$

10. $2x + 7y = 13,$
$5x + 4y = 10.$

11. $7s + 2t = 11,$
$5s + 3t = 4.$

12. $5u + 4v = 7,$
$7u + 3v = 6.$

Solve graphically.

13. $2x + y - 7 = 0,$
$x - 2y + 4 = 0.$

14. $5x + 2y = 18,$
$x - y = 5.$

15. $2x + 5y = 2,$
$4x + y = 13.$

16. $3x + 2y = 5,$
$8x + 3y = 18.$

Solve by any method. If the equations are inconsistent or dependent, show that the graphs are parallel or coincident.

17. $12x - 16y = 24,$
$15x - 20y = 30.$

18. $4x + 14y = 11,$
$12x + 42y = 7.$

19. $1.2x + 1.1y = 3.8,$
$1.6x + 2.3y = 3.4.$

20. $3.7x - 1.2y = 5.1,$
$0.7x + 1.3y = 8.6.$

21. $2(4x - y) - 3(5x + y) = 7,$
$9(2x + 5y) - 25(x + 2y) = 2.$

22. $4(3x + y) - 2x + 11y = 15,$
$4(x + y) + 2x + 5y = 9.$

23. $ax - by = a^2 + b^2,$
$bx + ay = a^2 + b^2.$

24. $mx + ny = m^2 - n^2,$
$mx - ny = m^2 + n^2.$

25. $a^2x + by = 2ab,$
$abx + ay = a^2 + b^2.$

26. $ax + by = c,$
$bx + ay = d.$

27. $\dfrac{3x + 5y - 7}{x - 3y + 1} + \dfrac{3}{2} = 0,$
$\dfrac{4x - y - 6}{x + 2y + 1} + \dfrac{2}{3} = 0.$

28. $\dfrac{x - 5y - 6}{2x + y - 3} = \dfrac{3}{4},$
$\dfrac{3x + 8y + 1}{x + y - 1} = \dfrac{5}{2}.$

Solve by first finding $1/x$ and $1/y$.

29. $\dfrac{5}{x}+\dfrac{3}{y}=4,$
$\dfrac{3}{x}-\dfrac{1}{y}=1.$

30. $\dfrac{4}{x}-\dfrac{1}{y}=2,$
$\dfrac{1}{x}+\dfrac{6}{y}=3.$

31. $\dfrac{a}{x}-\dfrac{b}{y}=2,$
$\dfrac{ab}{x}-\dfrac{ab}{y}=a+b.$

46. Linear Equations in Three Unknowns. To solve three simultaneous linear equations in three unknowns, we shall first eliminate one of the unknowns between each of two pairs of the given equations. We then solve the resulting two equations for the remaining two unknowns by any one of the methods of the preceding articles. When we have found the values of these two unknowns, we substitute their values in one of the given equations and solve for the third variable. As a check, we substitute the values found for the three variables in the remaining two equations to assure ourselves that these equations are also satisfied.

In special cases, the given equations may be **inconsistent,** and have no solution, or they may be **dependent** and have an unlimited number of solutions.

EXAMPLE. Solve:

$$2x+y+z=14, \quad (1)$$
$$3x+2y+6z=54, \quad (2)$$
$$8x+y-z=12. \quad (3)$$

By inspection, we find that y is the easiest variable to eliminate. Multiply equation (1) by 2 and subtract equation (2) from the result. We obtain

$$x-4z=-26. \quad (4)$$

Subtract equation (1) from equation (3). We have

$$6x-2z=-2. \quad (5)$$

On solving equations (4) and (5) for x and z, we find that $x=2$ and $z=7$.

Substitute these values of x and z in equation (1). We obtain

$$4+y+7=14.$$

Hence, $y=3$. Since these values of x, y, and z are found by substitution also to satisfy equations (2) and (3), the required solution is $x=2$, $y=3$, $z=7$.

Exercises

Solve the simultaneous equations and check your results.

1. $2x-3y+8z=19,$
$3x-y+z=6,$
$2x+4y-3z=7.$

2. $2x+y-z=5,$
$x+3y+2z=10,$
$3x+5y+4z=14.$

3. $6x + y + 5z = 5,$
$5x - y + 4z = 2,$
$3x + 2y + z = 1.$

4. $3x - 4y + z = 2,$
$x + 3y + 2z = 7,$
$5x + 5y + 11z = 1.$

5. $3A - B - 2C = 1,$
$4A + 3B - C = 1,$
$2A - B + 2C = 3.$

6. $u + v + w = 8,$
$5u - 5v + 4w = 3,$
$u - 6v + 3w = 1.$

7. $r - 2s = 7,$
$3s + 5t = 11,$
$5r - 2t = -3.$

8. $l - m + 6n = 3,$
$9l - 2m + 2n = 3,$
$4l + m + n = 0.$

9. $ax + by + cz = d,$
$bx + cy + az = d,$
$cx + ay + bz = d.$

10. $x + y + z = 1,$
$ax + by + cz = d,$
$a^2x + b^2y + c^2z = d^2.$

11. $\frac{4}{x} + \frac{5}{y} + \frac{3}{z} = -2,$
$\frac{2}{x} - \frac{1}{y} + \frac{6}{z} = 4,$
$\frac{6}{x} - \frac{4}{y} - \frac{9}{z} = 4.$

12. $\frac{6}{x} - \frac{4}{y} + \frac{3}{z} = 3,$
$\frac{15}{x} + \frac{8}{y} - \frac{7}{z} = 2,$
$\frac{3}{x} + \frac{4}{y} + \frac{1}{z} = 4.$

13. $2x + y + 2z + w = 10,$
$2x - 3y + z + w = 18,$
$5x + 2y + z - w = 7,$
$4x - y - 3z + 2w = 1.$

14. $3x + 7y - z + 2w = 13,$
$2x - 6y - 3z + w = 3,$
$x + 2y - 4z + w = 11,$
$4x + 3y + 4z - 2w = 5.$

47. Determinants of the Second Order. The symbol

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

is called a **determinant of the second order.** It is used to denote the expression $a_1b_2 - a_2b_1$; that is, by definition,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

The expression $a_1b_2 - a_2b_1$ is called the **expansion** of the determinant and the four numbers a_1, a_2, b_1, and b_2 are its **elements.**

Thus, from the definition, it follows that

$$\begin{vmatrix} 6 & -2 \\ 4 & 3 \end{vmatrix} = 6 \cdot 3 - 4(-2) = 26. \qquad \begin{vmatrix} 5 & 8 \\ 0 & -2 \end{vmatrix} = 5(-2) - 0 \cdot 8 = -10.$$

Exercises

Expand the determinants.

1. $\begin{vmatrix} 4 & 7 \\ 1 & 3 \end{vmatrix}.$ **2.** $\begin{vmatrix} 3 & 5 \\ -2 & 9 \end{vmatrix}.$ **3.** $\begin{vmatrix} 7 & 3 \\ 6 & 0 \end{vmatrix}$ **4.** $\begin{vmatrix} -2 & 7 \\ 0 & 6 \end{vmatrix}.$

5. $\begin{vmatrix} 5 & 4 \\ 6 & 7 \end{vmatrix}.$ **6.** $\begin{vmatrix} 0 & -1 \\ -3 & 7 \end{vmatrix}.$ **7.** $\begin{vmatrix} x & 2 \\ y & 5 \end{vmatrix}.$ **8.** $\begin{vmatrix} r & t \\ s & u \end{vmatrix}.$

48. Solution of Two Linear Equations by Determinants. With the aid of the determinant notation, we shall derive a set of formulas for the solution of two simultaneous linear equations.

If, from the equations

$$a_1x + b_1y = c_1,$$

and

$$a_2x + b_2y = c_2, \tag{6}$$

we eliminate first y, and then x, by the method of addition or subtraction (Art. 43), we obtain the two equations

$$(a_1b_2 - a_2b_1)x = c_1b_2 - c_2b_1,$$

and

$$(a_1b_2 - a_2b_1)y = a_1c_2 - a_2c_1. \tag{7}$$

If the number $a_1b_2 - a_2b_1 \neq 0$, we can divide each of these equations by it. We have

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \qquad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}. \tag{8}$$

It will be observed that the denominators in the two equations (8) are the expansions of the determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, that the numerator for the value of x differs from the denominator only in that a_1 and a_2 are replaced by c_1 and c_2, respectively, and that the numerator for the value of y differs only in that b_1 and b_2 are replaced by c_1 and c_1. Hence, we may write equations (8) in the form

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}. \tag{9}$$

These formulas give us the simultaneous solution of equations (6) *provided the value of the determinant* $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ *that appears in both denominators is different from zero.*

If the determinant in the denominators is equal to zero, and if at least one of the determinants in the numerators is different from zero, then the given equations are **inconsistent,** that is, they have no simultaneous solution (Art. 45).

If all three of the determinants that appear in equations (4) are equal to zero, the two equations (1) are **dependent,** that is, every solution of one equation is a solution of the other also (Art. 45).

EXAMPLE. Solve by determinants: $5x + 8y - 6 = 0,$
$4y - 3x + 2 = 0.$

We first write these equations in the form given in equations (6), with the x-terms first, then the y-terms, and the constant terms in the second member. We shall also multiply the second equation by -1. We then have

$$5x + 8y = 6,$$
$$3x - 4y = 2.$$

From equations (9), we have, as the required solution,

$$x = \frac{\begin{vmatrix} 6 & 8 \\ 2 & -4 \end{vmatrix}}{\begin{vmatrix} 5 & 8 \\ 3 & -4 \end{vmatrix}} = \frac{6 \cdot (-4) - 2 \cdot 8}{5 \cdot (-4) - 3 \cdot 8} = \frac{-24 - 16}{-20 - 24} = \frac{-40}{-44} = \frac{10}{11}.$$

$$y = \frac{\begin{vmatrix} 5 & 6 \\ 3 & 2 \end{vmatrix}}{\begin{vmatrix} 5 & 8 \\ 3 & -4 \end{vmatrix}} = \frac{5 \cdot 2 - 3 \cdot 6}{-44} = \frac{10 - 18}{-44} = \frac{-8}{-44} = \frac{2}{11}.$$

This solution should be checked by substitution in the given equations.

Exercises

1–22. Solve exercises 1 to 22, Art. 45, by determinants or prove them dependent or inconsistent.

Solve the following pairs of simultaneous equations by determinants or prove them inconsistent or dependent.

23. $5x + 3y = 1,$
$7x + 2y = 8.$

24. $3x + 2y = 18,$
$7x - 5y = 13.$

25. $5x - 2y = 7,$
$6x + 7y = 9.$

26. $x + 5y = 1,$
$11x - 7y = 9.$

27. $9x = 3y + 5,$
$6x - 2y = 7.$

28. $2x + 7y + 5 = 0,$
$3y - 4x = 8.$

49. Determinants of the Third Order. The symbol

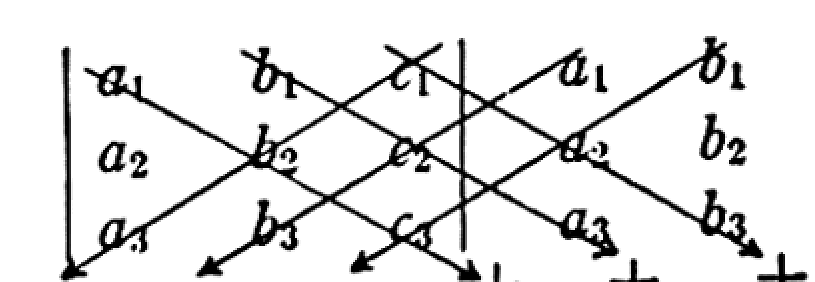
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

is called a **determinant of the third order.** It is used as an abbreviation for the following expression, which is its **expansion**

$$a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - c_1b_2a_3 - a_1c_2b_3 - b_1a_2c_3.$$

The nine numbers a_1, a_2, a_3, b_1, b_2, and so on, are the **elements** of the determinant.

The successive terms in the expansion of a determinant of the third order may be remembered by the aid of the following device: *rewrite the first two columns to the right of the determinant, as in the adjoining diagram, and draw the lines indicated. Multiply together each set of three numbers through which a line is drawn and prefix a plus sign to the product if the line extends downward to the right and a minus sign if it extends downward to the left.*



This device for finding the expansion holds *only* for determinants of the *third* order.

EXAMPLE. Expand the determinant: $\begin{vmatrix} 4 & -2 & 3 \\ 1 & 5 & 0 \\ 3 & 7 & 4 \end{vmatrix}$.

$$\begin{array}{ccc|cc} 4 & -2 & 3 & 4 & -2 \\ 1 & 5 & 0 & 1 & 5 \\ 3 & 7 & 4 & 3 & 7 \\ - & - & - & + \; + & + \end{array} \begin{aligned} &= 4\cdot 5\cdot 4 + (-2)\cdot 0\cdot 3 + 3\cdot 1\cdot 7 - 3\cdot 5\cdot 3 - 4\cdot 0\cdot 7 \\ &\quad - (-2)\cdot 1\cdot 4 = 80 + 0 + 21 - 45 - 0 + 8 = 64. \end{aligned}$$

Exercises

Expand the determinants.

1. $\begin{vmatrix} 2 & 3 & 5 \\ 1 & 8 & 3 \\ 7 & 9 & 11 \end{vmatrix}$.

2. $\begin{vmatrix} -5 & 3 & 7 \\ 3 & 0 & 4 \\ -2 & 4 & 1 \end{vmatrix}$.

3. $\begin{vmatrix} 3 & -2 & -7 \\ 9 & 5 & -4 \\ 2 & 0 & -1 \end{vmatrix}$.

4. $\begin{vmatrix} x & y & 1 \\ 2 & 3 & 6 \\ 4 & 1 & 3 \end{vmatrix}$.

5. $\begin{vmatrix} 2 & x & 6 \\ x & -1 & 1 \\ 5 & 0 & -3 \end{vmatrix}$.

6. $\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$.

50. Solution of Three Linear Equations by Determinants. Let

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1, \\ a_2x + b_2y + c_2z &= d_2, \\ a_3x + b_3y + c_3z &= d_3, \end{aligned} \tag{10}$$

be three linear equations which it is required to solve for x, y, and z.

If we solve these equations by the method of Art. 46, we obtain (provided the denominator common to all of the fractions is not equal to zero)

$$x = \frac{d_1b_2c_3 + b_1c_2d_3 + c_1d_2b_3 - c_1b_2d_3 - d_1c_2b_3 - b_1d_2c_3}{a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - c_1b_2a_3 - a_1c_2b_3 - b_1a_2c_3},$$

$$y = \frac{a_1d_2c_3 + d_1c_2a_3 + c_1a_2d_3 - c_1d_2a_3 - a_1c_2d_3 - d_1a_2c_3}{a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - c_1b_2a_3 - a_1c_2b_3 - b_1a_2c_3},$$

$$z = \frac{a_1b_2d_3 + b_1d_2a_3 + d_1a_2b_3 - d_1b_2a_3 - a_1d_2b_3 - b_1a_2d_3}{a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - c_1b_2a_3 - a_1c_2b_3 - b_1a_2c_3}.$$

It will be observed in this solution, first, that the denominator common to all of the fractions in the right members is precisely the expansion of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Further, the numerator of the expression for x differs from the denominator only in that each a is replaced by the corresponding d;

the numerator for y, in that each b is replaced by the corresponding d; and the numerator for z, in that each c is replaced by the corresponding d.

It follows that *we may write the simultaneous solution of equations* (10) *in the form*

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \tag{11}$$

provided that the value of the determinant in the denominators is not equal to zero.

If the determinant in the denominators is equal to zero, formulas (11) *fail.* It is proved in advanced mathematics that, if this determinant equals zero, the given equations are either **inconsistent** and have no solution, or they are **dependent** and have an unlimited number of solutions.

EXAMPLE. Solve by determinants:
$$2x + 9y + 3z = 7,$$
$$4x + 7y + z = 5,$$
$$3x + 4y = 2.$$

By equations (11), we have

$$x = \frac{\begin{vmatrix} 7 & 9 & 3 \\ 5 & 7 & 1 \\ 2 & 4 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & 9 & 3 \\ 4 & 7 & 1 \\ 3 & 4 & 0 \end{vmatrix}} = \frac{7\cdot 7\cdot 0 + 9\cdot 1\cdot 2 + 3\cdot 5\cdot 4 - 3\cdot 7\cdot 2 - 7\cdot 1\cdot 4 - 9\cdot 5\cdot 0}{2\cdot 7\cdot 0 + 9\cdot 1\cdot 3 + 3\cdot 4\cdot 4 - 3\cdot 7\cdot 3 - 2\cdot 1\cdot 4 - 9\cdot 4\cdot 0}$$

$$= \frac{0 + 18 + 60 - 42 - 28 - 0}{0 + 27 + 48 - 63 - 8 \ \ - 0} = \frac{78 - 70}{75 - 71} = \frac{8}{4} = 2.$$

Since we have computed the value of the denominator for x, it will not be necessary to recompute it for y and for z.

$$y = \frac{\begin{vmatrix} 2 & 7 & 3 \\ 4 & 5 & 1 \\ 3 & 2 & 0 \end{vmatrix}}{4} = \frac{2\cdot 5\cdot 0 + 7\cdot 1\cdot 3 + 3\cdot 4\cdot 2 - 3\cdot 5\cdot 3 - 2\cdot 1\cdot 2 - 7\cdot 4\cdot 0}{4}$$

$$= \frac{0 + 21 + 24 - 45 - 4 - 0}{4} = \frac{45 - 49}{4} = \frac{-4}{4} = -1.$$

$$z = \frac{\begin{vmatrix} 2 & 9 & 7 \\ 4 & 7 & 5 \\ 3 & 4 & 2 \end{vmatrix}}{4} = \frac{2\cdot 7\cdot 2 + 9\cdot 5\cdot 3 + 7\cdot 4\cdot 4 - 3\cdot 7\cdot 7 - 2\cdot 5\cdot 4 - 9\cdot 4\cdot 2}{4}$$

$$= \frac{28 + 135 + 112 - 147 - 40 - 72}{4} = \frac{16}{4} = 4.$$

The solution is $x = 2$, $y = -1$, $z = 4$. This result should be checked by substitution in the given equations.

Exercises

1–10. Solve exercises 1 to 10, Art. 46, by determinants.

51. Problems Leading to Simultaneous Linear Equations. In the following problems, as in those of Art. 21, the student must write out the equations to be solved, from the information given in the problem, before he can proceed to their solution. The method of attack on these problems is similar to that stated in the text of Art. 21, which should now be read again. The steps given there must be followed in solving these problems also, except that two or more of the unknown quantities are now to be indicated by letters and as many equations must be set up as there are unknown letters to be determined.

Problems

1. The sum of two numbers exceeds five times the smaller by unity. The difference between the numbers is 38 less than three times the larger number. Find the numbers.

2. A boy has \$1.25 in dimes and nickels. There are 18 coins in all. How many are dimes and how many are nickels?

3. If 5 is added to both the numerator and denominator of a fraction, the resulting fraction equals $\frac{11}{13}$. If 3 is added to the numerator and 6 is subtracted from the denominator, the resulting fraction equals $\frac{4}{3}$. Find the fraction.

4. The perimeter of a rectangle is 90 feet and its base exceeds twice its altitude by 3 feet. Find its dimensions.

5. An airplane made a trip of 504 miles against a head wind in 2 hr., 48 min. It returned with the wind in 2 hr. 24 min. Find its speed in still air and the velocity of the wind.

6. An airplane flew with a wind of 27 miles an hour from one city to another in 1 hr. 40 min. It returned against the wind in 2 hr. 20 min. Find the speed of the plane in still air and the distance between the cities.

7. If the larger of two numbers is divided by the smaller the quotient is 4 and the remainder 7. Five times the smaller number exceeds the larger by 12. Find the numbers.

8. A man has two investments, totaling \$6300. From one of these investments, which yields 5%, he receives \$39 more than from the other, which yields 3%. Find the amount of each investment.

9. A man bought some 4% bonds, for which he paid 90% of their face value, and some 5% bonds, for which he paid 105% of their face value. The total cost of the bonds was \$6780 and the annual yield was \$312. Find the face value of the bonds of each kind.

10. At a ball game, tickets for the bleachers were priced at 55 cents and, for the grandstand, at 85 cents. Total receipts were \$586. If tickets for the bleachers had been priced at 50 cents and, for the grandstand, at 75 cents, 200 more tickets, in all, would have been sold, twice as many grandstand tickets would have been sold, and total receipts would have been \$670. How many tickets of each kind were sold?

11. Three rectangles are equal in area. The second is 4 feet longer and 3 feet narrower than the first and the third is 14 feet longer and 8 feet narrower than the second. Find the dimensions of the first rectangle.

12. Gold loses about one-nineteenth, and silver about one-tenth, of its weight when immersed in water. If an alloy of gold and silver weighs 14.5 ounces in air and 13.41 ounces in water, find the weight, in air, of the gold and silver contained in it.

13. In the equation $y = mx + b$, find m and b, given that $y = 2$ when $x = 6$ and $y = 11$ when $x = 12$.

14. In the equation $y = ax^2 + bx + c$, find a, b, and c, given that $y = 3$ when $x = -1$, $y = -5$ when $x = 1$, and $y = 13$ when $x = 4$.

15. An estate of \$8400 is divided between a mother, son, and daughter. The mother received as much more than the son as the son received more than the daughter. Three times what the son received, plus \$1100, equals twice what the mother received plus what the daughter received. Find how much each received.

16. A mother is now the same age as the father was when their son was born. In 16 years, the sum of the ages of the father and mother will exceed 5 times the son's age by 2 years. In 3 years, three times the difference between the ages of the mother and son will exceed twice the father's age by 10 years. Find their ages.

Chapter 8

Quadratic Equations

52. Quadratic Equations. An equation that can be written in the form

$$ax^2 + bx + c = 0, \qquad a \neq 0 \tag{1}$$

where a, b, and c do not contain x, is a **quadratic equation** in x.

When a quadratic equation is written in the form shown in equation (1), it is in the **standard form.** It will frequently be necessary to simplify the equation in order to write it in the standard form.

EXAMPLE. Write the equation $(2x - 5)^2 + (4x + 3)^2 = (3x + 1)^2 + 17$ in the standard form.

By expanding the indicated squares, we obtain

$$4x^2 - 20x + 25 + 16x^2 + 24x + 9 = 9x^2 + 6x + 1 + 17,$$

or

$$11x^2 - 2x + 16 = 0.$$

Exercises

Reduce the quadratic equation to the standard form.

1. $9x^2 - 3x + 2 = 7x^2 + 5 - 6x$.
2. $(x + 3)(2x - 7) = 0$.
3. $4(x + 2)^2 + 3(x - 5)^2 = 4$.
4. $5(3x + 4)^2 = 61 + 3(2x - 3)^2$.
5. $(2y - 5)(9y + 8) = 3(y - 2)^2$.
6. $5z(2z - 1) = 4(z - 5)(z + 2)$.
7. $(ax + b)^2 = 4(cx + d)(ex + f)$.
8. $b^2x^2 - a^2(mx + k)^2 = a^2b^2$.

53. Solution by Factoring. If the first member of the standard form of a quadratic equation can be factored by inspection into two linear factors, the equation can be solved by equating each of these linear factors to zero and solving the resulting equations.

EXAMPLE 1. Solve the equation $x^2 + 5x = 24$.

First write the equation in the standard form: $x^2 + 5x - 24 = 0$.

Factor the first member: $(x + 8)(x - 3) = 0$.

This equation states that the product of the two numbers $x + 8$ and $x - 3$ is equal to zero. By Art. 4, *the product of two numbers equals zero if, and only if, at least one of the numbers is equal to zero.* Hence, the equation will be true if, and only if, either

$$x + 8 = 0, \quad \text{or} \quad x - 3 = 0.$$

The first of these equations is true only if $x = -8$ and the second, only if $x = 3$.

CHECK. $(-8)^2 + 5(-8) = 64 - 40 = 24$. $3^2 + 5 \cdot 3 = 9 + 15 = 24$.

The required roots are -8 and 3.

EXAMPLE 2. Solve the equation $15x^2 = 4x + 4$.

Write the equation in the standard form $15x^2 - 4x - 4 = 0$.
Factor the first member: $(3x - 2)(5x + 2) = 0$.
Equate the linear factors to zero and solve: $x = \frac{2}{3}$ or $x = -\frac{2}{5}$.

CHECK. $15(\frac{2}{3})^2 = 4(\frac{2}{3}) + 4$. $15(-\frac{2}{5})^2 = 4(-\frac{2}{5}) + 4$.
The required roots are $\frac{2}{3}$ and $-\frac{2}{5}$.

EXAMPLE 3. Solve the equation $9x^2 - 12x + 4 = 0$.

Factor the first member $(3x - 2)(3x - 2) = 0$.
By equating the first factor to zero, and solving, we obtain $x = \frac{2}{3}$. If we equate the second member to zero and solve, we again obtain $x = \frac{2}{3}$.

CHECK. $9(\frac{2}{3})^2 - 12(\frac{2}{3}) + 4 = 0$.
This equation has **two equal roots,** $\frac{2}{3}$ and $\frac{2}{3}$.

Solution by factoring is the easiest method of solving a quadratic equation provided the first member of the standard form of the equation can readily be factored by inspection. When the method of solving such an equation is not prescribed, it is customary, therefore, to try this method first. If the first member cannot be factored by inspection, one of the methods discussed in the next two articles should be used.

Exercises

Solve the following equations by factoring. Check your results by substitution.

1. $9x^2 - 25 = 0$.
2. $49x^2 = 81$.
3. $x^2 - 7x + 10 = 0$.
4. $4x^2 + 33 = 28x$.
5. $5z^2 = 8z$.
6. $25y^2 = 30y - 9$.
7. $6x^2 + 20 = 23x$.
8. $15x^2 + 2x - 8 = 0$.
9. $(3t - 1)^2 = (t - 7)^2$.
10. $(3z + 4)^2 = (2z + 5)^2 + 19$.
11. $6(s - 2)^2 + 11(s - 2) = 10$.
12. $3x^{-2} + 7x^{-1} - 6 = 0$.
13. $ax^2 + b = (ab + 1)x$.
14. $abx^2 + a^2x + b^2x + ab = 0$.
15. $x^3 - 4x^2 + 3x = 0$.
16. $x(x^2 - 4) - 3(x^2 - 4) = 0$.

54. Solution by Completing the Square. If the factors of the first member cannot easily be found, when the equation is in the standard form, the method of **completing the square** may be used.

This method depends on the fact that if, to the expression $x^2 + px$, we add *the square of half the coefficient of* x, that is, if we add $p^2/4$, the resulting expression is the square of $\left(x + \frac{p}{2}\right)$. For, we have

$$x^2 + px + \frac{p^2}{4} = \left(x + \frac{p}{2}\right)^2,$$

as may be verified by squaring the second member.

EXAMPLE 1. Solve by completing the square: $3x^2 - 8x + 4 = 0$.

Transpose the constant term: $3x^2 - 8x = -4$.
Divide by the coefficient of x^2: $x^2 - \frac{8}{3}x = -\frac{4}{3}$.
Add to both sides the square of half the coefficient of x:

$$x^2 - \tfrac{8}{3}x + \tfrac{16}{9} = \tfrac{16}{9} - \tfrac{4}{3} = \tfrac{16}{9} - \tfrac{12}{9} = \tfrac{4}{9}.$$

Write the first member as a square: $(x - \frac{4}{3})^2 = \frac{4}{9}$.

Extract the square roots, using both the plus and minus signs in the second member: $x - \frac{4}{3} = \pm \frac{2}{3}$.

If $x - \frac{4}{3} = \frac{2}{3}$, then $x = 2$; if $x - \frac{4}{3} = -\frac{2}{3}$, then $x = \frac{2}{3}$.

CHECK. $3 \cdot 2^2 - 8 \cdot 2 + 4 = 0$, $3(\frac{2}{3})^2 - 8(\frac{2}{3}) + 4 = 0$.
The roots are 2 and $\frac{2}{3}$.

EXAMPLE 2. Solve by completing the square: $2x^2 + 3x - 3 = 0$.

Transpose the constant term: $2x^2 + 3x = 3$.
Divide by the coefficient of x^2: $x^2 + \frac{3}{2}x = \frac{3}{2}$.
Add $(\frac{3}{4})^2$ to both sides: $x^2 + \frac{3}{2}x + \frac{9}{16} = \frac{9}{16} + \frac{3}{2} = \frac{9}{16} + \frac{24}{16} = \frac{33}{16}$,

or $$\left(x + \tfrac{3}{4}\right)^2 = \tfrac{33}{16}.$$

Extract the roots, using both signs: $x + \dfrac{3}{4} = \dfrac{\pm\sqrt{33}}{4}$.

Solve for x: $x = \dfrac{-3 + \sqrt{33}}{4}$, or $x = \dfrac{-3 - \sqrt{33}}{4}$.

CHECK, using $x = \dfrac{-3 + \sqrt{33}}{4}$:

$$2\left(\frac{-3+\sqrt{33}}{4}\right)^2 + 3\left(\frac{-3+\sqrt{33}}{4}\right) - 3$$
$$= \frac{9 - 6\sqrt{33} + 33}{8} + \frac{-9 + 3\sqrt{33}}{4} - 3$$
$$= \frac{9 - 6\sqrt{33} + 33 - 18 + 6\sqrt{33} - 24}{8} = 0.$$

The computation for checking the root $\dfrac{-3 - \sqrt{33}}{4}$ is left as an exercise.

Since $\sqrt{33} = 5.745$, approximately, the approximate decimal values of the roots are 0.686 and -2.186.

Exercises

Find the number that must be added to the given expression to complete the square and state the resulting square.

1. $x^2 - 6x$. **2.** $x^2 + 5x$. **3.** $x^2 + \frac{4}{5}x$. **4.** $x^2 - \frac{3}{7}x$.

Solve the following equations by completing the square. In Ex. 11 to 18,

express the roots using radicals then write them also, approximately, in decimal form.

5. $x^2 - 2x - 15 = 0$.
6. $x^2 + 7x - 44 = 0$.
7. $3y^2 - 26y + 35 = 0$.
8. $2t^2 - 5t - 3 = 0$.
9. $6w^2 + 2 = 7w$.
10. $10z^2 = 9z + 9$.
11. $4x^2 + 10x + 3 = 0$.
12. $5x^2 - 4x - 2 = 0$.
13. $(3k + 2)^2 - 4k^2 + 1 = 0$.
14. $x(2x - 7) + 4 = 0$.
15. $x^2 = 9x - 10$.
16. $(x + 5)^2 + 2(x + 1)^2 = 29$.
17. $3(2k + 3)^2 - 5(k + 2)^2 = 2$.
18. $y^2 = 6y + 10$.
19. $6x + x^2 + 13 = 0$.
20. $2x^2 + 2x + 5 = 0$.
21. $x^2 + x + 1 = 0$.
22. $3x^2 - 4x + 4 = 0$.
23. $x^2 - 4ax - 21a^2 = 0$.
24. $ax^2 + ax + x + 1 = 0$.
25. $bx^2 + x + b = 0$.
26. $x^2 + 2px = q$.

55. Solution by the Quadratic Formula. If we solve the quadratic equation

$$\mathbf{ax^2 + bx + c = 0, \qquad a \neq 0} \tag{2}$$

by the method of completing the square, the resulting expressions for the roots may be used as formulas for finding the roots of any quadratic equation whatever.

Let: $ax^2 + bx + c = 0$.

Transpose the constant term: $ax^2 + bx = - c$.

Divide by a: $x^2 + \frac{b}{a}x = - \frac{c}{a}$.

Add $\left(\frac{b}{2a}\right)^2$ to both sides: $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$,

or $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$.

Extract the square roots: $x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$.

Hence,

$$\mathbf{x = \frac{- b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{- b - \sqrt{b^2 - 4ac}}{2a},} \tag{3}$$

are the roots of the equation $ax^2 + bx + c = 0$.

Formulas (3) should be memorized. To solve any given equation by means of them, we must:

(1) *Reduce the given equation to the standard form.*
(2) *Find the values of a, b, and c.*
(3) *Substitute these values in the formulas, and simplify.*

Example. Solve by the formulas: $2x^2 = 7x - 4$.

First, write this equation in the standard form

$$2x^2 - 7x + 4 = 0.$$

Hence, $a = 2$, $b = -7$, and $c = 4$. Substitute these values in formulas (3)

$$x = \frac{7+\sqrt{49-32}}{4} = \frac{7+\sqrt{17}}{4} \quad \text{and} \quad \frac{7-\sqrt{49-32}}{4} = \frac{7-\sqrt{17}}{4}.$$

CHECK, using $x = \frac{7+\sqrt{17}}{4}$. $2\left(\frac{7+\sqrt{17}}{4}\right)^2 = 7\left(\frac{7+\sqrt{17}}{4}\right) - 4;$

$$\frac{49+14\sqrt{17}+17}{8} = \frac{49+7\sqrt{17}}{4} - \frac{16}{4} \text{ or } \frac{33+7\sqrt{17}}{4} = \frac{49+7\sqrt{17}-16}{4}.$$

Solve the following equations by the quadratic formula.

1. $x^2 - 8x - 33 = 0.$
2. $5x^2 - 9x - 2 = 0.$
3. $2x^2 = 5x + 3.$
4. $6x^2 = 7x + 20.$
5. $2w^2 - w - 10 = 0.$
6. $3y^2 - 16y + 20 = 0.$
7. $5n^2 + 3 = 11n.$
8. $11t^2 - 17t + 5 = 0.$
9. $3y^2 = 10y - 4.$
10. $4z^2 + 13z + 7 = 0.$
11. $3x^2 + 5x - 1 = 0.$
12. $6x^2 = 7x + 2.$
13. $7x^2 + 8x + 3 = 0.$
14. $2x^2 + 5x + 7 = 0.$
15. $5x^2 - 3x + 1 = 0.$
16. $11x^2 - 19x + 6 = 0.$
17. $x^2 + (x+1)^2 + (x+2)^2 = 0.$
18. $(2x+1)^2 + (x+3)^2 = 1.$
19. $\sqrt{2}x^2 - \sqrt{3}x + \sqrt{2} = 0.$
20. $2x^2 - 7ax + 3a^2 = 0.$
21. $x^2 + b^2 = 2bx + a^2.$
22. $abx^2 + 1 = ax + bx.$
23. $x^2 + kx + k = 1.$
24. $x^2 + c^2 = 2cx + c^2x^2.$

56. Equations in Fractional Form. The following equations, involving fractions in which the unknown appears in the denominator, can be solved by reducing them to quadratic equations in the standard form.

To clear of fractions, we multiply both members of the equation by the L.C.M. of the denominators. Since this operation involves multiplying by a polynomial involving x, it may introduce **extraneous roots**, that is, numbers which are roots of an equation obtained in the process of solution but which do not satisfy the given equation. Hence, *when an equation involving the unknown in the denominators is solved by clearing the equation of fractions, the roots found must be checked in the given eqaution and any extraneous ones rejected.*

EXAMPLE. Solve: $\frac{3x+8}{(x-2)(x+5)} = \frac{3x-4}{x-2} - \frac{2x+3}{x+5} + 2.$

Multiply both members by $(x-2)(x+5)$, which is the L.C.M. of the denominators:

$$3x + 8 = (3x-4)(x+5) - (2x+3)(x-2) + 2(x-2)(x+5).$$

or $$3x + 8 = 3x^2 + 11x - 20 - (2x^2 - x - 6) + 2x^2 + 6x - 20,$$

which reduces to $$x^2 + 5x - 14 = 0.$$

The roots of the last equation are -7 and 2.

The root -7 checks in the given equation, since

$$\frac{-13}{(-9)(-2)} = \frac{-25}{-9} - \frac{-11}{-2} + 2.$$

If we substitute $x = 2$ in the given equation, we obtain $\frac{14}{0 \cdot 7} = \frac{2}{0} - \frac{7}{7} + 2.$ This is not a true equation since neither member has a meaning.

The only root of the given equation is -7. The extraneous root 2 was introduced when we multiplied by $(x-2)(x+5)$ to clear of fractions.

Exercises

Solve the following equations and check the solutions in the given equation.

1. $\frac{4x}{3} + \frac{9}{x} = 7.$
2. $3x - 4 = \frac{5x+4}{x+5}.$
3. $\frac{4x^2 - 2x + 13}{x+3} = 2x + 1.$
4. $\frac{2x^2 - 5x + 16}{2x - 1} = 3x + 2.$
5. $\frac{x^2 + 15x - 16}{(x-1)(x+3)} = \frac{5x-2}{x-1} + \frac{4-3x}{x+3}.$
6. $\frac{2x+29}{x^2 - x - 2} = \frac{4x-3}{x+1} + \frac{2x+7}{x-2}.$
7. $\frac{2x+3}{x+1} + \frac{x-3}{x+3} = \frac{3x+3}{x+2}.$
8. $\frac{4x-9}{x-2} - \frac{6}{x} = \frac{4x-7}{x+2}.$
9. $\frac{2x^2 + 3x + 2}{5x^2 + 7x - 2} = 1.$
10. $\frac{5x^2 - 6x - 3}{2x^2 + 3x - 7} = \frac{6}{5}.$
11. $\frac{x + a^2}{x - a^2} + \frac{x + b^2}{x - b^2} = 0.$
12. $\frac{a}{x-a} + \frac{b}{x-b} + 2 = 0.$

57. Equations Involving Radicals. Equations in which the unknown appears under a radical sign can sometimes be rationalized. If the resulting equation can then be reduced to a quadratic equation in the standard form, it may be solved by the methods shown in the preceding articles.

To rationalize the given equation, first isolate, on one side of the equation, one radical containing the unknown, then raise both sides to a power sufficient to remove this radical. Repeat this process until the equation obtained is free from radicals containing the unknown.

The process of raising both sides of an equation to a power, in order to remove the radical, may introduce extraneous roots into the resulting equation. Hence, *if this operation has been performed, it is necessary that the roots obtained be checked by substitution in the given equation and any extraneous roots rejected.*

Example 1. Solve: $\sqrt{2x^2 - 2x + 5} - 2x + 1 = 0.$

Isolate the radical: $\sqrt{2x^2 - 2x + 5} = 2x - 1.$

Square both sides: $2x^2 - 2x + 5 = 4x^2 - 4x + 1.$

Simplify: $x^2 - x - 2 = 0.$

Solve: $x = 2, \quad \text{or} \quad x = -1.$

By substituting $x = 2$ in the given equation, we obtain $\sqrt{9} - 4 + 1 = 0.$ Hence, 2 is a root. If we substitute $x = -1$, we have $\sqrt{9} + 2 + 1 = 0.$ Hence, -1 is not a root.

The required solution is $x = 2$. The extraneous root $x = -1$ was introduced when we squared both sides of the equation.

If the equation contains expressions involving x that are raised to fractional powers, these may be replaced by their equivalent radicals.

EXAMPLE 2. Solve: $(3x-5)^{\frac{1}{2}} - (2x+3)^{\frac{1}{2}} + 1 = 0$.

Write the equation in radical form: $\sqrt{3x-5} - \sqrt{2x+3} + 1 = 0$.

Isolate the first radical: $\sqrt{3x-5} = \sqrt{2x+3} - 1$.

Square both sides: $3x - 5 = 2x + 3 - 2\sqrt{2x+3} + 1$.

Isolate the second radical: $x - 9 = -2\sqrt{2x+3}$.

Square both sides: $x^2 - 18x + 81 = 8x + 12$.

Simplify: $x^2 - 26x + 69 = 0$. Hence $x = 3$ or $x = 23$.

The first of these values of x satisfies the given equation; the second does not. Hence $x = 3$ is a solution and $x = 23$ is extraneous.

Exercises

Solve the following equations and check the results.

1. $\sqrt{11x-8} = 6$.
2. $\sqrt{17x+13} - 8 = 0$.
3. $\sqrt{6x-6} - \sqrt{4x+8} = 0$.
4. $\sqrt{9x+13} = \sqrt{13x-3}$.
5. $\sqrt{5x-6} = x$.
6. $\sqrt{x+11} + x - 1 = 0$.
7. $\sqrt{2x+3} - \sqrt{5x+1} + 1 = 0$.
8. $\sqrt{4x+1} - \sqrt{2x-3} = 2$.
9. $2(5x+1)^{\frac{1}{2}} - (12x-11)^{\frac{1}{2}} = 3$.
10. $(4x-7)^{\frac{1}{2}} - (2x-7)^{\frac{1}{2}} = 2$.
11. $(7+2y)^{\frac{1}{2}} = (5-y)^{\frac{1}{2}} + y^{\frac{1}{2}}$.
12. $(2z-14)^{\frac{1}{2}} + (z-6)^{\frac{1}{2}} = z^{\frac{1}{2}}$.
13. $\sqrt{x^2+7x-7} - \sqrt{x^2+3x} = 1$.
14. $\sqrt{2x^2-2x+1} - \sqrt{2x^2-10x+17} = 2$.
15. $\sqrt{\sqrt{1-x} - \sqrt{x+15}} = 2$.
16. $\sqrt{x^2-4x+6} = \sqrt{3}$.
17. $\sqrt{5x-6a^2} - \sqrt{x-2a^2} = 2a$.
18. $\sqrt{4x-4a^2} - \sqrt{2x-a^2} = a$.

58. Equations in Quadratic Form. The following equations are not quadratic equations in x. They can, however, be solved by the methods of quadratic equations provided a suitably chosen expression in x is denoted by a new variable.

EXAMPLE 1. Solve: $4x^4 - 109x^2 + 225 = 0$.

If we put $x^2 = y$, this equation becomes a quadratic equation in y:

$$4y^2 - 109y + 225 = 0.$$

The roots of this equation in y are $y = \frac{9}{4}$ and $y = 25$.

If, in these equations, we replace y by its value x^2, these equations become $x^2 = \frac{9}{4}$ and $x^2 = 25$. The roots of these equations are $x = \frac{3}{2}$, $x = -\frac{3}{2}$, $x = 5$, and $x = -5$. These four numbers are the required roots of the given equation.

EXAMPLE 2. Solve: $(x^2 - 4x)^2 - 2x^2 + 8x - 15 = 0$.

We may write this equation in the form $(x^2 - 4x)^2 - 2(x^2 - 4x) - 15 = 0$.

If we now put $\quad x^2 - 4x = y,$

the equation becomes $\quad y^2 - 2y - 15 = 0,$

the roots of which are $\quad y = 5 \quad \text{and} \quad y = -3.$

In these two solutions, replace y by its value $x^2 - 4x$. We then have the two equations in x

$$x^2 - 4x = 5, \quad \text{and} \quad x^2 - 4x = -3.$$

The roots of the first equation are found to be 5 and -1. Those of the second are 1 and 3. By substitution, we find that all of these values of x satisfy the given equation. Hence, the required roots are 5, -1, 1, and 3.

EXAMPLE 3. Solve: $x^2 - 3x - \sqrt{x^2 - 3x + 6} - 6 = 0$.

We first write this equation in the form:

$$x^2 - 3x + 6 - \sqrt{x^2 - 3x + 6} - 12 = 0.$$

If we now put $\quad y = \sqrt{x^2 - 3x + 6},$

we have the following quadratic equation in y

$$y^2 - y - 12 = 0,$$

the roots of which are $\quad y = 4 \quad \text{and} \quad y = -3.$

Replacing y by $\sqrt{x^2 - 3x + 6}$, we have the equations in x

$$\sqrt{x^2 - 3x + 6} = 4, \quad \text{and} \quad \sqrt{x^2 - 3x + 6} = -3.$$

The second of these equations we may reject at once since the *positive* square root of a number cannot equal -3.

By squaring the members of the first equation and simplifying, we obtain

$$x^2 - 3x - 10 = 0.$$

The roots of this equation, 5 and -2, also satisfy the given equation and are the required solutions.

Exercises

Solve the following equations.

1. $x^4 - 13x^2 + 36 = 0$.

2. $9x^{-4} - 37x^{-2} + 4 = 0$.

3. $x - 2x^{\frac{1}{2}} - 35 = 0$.

4. $x^{\frac{2}{3}} - 5x^{\frac{1}{3}} + 6 = 0$.

5. $x + 11 + 2\sqrt{x + 11} = 8$.

6. $4x - 4\sqrt{4x + 5} = 16$.

7. $(2x + 3)^2 - 4(2x + 3) = 5$.

8. $(5x - 1)^2 + 35x - 51 = 0$.

9. $(x^2 - 6x)^2 - 11(x^2 - 6x) = 80$.

10. $(2x^2 - 3x)^2 - 6(2x^2 - 3x) + 8 = 0$.

11. $2(x^2 - 5x)^2 - 7x^2 + 35x + 6 = 0$.

12. $4(x^2 + 3x)^2 + 8x^2 + 24x - 5 = 0$.

13. $2\left(x - \frac{3}{x}\right)^2 + 3\left(x - \frac{3}{x}\right) - 2 = 0.$
14. $\left(x - \frac{6}{x}\right)^2 + 4\left(x - \frac{6}{x}\right) - 5 = 0.$

15. $\left(\frac{x+3}{x-1}\right)^2 - 5\left(\frac{x+3}{x-1}\right) + 6 = 0.$
16. $\left(\frac{x+2}{2x-5}\right)^2 - \frac{x+2}{2x-5} - 2 = 0.$

17. $2x^2 - 3x - 5\sqrt{2x^2 - 3x + 2} + 6 = 0.$

18. $2x^2 - 4x - \sqrt{x^2 - 2x + 13} - 2 = 0.$

59. Graph of the Quadratic Function. The graph of the **quadratic function**

$$ax^2 + bx + c, \qquad a \neq 0 \tag{4}$$

may be drawn by the method outlined in Art. 40. We equate the given function to y,

$$y = ax^2 + bx + c,$$

assign values to x, compute the corresponding values of y, plot the points so determined, and draw a smooth curve through them.

The graph of a quadratic function is a **parabola.** If a is positive, the parabola opens upward (Figs. 13 and 15); if a is negative, it opens downward (Fig. 14). The lowest point on the parabola, if $a > 0$, or the highest if $a < 0$, is called its **vertex.** We shall find in Art. 195 that the coördinates of the vertex of this parabola are

$$\left(-\frac{b}{2a}, \ \frac{4ac - b^2}{4a}\right). \tag{5}$$

Since the vertex is the lowest (or highest) point on the given parabola, we have, from this expression for the coördinates of the vertex, the following property of the quadratic function: *the value of x for which $ax^2 + bx + c$ takes its least value if $a > 0$, or its greatest value if $a < 0$, is $x = -\frac{b}{2a}$. This extreme value of the function is $\frac{4ac - b^2}{4a}$.* This theorem is frequently useful when we want to find a value of one variable that makes another as large (or as small) as possible.

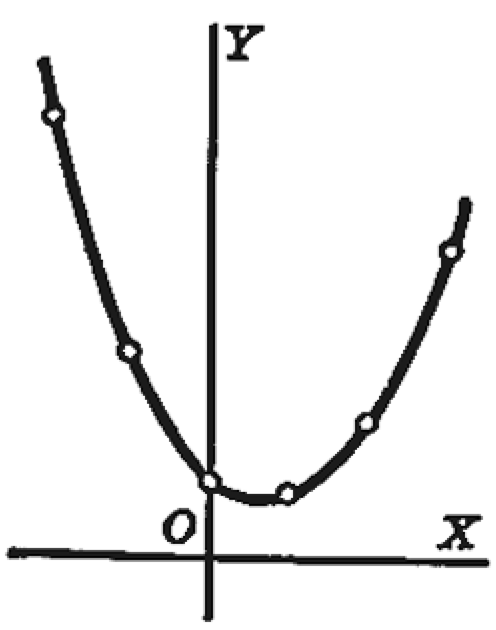


FIG 13

EXAMPLE. Draw the graph of the function: $2x^2 - 3x + 3$.

Equate the given function to y: $y = 2x^2 - 3x + 3$.

By assigning values to x and computing the corresponding values of y from the equation, we get the following table.

x	-2	-1	0	0.75	1	2	3
y	17	8	3	1.875	2	5	12

From (5), the coördinates of the vertex are (0.75, 1.875). Hence, the function $2x^2 - 3x + 3$ attains its least value when $x = 0.75$. This least value

of the function is 1.875. Since, in this case, the parabola opens upward and its lowest point is above the x-axis, this curve does not meet the x-axis.

60. Graphical Solution of Quadratic Equations. If the roots of the quadratic equation

$$ax^2 + bx + c = 0 \tag{6}$$

are real numbers, they can be found, at least approximately, by drawing the graph of the equation

$$y = ax^2 + bx + c, \tag{7}$$

and *measuring the abscissas of the points at which the graph crosses the x-axis.* For, since these points lie on the parabola, their coördinates satisfy equation (7) and, since they also lie on the x-axis, their y-coördinates are zero. Hence, if we substitute the coördinates of such a point in equation (7), we find that its abscissa makes $ax^2 + bx + c = 0$ and is consequently a root of equation (6).

If the vertex of the parabola lies on the x-axis, that is, from (5), if $4ac - b^2 = 0$, the two intersections of the parabola with the x-axis coincide and the roots of the quadratic equation (6) are **equal.**

If the parabola does not meet the x-axis, the roots of equation (6) are **imaginary.** The parabola shown in Figure 13, for example, does not meet the x-axis. Hence the roots of $2x^2 - 3x + 3 = 0$ are imaginary. They are, in fact, the imaginary numbers $\dfrac{3 + \sqrt{-15}}{4}$ and $\dfrac{3 - \sqrt{-15}}{4}$.

EXAMPLE 1. Solve graphically the quadratic equation $-x^2 + x + 2 = 0$.

We put $y = -x^2 + x + 2$, compute the following table, and draw the resulting graph (Fig. 14).

x	-2	-1	0	0.5	1	2	3
y	-4	0	2	2.25	2	0	-4

Since the graph crosses the x-axis at $(-1, 0)$ and at $(2, 0)$, the abscissas, -1 and 2, of these points must make the quadratic function $-x^2 + x + 2$ equal to zero. These numbers are, accordingly, the roots of the equation $-x^2 + x + 2 = 0$.

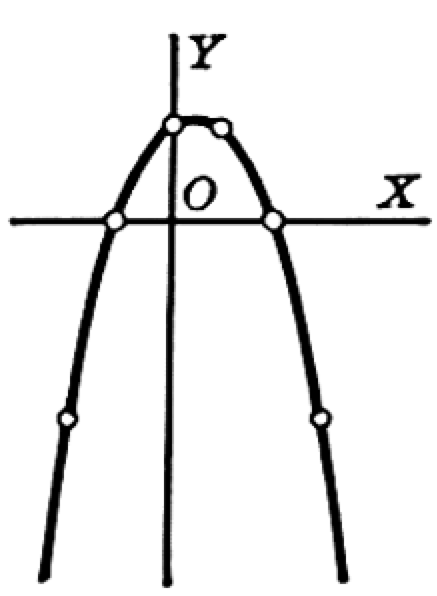


FIG. 14

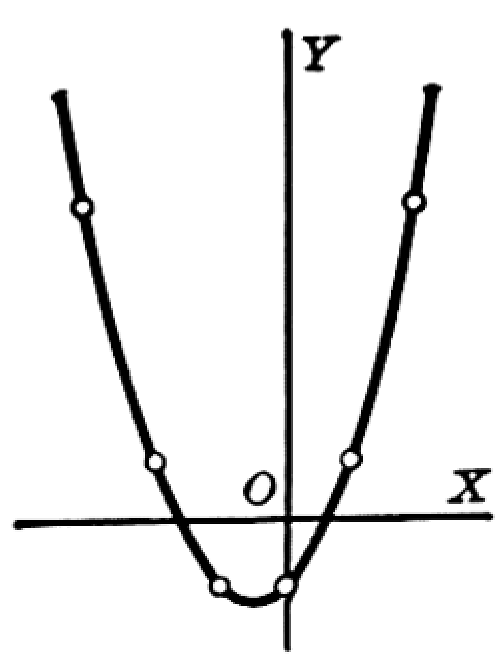


FIG. 15

EXAMPLE 2. Solve graphically: $x^2 + x - 1 = 0$.

Put $y = x^2 + x - 1$, tabulate pairs of values of x and y, and draw the graph (Fig. 15).

x	-3	-2	-1	-0.5	0	1	2
y	5	1	-1	-1.25	-1	1	5

The abscissas of the points of intersection of the graph with the x-axis are found approximately, by measurement, to be 0.6 and -1.6. It follows that these numbers are approximately the roots of $x^2 + x - 1 = 0$.

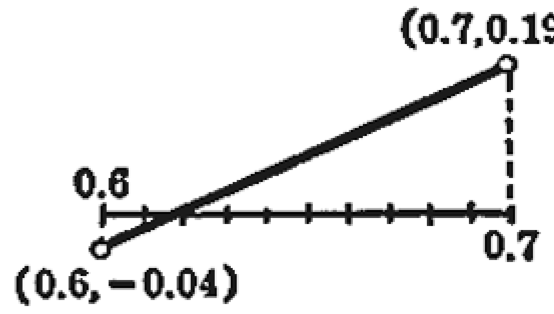


FIG. 15a

If a closer approximation to either of these roots is needed, it can be found graphically by drawing the graph, on an enlarged scale, in the neighborhood of the desired root. Thus, if we wish to determine the positive root to two decimal places, we substitute $x = 0.6$ in the given equation giving $y = -0.04$. By plotting this point on the original graph, we find that 0.6 is slightly less than the required root. Putting x equal to a slightly larger value, $x = 0.7$, we find $y = 0.19$. On an enlarged scale, draw the graph from $(0.6, -0.04)$ to $(0.6, 0.19)$, as in Figure 15. (In a short interval, such as this, we may draw the graph, approximately, as a straight line.) From this enlarged figure, we find that the graph crosses the x-axis at about $x = 0.62$. This is the required root to two decimal places.

Exercises

Draw the graphs of the first members of the following equations. Find the coördinates of the vertices of the parabolas. If the roots of the given equation are real, find them graphically to two decimal places.

1. $x^2 - 4 = 0$. **2.** $3x^2 - 5 = 0$.
3. $6 + x - x^2 = 0$. **4.** $2 + 3x - 2x^2 = 0$.
5. $x^2 - 2x + 1 = 0$. **6.** $x^2 + 4x + 4 = 0$.
7. $2x^2 + 3x - 3 = 0$. **8.** $2x^2 + 5x - 2 = 0$.
9. $x^2 + x - 5 = 0$. **10.** $x^2 - 2x - 5 = 0$.
11. $x^2 - 6x + 10 = 0$. **12.** $2x^2 - 7x + 7 = 0$.

13. Draw, on one set of axes, the graphs of $y = x^2 - 4x + c$ for $c = 0$, 4, and 8. What change is made in the graph when the constant term, only, is changed?

14. A ball is thrown from the origin with a velocity of about 57 feet per second in a direction making an angle of 45° with the x-axis which is assumed to be horizontal. Given that the equation of its path is $y = x - x^2/100$, find how high it will rise and the distance from the origin at which it will cross the x-axis.

15. Find the largest rectangular area that can be enclosed by a fence 100 rods long.

16. Solve Ex. 15 if only three sides of the rectangle are to be fenced.

17. The perimeter of a rectangle is 144 feet. Find its dimensions if the square of the length of a diagonal is a minimum.

61. Character of the Roots. We shall show, in this article and the following one, how one can determine certain useful facts about the roots of the equation

$$ax^2 + bx + c = 0, \tag{8}$$

without actually finding the roots themselves.

We saw, in Art. 55, that the roots, r_1 and r_2, of equation (8) are

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \text{and} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{9}$$

Suppose that a, b, and c are real numbers and consider the quantity

$$b^2 - 4ac,$$

which appears under the radical sign in the formulas (9).

If $b^2 - 4ac$ is positive, then $\sqrt{b^2 - 4ac}$ is a real number and the two roots, r_1 and r_2, of equation (1) are *real* numbers. Moreover, $r_1 \neq r_2$, since the denominators are the same and the numerators are unequal.

If $b^2 - 4ac = 0$, it follows from equations (9) that r_1 and r_2 are both equal to $-b/2a$. Hence, in this case, r_1 and r_2 are *real* and *equal.*

Finally, if $b^2 - 4ac$ is negative, then $\sqrt{b^2 - 4ac}$ is imaginary and r_1 and r_2 are *imaginary* and *unequal.*

These results may be summarized as follows: *If a, b, and c are real numbers, then the roots of $ax^2 + bx + c = 0$ are*

real and unequal	**if $b^2 - 4ac$ is positive,**
real and equal	**if $b^2 - 4ac = 0$,**
imaginary and unequal	**if $b^2 - 4ac$ is negative.**

The expression $b^2 - 4ac$ is called the **discriminant** of the quadratic equation $ax^2 + bx + c = 0$.

Furthermore, if a, b, and c are rational * numbers, it follows from equations (9) that r_1 and r_2 are rational numbers if, and only if, $\sqrt{b^2 - 4ac}$ is rational, that is, if $b^2 - 4ac$ is a perfect square. Hence, *if a, b, and c are rational numbers, the roots of the equation $ax^2 + bx + c = 0$ are*

rational	**if $b^2 - 4ac$ is a perfect square,**
irrational	**if $b^2 - 4ac$ is not a perfect square.**

* A rational number is defined, in Art. 28, as one that can be written in the form m/n, where m and n are both integers. The square root of a rational number, $\sqrt{N}$, is a rational number only if N is a perfect square.

Example 1. Find the character of the roots of $2x^2 - 5x + 2 = 0$.

The coefficients $a = 2$, $b = -5$, and $c = 2$ of this equation are rational numbers and $b^2 - 4ac = (-5)^2 - 4 \cdot 2 \cdot 2 = 9 = 3^2$ is a perfect square. Hence, the roots are real, rational, and unequal.

Example 2. Find the character of the roots of $25x^2 + 30x + 9 = 0$.

The coefficients are rational and $b^2 - 4ac = 30^2 - 4 \cdot 25 \cdot 9 = 0$. The roots are real, equal, and rational.

Example 3. Find the character of the roots of $5x^2 + 7x - 1 = 0$.

The coefficients are rational and $b^2 - 4ac = 7^2 - 4 \cdot 5 \cdot (-1) = 69$, which is positive but not a perfect square. The roots are real, unequal, and irrational.

Example 4. Find the character of the roots of $7x^2 - 2x + 1 = 0$.

Since $b^2 - 4ac = (-2)^2 - 4 \cdot 7 \cdot 1 = -24$, which is negative, the roots are imaginary and unequal.

Example 5. Find the values of k for which the quadratic equation in x, $2kx^2 + 5x^2 - 3kx + k - 1 = 0$, has equal roots.

We first write the equation in the standard form

$$(2k + 5)x^2 - 3kx + (k - 1) = 0.$$

Hence, $a = 2k + 5$, $b = -3k$, and $c = k - 1$. The roots of the given equation are equal if $b^2 - 4ac = (-3k)^2 - 4(2k + 5)(k - 1) = k^2 - 12k + 20 = 0$. By solving this equation, we find that k must equal 2 or 10.

As a check, we put $k = 2$ in the given equation. It becomes $9x^2 - 6x + 1 = 0$, which has the equal roots $\frac{1}{3}$ and $\frac{1}{3}$. Similarly, if $k = 10$, the given equation becomes $25x^2 - 30x + 9 = 0$, which has the equal roots $\frac{3}{5}$ and $\frac{3}{5}$.

Exercises

Find the character of the roots of the following equations.

1. $x^2 - 8x + 15 = 0$.

2. $2x^2 - 11x - 21 = 0$.

3. $3x^2 + 7x - 2 = 0$.

4. $6x^2 - 25x + 9 = 0$.

5. $9x^2 - 12x + 4 = 0$.

6. $4x^2 + 20x + 25 = 0$.

7. $5x^2 + 4x + 3 = 0$.

8. $x^2 - 5x + 7 = 0$.

9. $8x^2 = 2x + 15$.

10. $5x^2 + 3x = 1$.

11. $3x^2 - \frac{5}{2}x + \frac{1}{3} = 0$.

12. $x^2 - 1.7x - 0.84 = 0$.

Find the values of k for which the roots of the following quadratic equations in x are equal.

13. $5x^2 - 8x + k = 0$.

14. $12x^2 + 2kx + 3 = 0$.

15. $(3k + 1)x^2 + (k + 3)x = 4 - k$.

16. $(2k - 5)x^2 - 2(k - 1)x = 3 - k$.

17. $16x^2 + (3x + k)^2 = 16$.

18. $(x - 2)(x + 2k) + 16 = 0$.

62. The Sum and Product of the Roots. The sum of the roots, r_1 and r_2, of the equation $ax^2 + bx + c = 0$ is found, by adding the expressions found in equations (9), to be

$$r_1 + r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-2b}{2a} = \frac{-b}{a}.$$

The product of the roots is found, similarly, by multiplying these two expressions, to be

$$r_1 \cdot r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \cdot \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

Hence we have, for the sum and the product of the roots,

$$r_1 + r_2 = -\frac{b}{a}, \quad \text{and} \quad r_1 \cdot r_2 = \frac{c}{a}.$$

EXAMPLE 1. Find the sum and the product of the roots of the equation $(3x - 1)^2 - 2(x - 4)^2 + 35 = 0$.

Multiply out, and arrange the result in the standard form,

$$7x^2 + 10x + 4 = 0.$$

Hence, $$r_1 + r_2 = -\frac{b}{a} = -\frac{10}{7}; \quad r_1 \cdot r_2 = \frac{c}{a} = \frac{4}{7}.$$

EXAMPLE 2. Find the roots of the equation $6x^2 - 23x + c = 0$, and determine c, given that the difference between the roots is $\frac{5}{6}$.

The sum of the roots is: $r_1 + r_2 = \frac{23}{6}$.
Their difference is: $r_1 - r_2 = \frac{5}{6}$.
By solving these equations, we find that $r_1 = \frac{7}{3}$ and $r_2 = \frac{3}{2}$.

The product of the roots is $r_1 \cdot r_2 = \dfrac{c}{a} = \dfrac{c}{6} = \dfrac{7}{3} \cdot \dfrac{3}{2} = \dfrac{7}{2}$. Hence, $c = 21$.

The equation is $6x^2 - 23x + 21 = 0$ and its roots are $\frac{7}{3}$ and $\frac{3}{2}$.

EXAMPLE 3. Find the roots of the equation $3x^2 + 7x + k = (x + 1)^2$, given that one root is four times the other.

Write the equation in the standard form

$$2x^2 + 5x + k - 1 = 0,$$

and denote its roots by r_1 and r_2. Let $r_2 = 4r_1$.

Then $r_1 + r_2 = r_1 + 4r_1 = 5r_1 = -\dfrac{b}{a} = -\dfrac{5}{2}$. Hence, $r_1 = -\dfrac{1}{2}$ and $r_2 = 4r_1 = -2$.

The product of the roots is $r_1 \cdot r_2 = \dfrac{c}{a} = \dfrac{k-1}{2} = \left(-\dfrac{1}{2}\right)(-2) = 1$. Hence $k = 3$.

The equation is $2x^2 + 5x + 2 = 0$ and its roots are $-\frac{1}{2}$ and -2.

63. The Factored Form of $ax^2 + bx + c$. If r_1 and r_2 are the roots of $ax^2 + bx + c = 0$, we have, from the formulas for the sum and the product of the roots (Art. 62),

$$b = -a(r_1 + r_2), \quad \text{and} \quad c = ar_1r_2.$$

Substitute these values of b and c in the expression $ax^2 + bx + c$. We have

$$\begin{aligned} ax^2 + bx + c &= ax^2 - a(r_1 + r_2)x + ar_1r_2 \\ &= a[x^2 - (r_1 + r_2)x + r_1r_2] \\ &= a(x - r_1)(x - r_2). \end{aligned}$$

Hence,

$$\boldsymbol{ax^2 + bx + c = a(x - r_1)(x - r_2).}$$

It follows that *to factor a quadratic expression in x, equate the expression to zero and find the roots of the resulting equation. Then x minus each root is a factor of the given expression.*

EXAMPLE 1. Factor the expression $3x^2 - 8x + 2$.

The roots of $3x^2 - 8x + 2 = 0$ are found, by the quadratic formula, to be $r_1 = \dfrac{4 + \sqrt{10}}{3}$ and $r_2 = \dfrac{4 - \sqrt{10}}{3}$. Hence

$$3x^2 - 8x + 2 = 3\left(x - \frac{4 + \sqrt{10}}{3}\right)\left(x - \frac{4 - \sqrt{10}}{3}\right).$$

The student should verify this result by multiplying the factors together and simplifying the result.

EXAMPLE 2. Form a quadratic equation whose roots are $\frac{1}{3}$ and $-\frac{5}{4}$.

The required equation is $a(x - \frac{1}{3})(x + \frac{5}{4}) = 0$, where a may be any non-zero constant we choose. Putting $a = 12$, to avoid fractions, we have, as the required equation,

$$12(x - \tfrac{1}{3})(x + \tfrac{5}{4}) = (3x - 1)(4x + 5) = 12x^2 + 11x - 5 = 0.$$

Exercises

Without solving the equation, find the sum and product of its roots.

1. $3x^2 + 8x + 17 = 0$. **2.** $5x^2 - 7x - 13 = 0$.

3. $6.3x^2 + 4.9x - 3.6 = 0$. **4.** $3(2x - 7)^2 = 5(x - 3)^2 + 25$.

Form a quadratic equation having the given numbers as roots.

5. $4, -7$. **6.** $\frac{3}{5}, -\frac{2}{3}$. **7.** $1.7, 2.3$.

8. $2 + \sqrt{7}, 2 - \sqrt{7}$. **9.** $-3 + \sqrt{5}, -3 - \sqrt{5}$.

10. $4 + 5i, 4 - 5i$. **11.** $-5 + \sqrt{2}i, -5 - \sqrt{2}i$.

12. $\dfrac{-5+\sqrt{8}}{3}, \dfrac{-5-\sqrt{8}}{3}.$
13. $\dfrac{-1+\sqrt{3}i}{2}, \dfrac{-1-\sqrt{3}i}{2}.$

14. $a+\sqrt{b}, a-\sqrt{b}.$
15. $-p+\sqrt{p^2+q}, -p-\sqrt{p^2+q}.$

Factor:

16. $6x^2-13x-5.$
17. $15x^2+7x-2.$
18. $2x^2-6x+7.$
19. $7x^2+6x+3.$
20. $x^2-(2y+3)x-3y^2-11y-10.$
21. $2x^2+xy-15y^2+x+14y-3.$

Find the value of k, given that:

22. One root of $x^2-(3k-1)x+6k=0$ is 3.
23. The sum of the roots of $(k+4)x^2+(7-5k)x-k+3=0$ is 2.
24. The product of the roots of $(k-1)x^2+(3k-2)x+2k+1=0$ is $\frac{5}{2}$.

Form a quadratic equation whose roots are:

25. Equal numerically, but opposite in sign, to those of $3x^2-7x=6$.
26. Twice those of $2x^2-5x+4=0$.
27. Less by 1 than those of $3x^2-4x-1=0$.

64. Problems Involving Quadratic Equations. State each of the following verbal problems by means of a quadratic equation in one unknown. Solve this equation and check your result by comparison with the verbal statement.

1. Find two numbers whose difference is 3 and whose product is 154.

2. Three times the square of a number, less 28, equals 17 times the number. Find the number.

3. Find three consecutive odd, positive integers such that the sum of their squares equals 251.

4. The length of the hypotenuse of a right triangle is $3x+2$. The lengths of its sides are $2x+5$ and $x+3$. Find x.

5. Find a number that exceeds its reciprocal by $\frac{21}{10}$.

6. The sum of a number and its reciprocal is $\frac{25}{12}$. Find the number.

7. The sum of two numbers is 2 and the sum of their reciprocals is $\frac{25}{12}$. Find the numbers.

8. Two circles are tangent externally. The distance between their centers is 12 inches and the sum of their areas is 80π square inches. Find their radii.

9. A circular swimming pool is surrounded by a concrete walk 4 feet wide. The area of the walk is $\frac{11}{25}$ that of the pool. Find the radius of the pool.

10. A man bought some shares of stock for \$18,000. He sold all but 100 shares for the same amount, thereby gaining \$6 a share on the stock sold. How many shares did he buy?

11. A rectangular picture is surrounded by a rectangular frame 4 inches wide at the top and bottom and 2 inches wide at each side. The area of the

frame is twice that of the picture. Find the dimensions of the picture if the sum of its dimensions is 16 inches.

12. A closed cubical box is made of boards 1 inch thick. The volume of wood in the box is 1016 cubic inches. Find the length of the outside edge of the box.

13. A wholesaler adds a certain percentage to the manufacturer's price when he sells to the retailer. The retailer adds three times this percentage to the wholesaler's price when he sells to the consumer. If the price to the consumer is 75% more than the manufacturer's price, what percentage did the wholesaler add?

14. A man bought some land and sold it at a loss. With the proceeds of this sale, he bought some more land and sold it for the price he paid for the first land. His per cent gain of the second sale was 5% more than his loss on the first one. Find his per cent loss on the first sale.

15. An automobile, traveling east at 45 miles an hour, passed a certain intersection at noon. Another automobile, traveling north at 60 miles an hour, passed the same intersection 25 minutes later. Find, to the nearest minute, the times at which they were 40 miles apart.

16. In Ex. 15, denote the square of the distance between the automobiles by y and draw the graph of y as a function of the time. Find the time at which the square of the distance between them was least and find this least distance.

Chapter 9

Simultaneous Equations Involving Quadratics

65. Equation of Second Degree in Two Variables. An equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \tag{1}$$

in which A, B, and C are not all zero, is *an equation of second degree in x and y* and its graph is called a **conic section.** We shall study these curves in considerable detail in Chapters 23 and 25.

If we are given an equation of the type of equation (1), we can draw its graph by assigning values to one variable and computing the values of the other variable from the equation. Since this process is rather tedious, involving, usually, the solution of a quadratic equation, we shall, in Art. 66, draw the graphs of typical curves of the types defined by equation (1) so that the student may have a rather definite idea of the form of the curve he is trying to draw.

66. Graphs of Type Forms. (1) *The circle.*

The graph of the equation $x^2 + y^2 = a^2$ is a circle with center at the origin and of radius a.

EXAMPLE 1. Draw the graph of the equation $x^2 + y^2 = 10$.

The required curve is a circle with center at the origin and radius $\sqrt{10} = 3.16$ (Fig. 16).

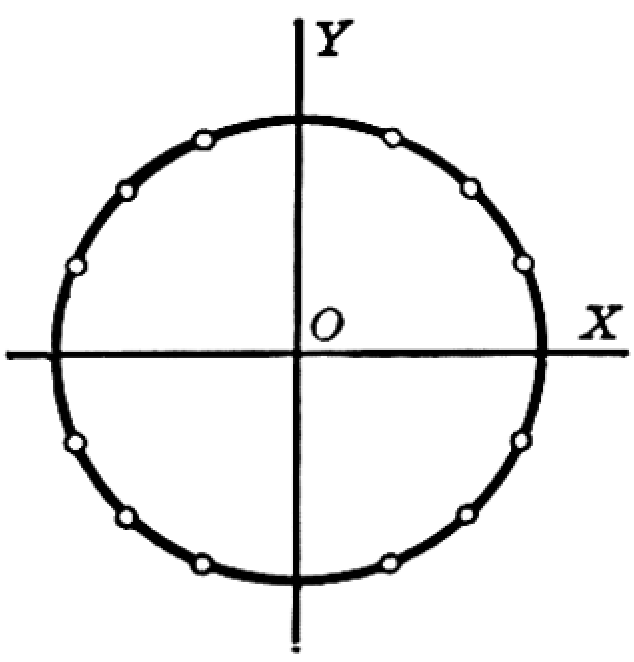


FIG. 16

Since the curve is symmetric with respect to both coördinate axes (Art. 41), we shall tabulate the values of the coördinates of points on it for the first quadrant, only. Coördinates of points on the curve in the other quadrants may be found by choosing points on it in the first quadrant and changing the signs of one or both coördinates.

x	0	1	2	3
y	3.2	3	2.4	1

If $x > \sqrt{10}$ (or $< -\sqrt{10}$), the values of y are imaginary and no corresponding points are obtained on the graph.

(2) *The ellipse. The graph of the equation*

$$Ax^2 + By^2 = C,$$

where A, B, and C are all positive, is an **ellipse.**

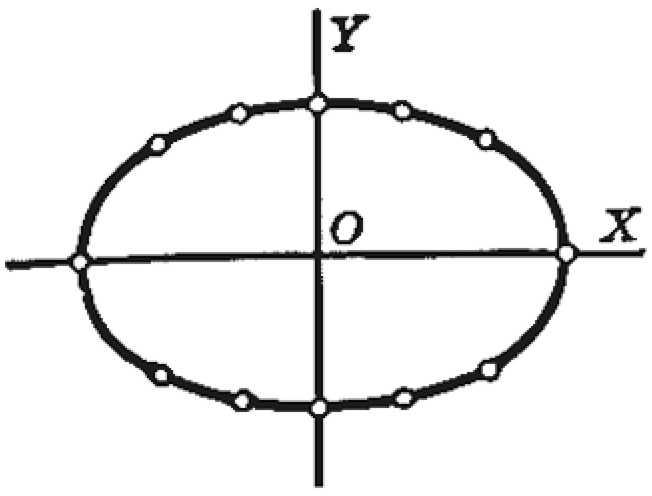


FIG. 17

EXAMPLE 2. Draw the ellipse: $4x^2 + 9y^2 = 36$.

The curve, also, is symmetric to both coördinate axes so we shall construct our table with respect to the first quadrant.

x	0	1	2	3
y	2	1.9	1.5	0

If $x > 3$ (or < -3), y is imaginary. The graph is shown in Figure 17.

(3) *The hyperbola. The graph of the equation*

$$Ax^2 - By^2 = C,$$

where A and B are positive and $C \neq 0$, is a **hyperbola.**

EXAMPLE 3. Draw the hyperbola: $x^2 - 4y^2 = 4$.

The curve is symmetric with respect to both axes. In this case, if x lies in the interval from -2 to 2, y is imaginary and there are no corresponding points on the curve. The values of y are real, however, for values of x indefinitely large. We shall tabulate values from $x = 2$ to $x = 5$.

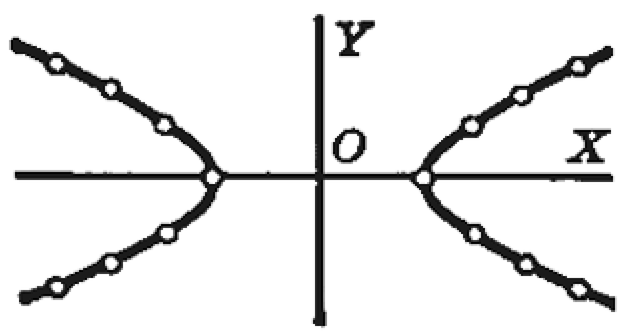


FIG. 18

x	2	3	4	5
y	0	1.1	1.7	2.3

The graph is shown in Figure 18.

The graph of the equation $xy = C$ is also a hyperbola in the position shown in Figure 19.

EXAMPLE 4. Draw the graph of the hyperbola $xy = 6$.

By assigning values to x, we obtain the following table.

x	-6	-3	-2	-1	1	2	3	6
y	-1	-2	-3	-6	6	3	2	1

There is no value of y corresponding to $x = 0$, since division by zero is impossible.

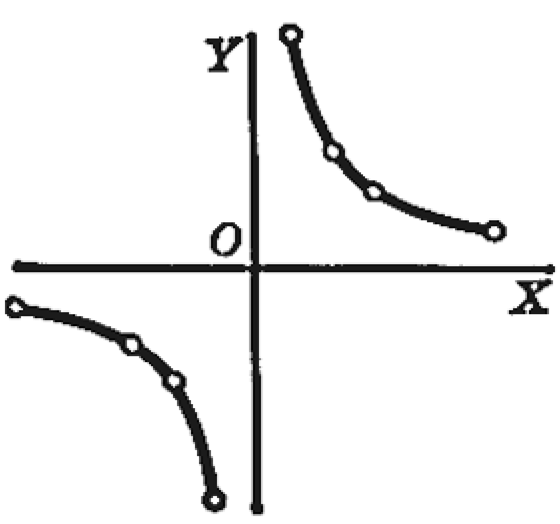


FIG. 19

The curve discussed in Ex. 4 is not symmetric with respect to either axis. It should be observed, however, that, if (x, y) are the coördinates of any point on this locus then $(-x, -y)$ are also the coördinates of a point on the locus. Since the line segment joining the points (x, y) and $(-x, -y)$ passes through the origin and is bisected by the origin, we say that *a curve that satisfies the above condition is symmetric with respect to the origin.*

(4) *The parabola.* It was seen in Art. 59 that the graph of the equation

$$y = ax^2 + bx + c, \qquad a \neq 0$$

that is, of a quadratic function of x, is a parabola opening upward if a is positive and downward if a is negative.

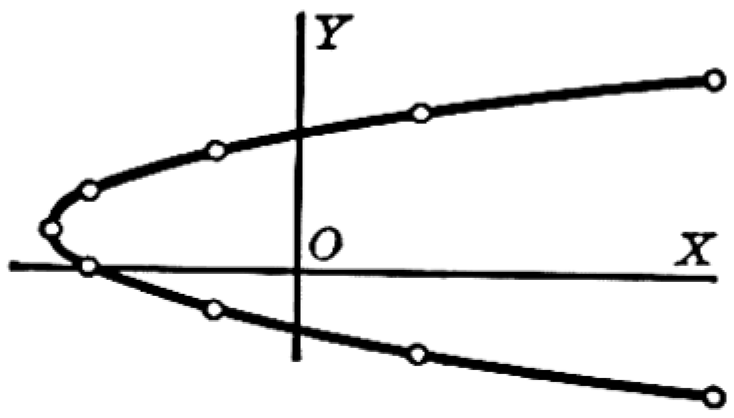


FIG. 20

Similarly, the graph of

$$x = ay^2 + by + c \qquad a \neq 0$$

is a parabola that opens to the right if a is positive and to the left if a is negative (Fig. 20).

EXAMPLE 5. Draw the graph of the parabola $x = y^2 - 2y - 5$.

In this case, we assign values to y and compute x from the equation. We have

x	10	3	-2	-5	-6	-5	-2	3	10
y	-3	-2	-1	0	1	2	3	4	5

If the given equation does not belong to one of the types illustrated in this article, the graph may be determined by plotting points on it. The form of the graph will usually be similar to one of the curves shown in the preceding examples but it will be differently placed with respect to the coördinate axes.

67. Graphical Solution of Simultaneous Equations. If the graphs of two equations are carefully drawn on the same set of axes, the real simultaneous solutions of the two equations can be found, at least approximately, as the coördinates of the points of intersection of the graphs of the two equations.

EXAMPLE. Solve graphically: $x^2 + y^2 = 20$,

$$xy = 8.$$

The graph of the first equation is a circle with center at the origin and radius $\sqrt{20}$. That of the second is a hyperbola (Fig. 21).

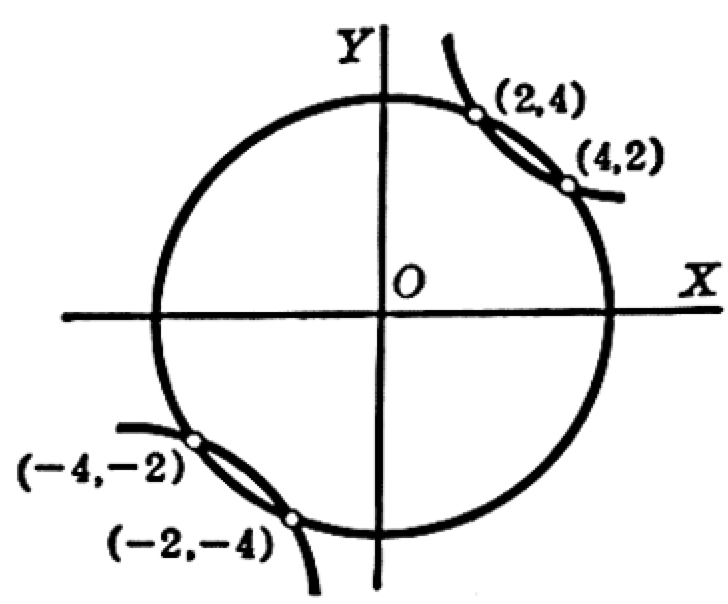


FIG. 21

These graphs intersect at the points whose coördinates are (4, 2) (2, 4), $(-2, -4)$, and $(-4, -2)$. It follows that these pairs of numbers are, at least approximately, the simultaneous solutions of the two equations. By substitution, we find that each pair exactly satisfies both equations and hence is precisely a simultaneous solution of the two equations.

Exercises

Draw the graph of each equation and state the name of the curve.

1. $x^2 + y^2 = 36.$

2. $x^2 + y^2 = 29.$

3. $4x^2 + 25y^2 = 100.$

4. $49x^2 + 9y^2 = 441.$

5. $y + 2x^2 - 5x + 3 = 0.$

6. $x - y^2 + 3y + 8 = 0.$

7. $9x^2 - 16y^2 = 144.$

8. $25y^2 - 16x^2 = 400.$

9. $x^2 + y^2 - 4x + 6y - 3 = 0.$

10. $xy + 3x - 2y = 0.$

Solve the following simultaneous equations graphically to one decimal place.

11. $x^2 + y^2 = 25,$
$x + y = 1.$

12. $x^2 + y^2 = 34,$
$x + y = 8.$

13. $x^2 - y^2 = 24,$
$2x - y = 9.$

14. $xy = 10,$
$x + y = 7.$

15. $x^2 + y^2 = 40,$
$xy + 12 = 0.$

16. $x^2 + 2y^2 = 11,$
$x^2 - y^2 = 2.$

17. $y = 3x^2 + 2x - 7,$
$y = -2x + 3.$

18. $3x^2 - 2y^2 = 6,$
$xy = 7.$

19. $x^2 + y^2 + 6x = 8y,$
$x + 2y - 1 = 0.$

20. $xy - x - y = 2,$
$x^2 - xy = 4.$

68. Algebraic Solution of Simultaneous Equations. The algebraic solution of two simultaneous equations, of which one is of the first degree and the other of the second, can always be effected by quadratics (Art. 69).

If both given equations are of the second degree, the algebraic solution usually leads to an equation of the fourth degree in one of the variables which cannot, in most cases, be solved by quadratics. In Arts. 70 to 72 we shall consider a number of special cases of two such equations that can be solved by quadratics.

In all these cases, it will be found useful to draw the graphs in connection with the algebraic solution, not only as a means of checking, but also as a method of interpreting the algebraic processes and the results.

69. Simultaneous Equations of the First and Second Degrees. To find the simultaneous solutions of two equations

$$ax + by + c = 0,$$

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

of the first and second degrees, respectively, we solve the linear equation for one variable in terms of the other, substitute the value so found in the other given equation, and solve the resulting quadratic equation. We then substitute each root of the quadratic equation in the given

linear equation and find the corresponding values of the other variable. Each of the pairs of corresponding values of x and y obtained in this way is a simultaneous solution of the two given equations.

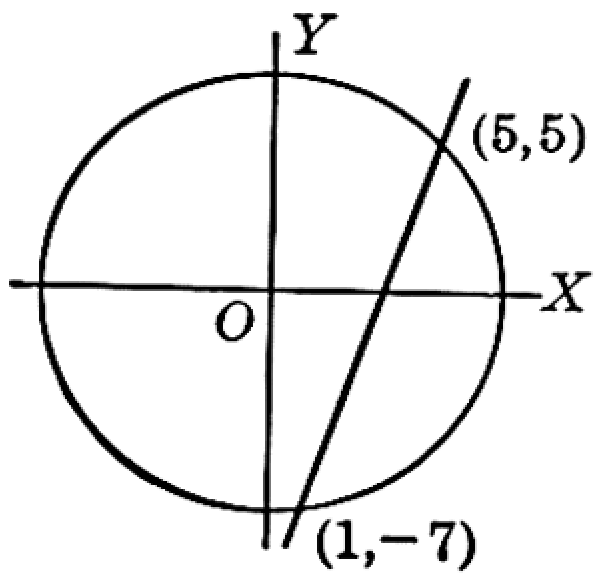


FIG. 22

EXAMPLE 1. Solve the simultaneous equations $x^2 + y^2 = 50$ and $y = 3x - 10$.

The graph of the first equation is a circle with center at the origin and radius $5\sqrt{2}$. The graph of the second is a line (Fig. 22).

Since the coördinates of the points of intersection of the line and the circle lie on both curves, their coördinates satisfy both equations and are thus the required simultaneous solutions.

Substitute the value of y from the second equation in the first. The result is

$$x^2 + (3x - 10)^2 = 50, \quad \text{or} \quad x^2 - 6x + 5 = 0.$$

The roots of this equation, 5 and 1, are the abscissas of the points of intersection. To find the ordinates, substitute these values of x, successively, in the equation of the line. If $x = 5$, we find $y = 5$ and, if $x = 1$, $y = -7$. The required solutions are $(5, 5)$ and $(1, -7)$. These are also the coördinates of the intersections of the line with the circle.

EXAMPLE 2. Solve simultaneously: $x^2 + y^2 = 45$ and $y = 2x + 15$.

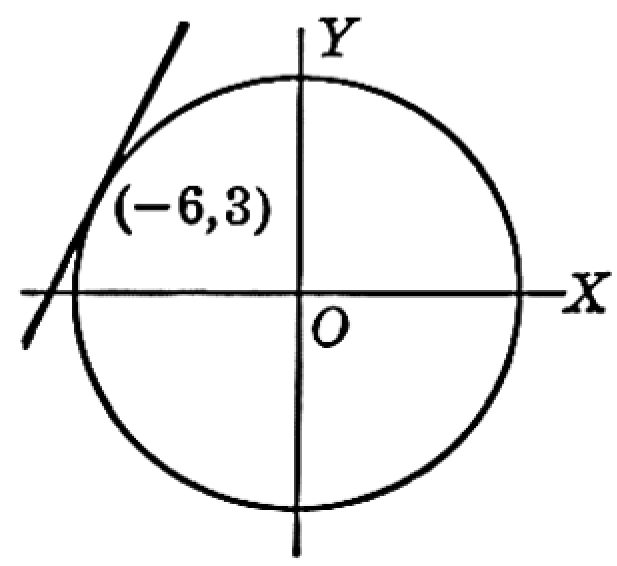


FIG. 23

From Figure 23, we see that the line is tangent to the circle. The values of x and of y for the two solutions should therefore be respectively equal.

Substitute the value of y from the second equation in the first and simplify. We get

$$x^2 + 12x + 36 = 0.$$

The roots are -6 and -6. By substituting $x = -6$ in the equation $y = 2x + 15$, we find $y = 3$. The simultaneous solutions are $(-6, 3)$ and $(-6, 3)$.

EXAMPLE 3. Solve simultaneously: $x^2 + y^2 = 25$ and $x + 2y = 15$.

Substitute the value of x from the second equation in the first and simplify. We have

$$y^2 - 12y + 40 = 0.$$

Hence $y = 6 + 2i$ and $y = 6 - 2i$. The corresponding values of x are $x = 3 - 4i$ and $x = 3 + 4i$, giving the simultaneous solutions $(6 + 2i, 3 - 4i)$ and $(6 - 2i, 3 + 4i)$.

The student should draw the graphs and show that the line and circle do not intersect.

Exercises

Solve the pairs of simultaneous equations algebraically and check graphically.

1. $x^2 + y^2 = 40,$
$2x + y = 10.$

2. $x^2 + y^2 = 65,$
$3x + 2y = 26.$

3. $y = 2x^2 - 4x - 13,$
$y = 2x + 7.$

4. $xy = 12,$
$3x - 5y = 3.$

5. $4x^2 + y^2 = 68,$
$2x + y = 10.$

6. $3x = y^2 - 5y + 3,$
$x + 5y + 6 = 0.$

7. $x^2 - y^2 = 24,$
$2x - y = 9.$

8. $7x^2 - 4y^2 = 3,$
$3x + 2y = 1.$

9. $3x^2 + 2y^2 = 11,$
$3x - 4y = 11.$

10. $2x^2 + 6x + 5y + 1 = 0,$
$2x + y + 3 = 0.$

11. $4x^2 - 3y^2 = 5,$
$y = 2x - 1.$

12. $x^2 + 3xy + y^2 = 4,$
$x - y - 7 = 0.$

Find the values of k that make the line tangent to the curve.

13. $4x^2 - 3y^2 = 24,$
$2x + y + k = 0.$

14. $5x^2 + 2y^2 = 14,$
$5x - 2y + k = 0.$

70. Systems of the Form $Ax^2 + By^2 = C$. Two simultaneous equations of this form should first be solved as linear equations in x^2 and y^2. The values of x and y can then be found by extracting the square roots of the values of x^2 and y^2.

EXAMPLE. Solve: $x^2 + y^2 = 13,$
$3x^2 + 2y^2 = 30.$

Eliminate y^2 by multiplying the first equation by 2 and subtracting from the second

$$\begin{array}{l} 2x^2 + 2y^2 = 26 \\ \underline{3x^2 + 2y^2 = 30} \\ x^2 \qquad\;\; = 4. \end{array}$$

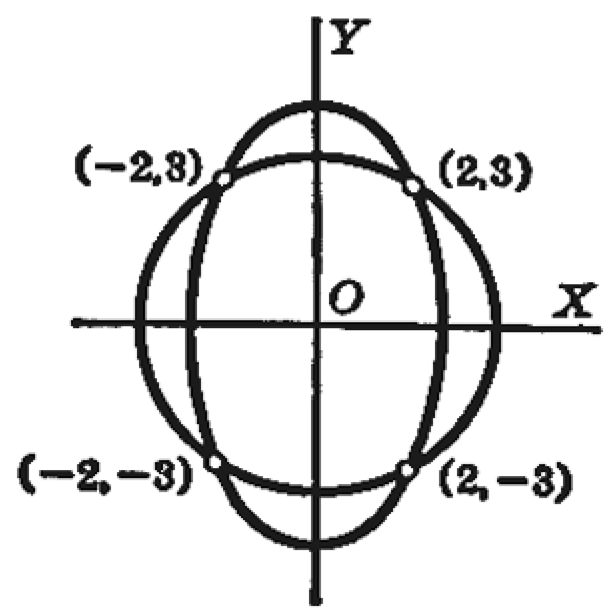


FIG. 24

Hence, $x = 2$ or $x = -2$. If we substitute either of these values of x in the first equation, we obtain $y^2 = 9$, from which $y = 3$ or $y = -3$.

Since either value of x, paired with either value of y, satisfies both of the given equations, there are four solutions $(2, 3)$, $(-2, 3)$, $(-2, -3)$, and $(2, -3)$ (Fig. 24).

Exercises

Solve the simultaneous equations.

1. $x^2 + y^2 = 25,$
$5x^2 - 2y^2 = 13.$

2. $4x^2 + 3y^2 = 48,$
$x^2 + y^2 = 13.$

3. $4x^2 + y^2 = 61,$
$2x^2 + 3y^2 = 93$

4. $x^2 - 5y^2 = 4,$
$2x^2 + 3y^2 = 21.$

5. $2x^2 + 5y^2 = 95,$
$x^2 + 3y^2 = 52.$

6. $7x^2 - 3y^2 = 2,$
$3x^2 + y^2 = 42.$

7. $7x^2 - 2y^2 = 11,$
$2x^2 + y^2 = 11.$

8. $3x^2 + 2y^2 = 23,$
$2x^2 + 3y^2 = 27.$

9. $3x^2 + 2y^2 = 18,$
$4x^2 + 5y^2 = 45.$

10. $7x^2 - 3y^2 = 7,$
$11x^2 - 5y^2 = 10.$

11. $3x^2 + 4y^2 = 11,$
$4x^2 + 7y^2 = 8.$

12. $4x^2 - 3y^2 = 9,$
$5x^2 - 8y^2 = 41.$

13. Show that the circle $x^2 + y^2 = 4$ touches the ellipse $5x^2 + 2y^2 = 20$ at two points.

71. Systems of the Form $Ax^2 + Bxy + Cy^2 = D$. Two methods are available for the solution of two equations both of which are of this type. These two methods are illustrated by solutions A and B of the following example.

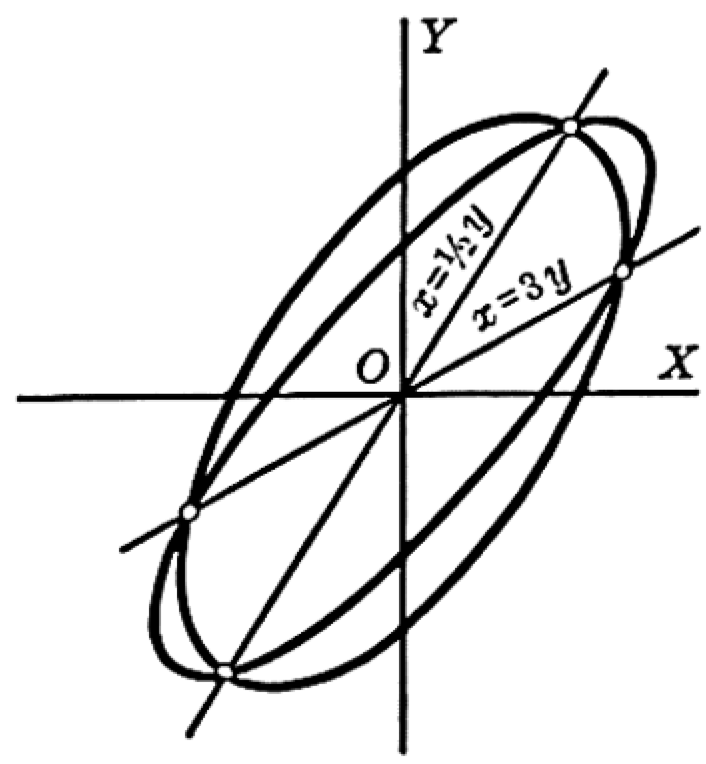


FIG. 25

EXAMPLE. Solve: $2x^2 - xy + y^2 = 4,$
$4x^2 - 5xy + 3y^2 = 6.$

A. *Solution by eliminating the constant term.*

Multiply the first equation by 3, the second by 2, and subtract the first from the second.

$$\begin{array}{l} 6x^2 - 3xy + 3y^2 = 12 \\ \underline{8x^2 - 10xy + 6y^2 = 12} \\ 2x^2 - 7xy + 3y^2 = 0. \end{array}$$

Solve the resulting equation as a quadratic equation in x. We obtain

$$x = \tfrac{1}{2}y, \quad \text{or} \quad x = 3y.$$

Each of these equations defines a line passing through two of the required intersections (Fig. 25).

If we substitute $x = \frac{1}{2}y$ in the first given equation, we have

$$\tfrac{1}{2}y^2 - \tfrac{1}{2}y^2 + y^2 = 4, \quad \text{or} \quad y^2 = 4.$$

Hence, $y = 2$, or $y = -2$.

Since $x = \frac{1}{2}y$, we have the solutions

$$x = 1, y = 2, \quad \text{and} \quad x = -1, y = -2.$$

If we substitute $x = 3y$ in the first given equation, we have

$$18y^2 - 3y^2 + y^2 = 4, \quad \text{or} \quad 4y^2 = 1.$$

Hence, $y = \frac{1}{2}$, or $y = -\frac{1}{2}$.

Since $x = 3y$, we have the solutions

$$x = \tfrac{3}{2}, y = \tfrac{1}{2}, \quad \text{and} \quad x = -\tfrac{3}{2}, y = -\tfrac{1}{2}.$$

B. *Solution by putting $y = mx$.* If we put $y = mx$ in the given equations, we have, respectively

$$(2 - m + m^2)x^2 = 4, \quad \text{and} \quad (4 - 5m + 3m^2)x^2 = 6.$$

Solve these equations for x^2.

$$x^2 = \frac{4}{2 - m + m^2}, \quad \text{and} \quad x^2 = \frac{6}{4 - 5m + 3m^2}. \tag{2}$$

Equate these values of x^2 and clear of fractions:

$$4(4 - 5m + 3m^2) = 6(2 - m + m^2).$$

Simplify: $$3m^2 - 7m + 2 = 0.$$

The roots of this equation are $m = 2$ and $m = \frac{1}{3}$.

If we put $m = 2$ in either of equations (2), we have

$$x^2 = 1.$$

Hence, $x = 1$, or $x = -1$.

Since $y = mx$, where $m = 2$, we have the solutions

$x = 1, y = 2$, and $x = -1, y = -2$.

If we put $m = \frac{1}{3}$ in either of equations (2), we have

$$x^2 = \tfrac{9}{4}.$$

Hence, $x = \frac{3}{2}$, or $x = -\frac{3}{2}$.

Since $y = mx$, where $m = \frac{1}{3}$, we have the solutions

$x = \frac{3}{2}, y = \frac{1}{2}$, and $x = -\frac{3}{2}, y = -\frac{1}{2}$.

Exercises

Solve the simultaneous equations.

1. $4x^2 - 4xy - y^2 = 7$, $7x^2 - xy + 2y^2 = 28$.
2. $4x^2 - 5xy + y^2 = 10$, $3x^2 - 3xy - y^2 = 5$.
3. $3x^2 - 6xy + 2y^2 = 11$, $2x^2 + xy + y^2 = 22$.
4. $4x^2 + xy + y^2 = 34$, $x^2 - 5xy + 3y^2 = 51$.
5. $6x^2 + 10xy - 5y^2 = 39$, $x^2 + 11xy + 26y^2 = 52$.
6. $x^2 + 4y^2 = 5$, $x^2 + 2xy + 4y^2 = 3$.
7. $2x^2 - xy = 12$, $xy + y^2 = 6$.
8. $2x^2 + 3xy + y^2 = 24$, $x^2 + 6xy - y^2 = 18$.
9. $8x^2 - 5xy + 42 = 0$, $3xy - 8y^2 = 14$.
10. $2x^2 + xy - y^2 + 5 = 0$, $2x^2 - xy - y^2 = 7$.
11. $x^2 + 5xy + 3y^2 = 12$, $2y^2 + xy - 3x^2 = 12$.
12. $2x^2 - 3xy = 5$, $2xy - 3y^2 = 2$.

72. Special Devices. Many simultaneous equations, some of them of degree higher than two, can be solved by quadratics if certain devices appropriate to the problem are used. A few of these devices are suggested in the following examples and exercises.

EXAMPLE 1. Solve:
$$2x^2 + y^2 - 4y - 23 = 0,$$
$$5x^2 - y^2 - 3y - 5 = 0.$$

If we eliminate x^2, the resulting equation will not contain the variable x. Multiply the first equation by 5, the second by 2, and subtract.

$$\begin{array}{r} 10x^2 + 5y^2 - 20y - 115 = 0 \\ \underline{10x^2 - 2y^2 - 6y - 10 = 0} \\ 7y^2 - 14y - 105 = 0. \end{array}$$

Hence, $y = 5$ or $y = -3$. If we substitute $y = 5$ in the first equation, we obtain $2x^2 - 18 = 0$. Hence $x = 3$ or $x = -3$. It follows that $(3, 5)$ and $(-3, 5)$ are solutions. If we put $y = -3$ in the first equation, we have $2x^2 - 2 = 0$, giving $x = 1$ or $x = -1$. Hence, $(1, -3)$ and $(-1, -3)$ are also solutions.

The required solutions are, accordingly, $(3, 5)$, $(-3, 5)$, $(1, -3)$, and $(-1, -3)$.

An equation of the form

$$A(x^2 + y^2) + Bxy + C(x + y) + D = 0,$$

is said to be **symmetric** in x and y. It has the property that it remains the same equation if x and y are interchanged throughout.

If both given equations are symmetric, the system can be solved by putting $x = u + v$, $y = u - v$ and eliminating v^2 from the resulting equations.

EXAMPLE 2. Solve: $x^2 + y^2 - 3xy - 2x - 2y - 15 = 0,$
$xy + 2x + 2y - 1 = 0.$

Since both of these equations are symmetric in x and y, we substitute $x = u + v$ and $y = u - v$, giving

$$5v^2 - u^2 - 4u - 15 = 0,$$
$$-v^2 + u^2 + 4u - 1 = 0.$$

By eliminating v^2, we obtain $4u^2 + 16u - 20 = 0$. Hence, $u = 1$ or $u = -5$. If we substitute these values for u in the preceding equations and solve for v, we obtain the following sets of solutions for u and v:

$$u = 1, v = 2; \; u = 1, v = -2; \; u = -5, v = 2; \; u = -5, v = -2.$$

By substituting these pairs of values of u and v successively in the equations $x = u + v$, $y = u - v$, we obtain, as the required pairs of values of x and y, $(3, -1)(-1, 3)$, $(-3, -7)$, and $(-7, -3)$.

Exercises

Solve the simultaneous equations by first eliminating x^2 or y^2.

1. $y^2 - 2x^2 + 11x - 30 = 0,$
$2y^2 - x^2 + 13x - 114 = 0.$

2. $y^2 - x^2 - 2x + 5 = 0,$
$5y^2 + x^2 - x - 2 = 0.$

3. $3x^2 + y^2 + 4y - 24 = 0,$
$x^2 - y^2 - 4y + 20 = 0.$

4. $3x^2 + y^2 - 2y - 2 = 0,$
$4x^2 + 2y^2 - y - 11 = 0.$

Solve the simultaneous symmetric equations.

5. $3x^2 + 3y^2 - 2xy - x - y = 8,$
$4x^2 + 4y^2 - 5x - 5y - 5 = 0.$

6. $x^2 + y^2 - 2x - 2y - 11 = 0,$
$xy - 3x - 3y + 9 = 0.$

7. $2x^2 + 2y^2 - xy + x + y = 11,$
$x^2 + y^2 + xy + 2x + 2y = 13.$

8. $x^2 + y^2 - 2x - 2y - 23 = 0,$
$xy + 2x + 2y + 10 = 0.$

In Exercises 9 to 12, factor the first member of the second equation and replace one factor by its value from the first equation. Solve the resulting equation with the first given one.

9. $x + y = 3,$
$x^3 + y^3 = 9.$

10. $x^2 + xy + y^2 = 13,$
$x^3 - y^3 = 26.$

11. $x - 3y = 2,$
$x^3 - 27y^3 = 98.$

12. $x - y = 3,$
$x^3 - y^3 = 3x^2 + 15y^2.$

In Exercises 13 to 16, first find the values of $1/x$ and $1/y$.

13. $\dfrac{6}{x} + \dfrac{1}{y} = 2,$
$\dfrac{8}{x^2} + \dfrac{2}{y^2} = 1.$

14. $\dfrac{8}{x^2} - \dfrac{9}{y^2} = 1,$
$\dfrac{12}{x^2} + \dfrac{18}{y^2} = 5.$

15. $\dfrac{1}{x^2} - \dfrac{1}{xy} = 3,$
$\dfrac{3}{xy} + \dfrac{1}{y^2} = -2.$

16. $\dfrac{1}{x^2} + \dfrac{1}{y^2} = 25,$
$\dfrac{1}{xy} + 12 = 0.$

Solve by any method.

17. $3x + 5y = 17,$
$9x^2 + 25y^2 = 169.$

18. $x + 2y = 5,$
$x^2 - 2y = 7.$

19. $6x^2 - y^2 = 7,$
$x^2 + 2y^2 = 12.$

20. $x^2 - xy + y^2 = 39,$
$x^2 - 2xy = 24.$

21. $x + y = 2,$
$2y = 2x^2 + 3x + 1.$

22. $(3x + 2y)^2 + 2(x + y)^2 = 17,$
$2(3x + 2y)^2 + (x + y)^2 = 22.$

23. $y = 3x^2 - 4x + 9,$
$y = 2x^2 + x + 3.$

24. $4\sqrt{x} - 3\sqrt{y} = 8,$
$3\sqrt{x} - 2\sqrt{y} = 7.$

73. Problems Involving Simultaneous Quadratics. In each of the following problems, denote two of the unknowns by x and y. From the statement of the problem, set up two equations in these unknowns and solve them for x and y.

1. The perimeter of a rectangle is 22 feet and its area is 24 square feet. Find its dimensions.

2. The area of a rectangle is 48 square feet and the length of its diagonal is 10 feet. Find its dimensions.

3. The area of a rectangle is 60 square feet and the square of its longer side exceeds the square of its shorter side by 119 square feet. Find its dimensions.

4. Find two numbers whose product is 432 and whose quotient is $\frac{3}{4}$.

5. The altitude of an isosceles triangle is 12 inches and its perimeter is 36 inches. Find the lengths of its sides.

6. If the numerator of a simple fraction is increased by 1 and the denominator by 10, the resulting fraction equals the reciprocal of the given fraction. If both numerator and denominator are increased by 5, the resulting fraction equals $\frac{4}{3}$. Find the fraction.

7. Find two numbers such that their sum, their product, and the difference of their squares are all equal.

8. A motor boat can go 45 miles upstream and return in 8 hours. If it goes 54 miles upstream and returns, it is still 24 miles from its destination at the end of 8 hours. Find its speed in still water and the rate of the current.

9. A pedestal 10 feet high is formed of two cubical blocks. The sum of the volumes of these blocks is 370 cubic feet. Find the lengths of the edges of the blocks.

10. Twenty minutes after A started on a journey, B was sent after him with a message. B traveled 35 miles an hour, delivered his message, and returned to the starting point at the instant A was 130 miles away. How fast did A travel?

11. The area of a rectangle is 60 square feet. If each side is increased by 3 feet, the area of the square on the diagonal is increased by 120 square feet. Find the dimensions of the original rectangle.

12. Two circles are tangent externally and both are tangent internally to a larger circle. The distance from the center of one small circle to the center of each of the others is 8 inches and the sum of the areas of the two small circles is two-ninths that of the large circle. Find the radii of the three circles.

13. A man has $6400 invested in two securities. From one he receives annually $162 and from the other, on which the interest rate is one per cent greater, he receives annually $154. Find the amount of each investment and the interest rate on each.

14. The length of the diagonal of a rectangle is a and its area is b^2. Find the lengths of its sides.

Chapter 10

Logarithms

74. Definition. *The logarithm of a positive* number N to a base a $(a > 1)$ is the exponent of the power to which a must be raised to equal N.*

We shall write the expression, "the logarithm of N to the base a" in the abbreviated form

$$\log_a N.$$

It should be read, however, without abbreviation.

It follows from the above definition that, if N is positive and a is greater than unity, *the two statements*

$$\log_a N = x, \quad \text{and} \quad N = a^x, \tag{1}$$

are equivalent. If either is true, the other is also true.

Thus, the following pairs of statements are equivalent.

$5^3 = 125$ and $\log_5 125 = 3$; $\quad 3^{-4} = \frac{1}{81}$ and $\log_3 (\frac{1}{81}) = -4$;

$2^{1.2} = \sqrt[5]{64}$ and $\log_2 \sqrt[5]{64} = 1.2$; $\quad 10^{0.30103} = 2$ and $\log_{10} 2 = 0.30103$.

It will frequently be necessary, throughout this chapter, to transform one of the equations (1) to the other form. The student should therefore familiarize himself with the equivalence of the two forms of equations (1) so that, when either equation is given, he can write the other form immediately.

We shall assume the truth of the following theorem: *Given any positive number N, there exists one, and only one, positive number x such that $\log_a N = x$, and conversely.*

Exercises

Write the following equations in the logarithmic form.

1. $2^5 = 32$. **2.** $13^2 = 169$. **3.** $(49)^{\frac{1}{2}} = 7$.
4. $(121)^{0.5} = 11$. **5.** $(81)^{0.25} = 3$. **6.** $9^{-1.5} = \frac{1}{27}$.
7. $\sqrt{25} = 5$. **8.** $\sqrt[3]{8^2} = 4$. **9.** $17^0 = 1$.

Write the following equations in the exponential form.

10. $\log_3 81 = 4$. **11.** $\log_{36} 6 = 0.5$. **12.** $\log_9 243 = 2.5$.
13. $\log_2 0.0625 = -4$. **14.** $\log_{1.2} 1.44 = 2$. **15.** $\log_{32} 0.5 = -0.2$.
16. $\log_{625} 0.2 = -0.25$. **17.** $\log_a a^2 = 2$. **18.** $\log_a 1 = 0$.

* The logarithms of negative and imaginary numbers also exist but, since they are imaginary if a is positive, we shall not consider them in this book.

Find the values of the following logarithms.

19. $\log_{10} 1000$. **20.** $\log_3 81$. **21.** $\log_4 8$.
22. $\log_{10} 0.001$. **23.** $\log_5 0.04$. **24.** $\log_{16} 0.125$.
25. $\log_{100} 0.1$. **26.** $\log_a a$. **27.** $\log_a a^{-1}$.

Find x, given:

28. $\log_3 x = 4$. **29.** $\log_2 x = -3$. **30.** $\log_{81} x = 0.25$.
31. $\log_{49} x = -0.5$. **32.** $\log_9 x = -2$. **33.** $\log_7 x = 0$.
34. $\log_x 5 = 0.5$. **35.** $\log_x 7 = \frac{1}{3}$. **36.** $\log_x 0.008 = -3$.
37. $\log_x 4 = 0.4$. **38.** $\log_x a = 1$. **39.** $\log_x a = 0.5$.

Show that:

40. $\log_a a^x = x$. **41.** $a^{\log_a x} = x$. **42.** $\log_a (1/a^2) = -2$.

75. Properties of Logarithms. Since logarithms have been defined as exponents, the laws of operation with them will be derived from the laws of exponents. We have seen (Art. 29) that, if m and n are rational numbers

$$a^m \cdot a^n = a^{m+n}, \quad a^m \div a^n = a^{m-n}, \quad \text{and} \quad (a^m)^n = a^{mn}. \tag{2}$$

Since most of the exponents with which we shall deal are irrational, we now assume that *these laws still hold if m and n are irrational numbers.*

From each of the equations (2), we shall now derive a corresponding law for logarithms. It is upon these laws that the usefulness of logarithms in numerical computation depends.

I. *The logarithm of the product of two numbers equals the sum of the logarithms of the numbers*, that is,

$$\mathbf{\log_a M \cdot N = \log_a M + \log_a N.}$$

For, let

$$\log_a M = x, \quad \text{and} \quad \log_a N = y.$$

By the definition of a logarithm, these equations are respectively equivalent to

$$M = a^x, \quad \text{and} \quad N = a^y.$$

The product MN is found, by the laws of exponents, to be

$$M \cdot N = a^x \cdot a^y = a^{x+y}.$$

Write the equation $M \cdot N = a^{x+y}$ in the logarithmic form and replace x and y by their values, as stated above. We have

$$\log_a M \cdot N = x + y = \log_a M + \log_a N,$$

which is the required formula.

It can be proved in the same way that

$$\log_a M \cdot N \cdot P = \log_a M + \log_a N + \log_a P,$$

and similarly for any number of factors.

Thus, $\log_2 2048 = \log_2 128 \cdot 16 = \log_2 128 + \log_2 16 = 7 + 4 = 11.$
$\log_{10} 77 = \log_{10} 7 \cdot 11 = \log_{10} 7 + \log_{10} 11.$
$\log_{10} (53.12)(46.35)(9.643) = \log_{10} 53.12 + \log_{10} 46.35 + \log_{10} 9.643.$

II. *The logarithm of a fraction equals the logarithm of the numerator minus the logarithm of the denominator*, that is,

$$\log_a \frac{M}{N} = \log_a M - \log_a N.$$

For, let

$$\log_a M = x, \quad \text{and} \quad \log_a N = y.$$

Write these two equations in the equivalent exponential form

$$M = a^x, \quad \text{and} \quad N = a^y.$$

For the fraction M/N, we have, by the laws of exponents,

$$\frac{M}{N} = \frac{a^x}{a^y} = a^{x-y}.$$

Write the equation $M/N = a^{x-y}$ in the logarithmic form and replace x and y by their values. We have

$$\log_a \frac{M}{N} = x - y = \log_a M - \log_a N.$$

Thus, $\log_3 \frac{243}{9} = \log_3 243 - \log_3 9 = 5 - 2 = 3.$

$$\log_{10} \frac{586.2}{41.76} = \log_{10} 586.2 - \log_{10} 41.76.$$

III. *The logarithm of the nth power of a number equals n times the logarithm of the number*, that is,

$$\log_a N^n = n \log_a N.$$

Let $\log_a N = x$, from which $N = a^x$.
Raise both sides of this equation to the power n. We have

$$N^n = (a^x)^n = a^{nx}.$$

Write the equation $N^n = a^{nx}$ in the logarithmic form and replace x by its value. We have

$$\log_a N^n = nx = n \log_a N.$$

Thus, $\log_5 (25)^3 = 3 \log_5 25 = 3 \cdot 2 = 6,$

$$\log_{10} (716)^7 = 7 \log_{10} 716.$$

In Theorem III, n need not be an integer. In fact, since $\sqrt[m]{N} = N^{\frac{1}{m}}$, we have, on replacing n by $1/m$ in Theorem III, the following theorem.

IV. *The logarithm of the mth root of a number equals the result obtained by dividing the logarithm of the number by m*, that is,

$$\log_a \sqrt[m]{N} = \frac{1}{m} \log_a N.$$

Thus,
$$\log_2 \sqrt[3]{64} = \tfrac{1}{3} \log_2 64 = \tfrac{1}{3} \cdot 6 = 2.$$
$$\log_{10} \sqrt[4]{7.842} = \tfrac{1}{4} \log_{10} 7.842.$$

EXAMPLE 1. Express $\log_{10} \dfrac{4^3 \cdot \sqrt[3]{31}}{7^{\frac{1}{4}} \cdot (38)^5}$ as an algebraic sum of the logarithms of integers.

$$\begin{aligned}\log_{10} \frac{4^3 \cdot \sqrt[3]{31}}{7^{\frac{1}{4}} \cdot (38)^5} &= \log_{10} 4^3 \cdot \sqrt[3]{31} - \log_{10} 7^{\frac{1}{4}} \cdot (38)^5 \\ &= \log_{10} 4^3 + \log_{10} \sqrt[3]{31} - (\log_{10} 7^{\frac{1}{4}} + \log_{10} 38^5) \\ &= 3 \log_{10} 4 + \tfrac{1}{3} \log_{10} 31 - \tfrac{1}{4} \log_{10} 7 - 5 \log_{10} 38.\end{aligned}$$

EXAMPLE 2. Express $2 \log_{10} 15 + \frac{1}{3} \log_{10} 11 + \frac{2}{5} \log_{10} 8 - 5 \log_{10} 41 - \frac{3}{7} \log_{10} 53$ as the logarithm of a single number.

$$\begin{aligned}&2 \log_{10} 15 + \tfrac{1}{3} \log_{10} 11 + \tfrac{2}{5} \log_{10} 8 - 5 \log_{10} 41 - \tfrac{3}{7} \log_{10} 53 \\ &\quad = \log_{10} (15)^2 + \log_{10} (11)^{\frac{1}{3}} + \log_{10} 8^{\frac{2}{5}} - [\log_{10} (41)^5 + \log_{10} (53)^{\frac{3}{7}}] \\ &\quad = \log_{10} 15^2 \cdot (11)^{\frac{1}{3}} \cdot 8^{\frac{2}{5}} - \log_{10} (41)^5 \cdot (53)^{\frac{3}{7}} \\ &\quad = \log_{10} \frac{(15)^2 \cdot (11)^{\frac{1}{3}} \cdot 8^{\frac{2}{5}}}{(41)^5 \cdot (53)^{\frac{3}{7}}}.\end{aligned}$$

Exercises

Express as an algebraic sum of the logarithms of integers:

1. $\log_{11} (51)(896)(743)$.

2. $\log_3 \sqrt[3]{6^7} \cdot \sqrt[4]{9685}$.

3. $\log_7 \dfrac{43^2\ (695)^{\frac{2}{3}}}{\sqrt[3]{71} \cdot \sqrt{563}}$.

4. $\log_{10} \dfrac{3^5\ (671)^{\frac{2}{3}} \cdot \sqrt[4]{591}}{17^2 \sqrt[3]{96} \cdot \sqrt[5]{4817}}$.

Express as the logarithm of a single quantity:

5. $2 \log_7 76 + 3 \log_7 48 - 5 \log_7 59$.

6. $3 \log_6 47 + \frac{1}{2} \log_6 34 - 4 \log_6 71 - \frac{3}{5} \log_6 85$.

7. $\log_{10} 16 + 2 \log_{10} t$.

8. $\log_{10} P + n \log_{10} (1 + i)$.

9. $\log_{10} 4 - \log_{10} 3 + \log_{10} \pi + 3 \log_{10} r$.

10. $\log_{10} k + \log_{10} b + 2 \log_{10} d - \log_{10} l$.

11. $\frac{1}{2}[\log_{10} (s - a) + \log_{10} (s - b) + \log_{10} (s - c) - \log_{10} s]$.

Find the numerical value of each of the following expressions.

12. $\log_5 \dfrac{125 \cdot 625}{25}$.

13. $\log_9 \sqrt[3]{81}$.

14. $\log_7 \dfrac{49\sqrt{7}}{7^3}$.

15. $\log_2 \dfrac{32\ (64)^{\frac{2}{3}}}{\sqrt[3]{128}}$.

16. $\log_8 \dfrac{\sqrt{2} \cdot \sqrt[3]{256}}{\sqrt[6]{32}}$.

17. $\log_{13} \dfrac{169\,(13)^{\frac{4}{3}}}{\sqrt[3]{13^2} \cdot \sqrt{13^3}}$.

Given that $\log_{10} 2 = 0.30103$ and $\log_{10} 3 = 0.47712$, approximately, find:

18. $\log_{10} 12$. **19.** $\log_{10} 60$. **20.** $\log_{10} \frac{3}{2}$.
21. $\log_{10} 5$. **22.** $\log_{10} \frac{1}{72}$. **23.** $\log_{10} 450$.
24. $\log_{10} \sqrt{15}$. **25.** $\log_{10} \sqrt[4]{18}$. **26.** $\log_{10} (\frac{27}{16})^{\frac{4}{5}}$.

76. Common Logarithms. In numerical computations with logarithms, it is customary to use the base 10. Logarithms to this base are called **common logarithms.** From now on, unless a different base is stated, the logarithms we shall use will be common logarithms and the base will not be indicated; that is, instead of $\log_{10} N$, we shall write $\log N$.

The common logarithm of a number is usually considered as consisting of two parts, the **characteristic** (Art. 78) and the **mantissa** (Art. 80). The reason for this division lies in the fact that the value of the characteristic depends only on the position of the decimal point and the value of the mantissa on the sequence of significant figures in the number.

77. Logarithms of Integral Powers of Ten. If a number is an integral power of ten,

$$N = 10^k,$$

where k is an integer or zero, then, by the definition of a logarithm,

$$\log N = k.$$

In this special case, we say that the characteristic of the logarithm is k and that its mantissa is zero.

The reader should verify the correctness of the logarithms given in the following table.

N	.0001	.001	.01	.1	1	10	100	1000	10000
$\log N$	-4	-3	-2	-1	0	1	2	3	4

78. The Characteristic. *The characteristic of the logarithm of a number not an integral power of 10 is the integer, or zero, next less than the logarithm of the number.*

It is proved in advanced mathematics that

$$\text{if} \quad M < N, \quad \text{then} \quad \log M < \log N.$$

It follows that the characteristic of $\log N$ is the logarithm of the integral power of ten next less than N. For,

$$\text{if} \quad 10^k < N < 10^{k+1}, \quad \text{then} \quad k < \log N < k + 1.$$

Hence, if k is zero or an integer, then k is the characteristic of $\log N$. If the characteristic k is a negative number $-k'$, it is customary to write it in the form $(10 - k') - 10$.

EXAMPLE 1. Find the characteristic of log 579.3.

Since $100 < 579.3 < 1000$, it follows that $\log 100 < \log 579.3 < \log 1000$, or $2 < \log 579.3 < 3$. Hence,

$$\log 579.3 = 2 + \text{a positive decimal less than } 1.$$

It follows from this equation that the characteristic of log 579.3 is 2.

EXAMPLE 2. Find the characteristic of log 0.0025438.

Since $0.001 < 0.0025438 < 0.01$, we have $\log 0.001 < \log 0.0025438 < \log 0.01$, or $-3 < \log 0.0025438 < -2$. It follows that

$$\log 0.0025438 = -3 + \text{a positive decimal less than } 1.$$

Hence, the characteristic of log 0.0025438 is -3 or $7 - 10$.

By the reasoning used in these examples, we obtain the following rule for finding the characteristic of $\log N$.

If $N > 1$, the characteristic is the positive integer 1 *less than the number of digits to the left of the decimal point.*

If $0 < N < 1$, and the first non-zero figure of N is in the kth decimal place, then the characteristic is $-k$, or $(10 - k) - 10$.

Exercises

Write the characteristics of the logarithms of the following numbers.

1. 4863.2. **2.** 76.352. **3.** 5.7843. **4.** 9652700.
5. 0.71643. **6.** 0.00721. **7.** 0.000092. **8.** 0.000009.

Given the sequence of digits 46739. Place the decimal point (adding zeros if necessary), given that the characteristic of its logarithm is:

9. 3. **10.** 0. **11.** 4. **12.** 7.
13. $9 - 10$. **14.** $7 - 10$. **15.** $5 - 10$. **16.** $3 - 10$.

79. Significant Figures. Two numbers that differ only in the position of the decimal point (together with zeros that may be added at either end solely to fix the position of the decimal point) are said to have *the same sequence of significant figures.*

Thus, .0053076, 53.076, and 53076000. have the same sequence of significant figures, namely, 5, 3, 0, 7, and 6.

Since two numbers, M and N, having the same sequence of significant figures, differ at most only in the position of the decimal point, it follows that

$$M = 10^k N, \tag{3}$$

where k is an integer or zero, and conversely.

The first significant figure of a number is the first digit, other than zero, of the number, reading from the left. The number of significant

figures is the number of digits, counting from the first significant figure but excluding zeros added at the right only to fix the decimal point. Zeros at the right of a number are significant if their values are affirmed to be zero; they are not significant if they are added only to fix the decimal point.

In computations involving the use of logarithmic (and other) tables, the results are usually valid, at most, to a definite number of significant figures. Thus, we shall be able to find, from Table I, the logarithms of numbers of five significant figures only. If a number arises having more than five significant figures, we shall "round it off" to the nearest five significant figures.

Thus, to five figures, 21.5837 and 503.862 are rounded off to 21.584 and 503.86, respectively; to four figures, they are rounded off to 21.58 and 503.9, respectively. To round off, to five significant figures, a number such as 0.836475, we shall, as a matter of convention, customarily make the last figure an even number. Thus, 0.836475 becomes 0.83648 but 609.745 becomes 609.74.

80. The Mantissa. *The mantissa of log N is the positive or zero decimal, less than* 1, *that must be added to the characteristic to equal log N; that is,*

$$\log N = \textbf{characteristic} + \textbf{mantissa}.$$

If two numbers, M and N, have the same sequence of significant figures, their mantissas are equal, and conversely. For, if M and N have the same sequence of significant figures, then, from equation (3),

$$\log M = \log (10^k N) = \log 10^k + \log N = k + \log N.$$

Since k is an integer or zero, it does not affect the decimal part of the logarithm; that is, it does not affect the mantissa.

Conversely, if the mantissas of M and N are equal, then

$$\log M = k + \log N = \log (10^k N),$$

so that $M = 10^k N$ and M and N have the same sequence of significant figures.

Because of this theorem, tables of common logarithms give only the mantissas of sequences of significant figures. Table I, at the back of the book, gives, to five decimal places, the mantissas of all sequences of four significant figures.

81. Antilogarithms. The number N which has a given number x as its logarithm is called the **antilogarithm** of x; that is, if

$$\log N = x,$$

then N is the antilogarithm of x.

The processes involved in finding the logarithms and antilogarithms of numbers are explained in the Introduction to the Tables, pages 427–428.

Exercises

Find the logarithms of the following numbers.

1. 57382.	**2.** 21.469.	**3.** 0.0017365.
4. 913.74.	**5.** 8324.3.	**6.** 461290.
7. 726.94.	**8.** 9.1285.	**9.** 0.34627.
10. 62.571.	**11.** 152.45.	**12.** 0.084353.

Find the antilogarithms of the following numbers.

13. 0.36491.	**14.** 2.78763.	**15.** $9.52436 - 10$.
16. 1.56273.	**17.** 4.68432.	**18.** $7.21672 - 10$.
19. 1.45596.	**20.** 0.41332.	**21.** $8.66397 - 10$.
22. 1.97258.	**23.** $5.67430 - 10$.	**24.** $6.07052 - 10$.

82. Computation by Means of Logarithms. Computation by means of logarithms is based on the four theorems stated in Art. 75. If, for example, we wish to find the product of two or more numbers, it follows from Theorem I that we should add the logarithms of the numbers and find the antilogarithm of the sum. *A form for the entire computation should always be written out in full before any logarithms are looked for.*

Example 1. Find, to five significant figures: $N = \dfrac{7.1863 \times 27.374}{0.04584 \times 6491.7}$.

Denote the numerator of this fraction by A and the denominator by B.

By Theorem I of Art. 75,

$$\log A = \log 7.1863 + \log 27.374 \quad \text{and} \quad \log B = \log 0.04584 + \log 6491.7.$$

Since $N = A/B$, it follows from Theorem II that

$$\log N = \log A - \log B = (\log 7.1863 + \log 27.374) - (\log 0.04584 + \log 6491.7).$$

Form for the computation

$$\begin{array}{rll}
\log 7.1863 = & & \\
\log 27.374 = & \underline{\qquad\qquad} & + \\
\log \text{numerator} = & & \\
& & \\
\log 0.04584 = & & \\
\log 6491.7 = & \underline{\qquad\qquad} & + \\
\log \text{denominator} = & \underline{\qquad\qquad} & - \\
\log N = & & \\
N = & &
\end{array}$$

Completed computation

$$\begin{array}{rll}
\log 7.1863 = & 0.85651 & \\
\log 27.374 = & \underline{1.43733} & + \\
\log \text{numerator} = & 12.29384 & -\ 10 \\
& & \\
\log 0.04584 = & 8.66124 & -\ 10 \\
\log 6491.7 = & \underline{3.81236} & + \\
\log \text{denominator} = & \underline{2.47360} & - \\
\log N = & 9.82024 & -\ 10 \\
N = & 0.66106 &
\end{array}$$

In this example, the logarithm of the denominator is larger than the logarithm of the numerator. To obtain the decimal part of their difference as a *positive* number, we have written the logarithm of the numerator, before making the subtraction, in the form $12.29384 - 10$.

EXAMPLE 2. Find: $\sqrt[7]{0.0867538}$.

First round the given number off to five significant figures, 0.086754.

By Theorem IV, Art. 75, $\log \sqrt[7]{0.086754} = \frac{1}{7} \log 0.086754$.

Form for the computation

$\log 0.086754 =$	$\frac{1}{7} \log =$	$N =$

Completed computation

$\log 0.086754 = 68.93829 - 70$	$\frac{1}{7} \log = 9.84833 - 10$	$N = 0.70523.$

One would ordinarily write $\log 0.086754 = 8.93829 - 10$. In this case, since we are to divide by 7, and since the negative part of the quotient should be -10, we add and subtract 60 and write the logarithm in the form $68.93829 - 70$. *This device should always be used when it is required to find the logarithm of a root of a number less than one.*

EXAMPLE 3. Find $N = \sqrt[3]{\dfrac{(46.358)^2 \cdot \sqrt[4]{197.82}}{(2698.6)^{\frac{5}{3}} \cdot \sqrt{0.81496}}}$.

Since logarithms of powers and roots of the given numbers are to be found, we arrange the work in the following way.

$$
\begin{array}{lrr}
\log 46.358 = 1.66612 & 2 \log = & 3.33224 \\
\log 197.82 = 2.29627 & \frac{1}{4} \log = & \underline{0.57407} \quad + \\
 & \text{log numerator} = & 13.90631 - 10 \\
\log 2698.6 = 3.43114 & \frac{5}{3} \log = & 5.71857 \\
\log 0.81496 = 19.91114 - 20 & \frac{1}{2} \log = & \underline{9.95557 - 10} + \\
 & \text{log denominator} = & \underline{5.67414} \quad - \\
 & \log N^3 = & 3\overline{)28.23217 - 30} \\
 & \log N = & 9.41072 - 10 \\
 & N = & 0.25746.
\end{array}
$$

Exercises

Find the values of the following expressions to five significant figures, using Table I. Write out a form for each exercise before looking up the logarithms.

1. 285.73×0.091362.

2. 486370×1.9675.

3. 6813.2×416.93.

4. 0.031865×0.57841.

5. $8375.2 \div 21.586$.

6. $0.19354 \div 43.769$.

7. $29.424 \div 79.527$.

8. $4.9532 \div 0.31586$.

9. $93.872 \times 4.1645 \times 14.838$.

10. $2.8473 \times 47.239 \times 8.5943$.

11. $432.76 \times 5.7938 \div 92.359$.

12. $86.931 \times 3.4765 \div 1937.4$.

13. $\dfrac{157.36 \times 53.892}{942.63 \times 764.83}$.

14. $\dfrac{573.18 \times 3.2967 \times 884.35}{29.521 \times 632.47 \times 2.1843}$.

15. $(51.486)^3$.

16. $(1.8741)^7$.

17. $\sqrt{9564.3}$.

18. $\sqrt[6]{0.000021639}$.

19. $(9.1574)^4 \cdot \sqrt[3]{0.71639}$.

20. $(6.1853)^3 \cdot \sqrt{21.486}$.

21. $\dfrac{\sqrt[3]{-183.72} \cdot \sqrt{86.493}}{51.586\,(-2.3769)^2}$.

22. $\dfrac{\sqrt{438.55}\,(-2.5386)^3}{-72.135\,(4.8357)^2}$.

HINT. Compute as if all the numbers were positive; then determine the sign of the result by inspection. As a reminder to consider the signs, put a letter (n) in parentheses after each logarithm of a negative number.

23. $\sqrt[3]{\dfrac{(-413.86)^{\frac{1}{3}} \cdot (-82.748)^{\frac{1}{5}}}{\sqrt{24.689} \cdot (-3.2965)^{\frac{2}{3}}}}$.

24. $\left[\dfrac{(-46.834)^{\frac{3}{5}} \cdot (18.647)^{\frac{1}{4}}}{(-4.9321)^{\frac{1}{3}} \cdot (-216.43)^{\frac{1}{5}}}\right]^{\frac{5}{3}}$

25. $6.3731 \log 4.9478$.

26. $\log 7482.8 \div 12.593$.

HINT. Find the value of log 4.9478. The number so found is to be multiplied by 6.3731. The final multiplication may be performed by logarithms.

27. $\log 767.84 \cdot \log 294.57$.

28. $\log 186.34 \cdot \log 0.51927$.

29. $\sqrt{39.576} + (3.852)^2$.

30. $(5.8162)^2 - \sqrt[3]{6832.5}$.

HINT. Find the value of each term and add (or subtract) the results.

Find N, given:

31. $N = (2.5416)^{5.39}$.

32. $N = (7.3485)^{2.4595}$.

HINT. If $N = (2.5416)^{5.39}$, then $\log N = 5.39 \log 2.5416 = 5.39 \times 0.40511$. Find the last result by logarithms and equate the result to $\log N$. The value of N can then be found from Table I.

33. $N = (4.1965)^{5.361}$.

34. $N = (425.54)^{2.1843}$.

35. If $s = \frac{1}{2}gt^2$, find s when $g = 32$ and $t = 13.472$.

36. If $V = \frac{4}{3}\pi r^3$, find V when $\pi = 3.1416$ and $r = 7.3845$.

37. If $w = \dfrac{kbd^2}{l}$, find w when $k = 2.3674$, $b = 2.6431$, $d = 4.1659$, and $l = 58.329$.

38. If $T = 2\pi\sqrt{\dfrac{I}{mgh}}$, find T when $\pi = 3.1416$, $I = 25{,}376$, $m = 431.2$, $g = 980$, and $h = 5.1627$.

39. If $H = ki^2Rt$, find H when $k = 7.1652$, $i = 6.3485$, $R = 17.658$, and $t = 16$.

40. If $S = \dfrac{kmgl}{\pi r^2}$, find S when $k = 7.8347 \times 10^{-11}$, $m = 486.47$, $g = 980$, $l = 537$, $\pi = 3.1416$, and $r = 0.2$.

83. Cologarithms. The **cologarithm** of a number is the logarithm of the reciprocal of the number. We denote the cologarithm of N by the symbol colog N. We have

$$\text{colog } N = \log 1/N = \log 1 - \log N = -\log N.$$

Thus,
$$\text{colog } 58.732 = \log 1/58.732 = -1.76887 = 8.23113 - 10$$
$$\text{colog } 0.76495 = \log 1/0.76495 = -(9.88363 - 10) = 0.11637.$$

It will be observed that *the positive part of colog N can be found by subtracting mentally the last non-zero digit of the positive part of log N from* 10 *and each of the other digits from* 9.

Cologarithms are convenient to use in computation because, if we find the cologarithms of all factors that appear as divisors, all subtractions of logarithms are replaced by the additions of the corresponding cologarithms.

EXAMPLE. Find $N = \dfrac{93.465 \times 527.82}{0.86578 \times 254.74}$.

Since $N = 93.465 \times 527.82 \times \dfrac{1}{0.86578} \times \dfrac{1}{254.74}$,

$$\log N = \log 93.465 + \log 527.82 + \operatorname{colog} 0.86578 + \operatorname{colog} 254.74.$$

The work may be arranged in the following way.

$$\begin{aligned} \log 93.465 &= 1.97065 \\ \log 527.82 &= 2.72249 \\ \operatorname{colog} 0.86578 &= 0.06259 \\ \operatorname{colog} 254.74 &= \underline{7.59390} - 10 + \\ \log N &= 2.34963 \\ N &= 223.68 \end{aligned}$$

Exercises

1–15. Do Ex. 5–8, 11–14, 21–24, 37–38, and 40, Art. 82, using cologarithms.

84. Exponential and Logarithmic Equations. An equation which contains the unknown in an exponent is an *exponential equation;* one that contains the logarithm of an unknown is a *logarithmic equation.*

EXAMPLE 1. Find x, given: $(2.4643)^{x+5} = 59362$.

Equate the logarithms of the two members.

$$\log (2.4643)^{x+5} = \log 59362,$$

or

$$(x + 5) \log 2.4643 = \log 59362.$$

Hence,

$$x + 5 = \frac{\log 59362}{\log 2.4643} = \frac{4.77351}{0.39169} = 12.187.$$

The last division may be performed by logarithms.

We have $x + 5 = 12.187$. Hence $x = 7.187$.

EXAMPLE 2. Find x, given:

$$3.21 \log (x - 2) + 8.72 \log 4.9376 = 4.281 \log 573.97.$$

On substituting the values of the logarithms of the given numbers, we have:

$$3.21 \log (x - 2) + 8.72 \times 0.69351 = 4.281 \times 2.75889,$$

or $$3.21 \log (x-2) + 6.0476 = 11.811,$$

$$\log (x-2) = \frac{5.763}{3.21} = 1.7953.$$

Hence, $$x - 2 = 62.4, \quad \text{and} \quad x = 64.4.$$

EXAMPLE 3. Find x, given: $x^{7.3641} = 52857$.

Write the equation in logarithmic form: $7.3641 \log x = \log 52857$.

Hence, $$\log x = \frac{\log 52857}{7.3641} = \frac{4.7231}{7.3641} = 0.64137.$$

Since $\log x = 0.64137$, we have $x = 4.3790$.

Exercises

Solve the following equations for x.

1. $(1.05)^x = 2$.
2. $(4.76)^x = 927.5$.
3. $(6.7394)^{x-1} = 4368.4$.
4. $3(129.48)^{x+3} = 8165.9$.
5. $\dfrac{(1.06)^x - 1}{0.06} = 127.14$.
6. $\dfrac{(1.045)^x - 1}{0.045} = 2138.2$.
7. $x^{4.38} = 517.92$.
8. $(x-2)^{13.792} = 84542$.
9. $(2x+1)^{7.1838} = 5912.6$.
10. $21.7(3x)^{2.536} = 61493$.
11. $12.537 \log x = \log 819.65$.
12. $4.3728 \log (x-4) = \log 53761$.
13. Given $y = e^{-\frac{x^2}{2}}$, find x when $y = 0.164$ and $e = 2.7183$.

85. Graphs of the Logarithmic and Exponential Functions. The graph of the logarithmic function, $f(x) = \log x$, is shown in Figure 26.

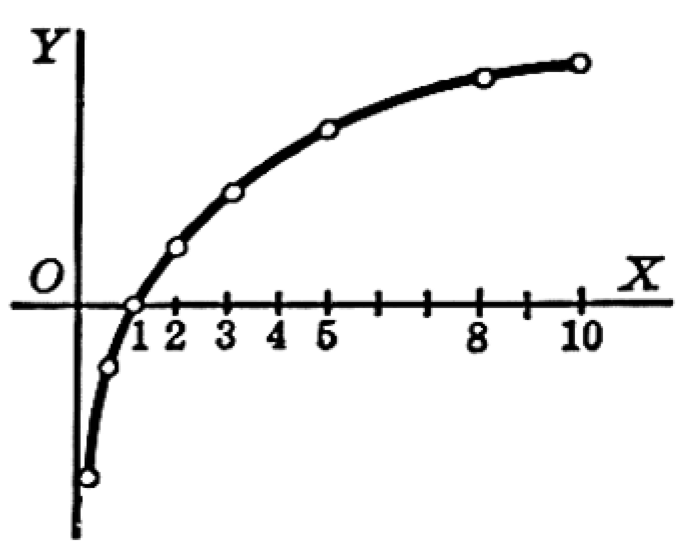


FIG. 26

The following table of coördinates of points on the curve $y = \log x$ was formed by assigning values to x and finding the corresponding values of y from Table I.

x	0.1	0.5	1	2	3	5	9	10
y	-1	-0.3	0	0.3	0.5	0.7	0.9	1

Since, from the definition of a logarithm, the equation $y = \log x$ is equivalent to $x = 10^y$, it follows that Figure 26 is also the graph of the exponential equation $x = 10^y$.

The graph of the equation

$$y = \log_a x, \quad \text{or} \quad x = a^y,$$

where a is any other base greater than unity, can be determined by equating the logarithms to the base 10 of the two sides of the equation $x = a^y$. This gives

$$\log x = y \log a, \quad \text{or} \quad y = \frac{\log x}{\log a}.$$

By assigning values to x and computing the corresponding values of y we can now determine as many points as we please on the required graph. The resulting graph will differ from the one shown in Figure 26 only in that each ordinate is divided by log a.

We find from the resulting graphs that, for all values of a greater than unity,

1. *if x is negative, y is imaginary.*
2. $\log_a 1 = 0$, *and* $\log_a a = 1$.
3. *if x lies between* 0 *and* 1, $\log_a x$ *is negative and decreases without limit as x approaches zero.*
4. *if x is greater than unity,* $\log_a x$ *is positive and increases without limit as x increases without limit.*

Exercises

1. Draw the graph of $y = 10^x$ by putting $x = \log y$ and assigning values to y. Compare the result with Figure 26.

2. Draw the graph of (*a*) $y = \log_2 x$, (*b*) $y = 2^x$.

3. Draw the graph of $y = \log_e x$, where $e = 2.7183$.

4. Draw the graph of $y = 10^{x-2}$.

5. Draw the graph of $y = 10^{\frac{x}{2}}$.

86. Logarithms to Bases Other than Ten. For numerical computation, the most convenient logarithms to use, in most cases, are logarithms to the base ten. For certain other purposes, however, it is much more convenient to use other bases.

The most frequently used base, other than 10, is the base $e = 2.71828^+$. Logarithms to this base are called **natural,** or **Napierian, logarithms.** In textbooks on calculus, and in most advanced works in mathematics, the logarithms usually used are natural logarithms.

If we have a table of logarithms to one base a, we can find the logarithm (or the antilogarithm) of a number to any other base b by means of the formula

$$\log_b N = \frac{\log_a N}{\log_a b}. \tag{4}$$

To show that this formula is true, let

$$\log_a N = x, \quad \text{and} \quad \log_b N = y.$$

By the definition of a logarithm (Art. 74), these equations are respectively equivalent to

$$N = a^x, \quad \text{and} \quad N = b^y.$$

Hence,

$$a^x = b^y,$$

because both members are equal to N. Take the logarithms to the base a of both sides of this equation. We have

$$\log_a a^x = \log_a b^y, \quad \text{or} \quad x = y \log_a b.$$

Solve for y and substitute the values $y = \log_b N$ and $x = \log_a N$.

$$y = \frac{x}{\log_a b}, \quad \text{or} \quad \log_b N = \frac{\log_a N}{\log_a b}.$$

This formula enables us, if we have available a table of logarithms to some one base a, to compute the logarithm of N to any other desired base b.

Suppose, for example, we have a table of common logarithms and wish to find the natural logarithm of a number N. We take, in equation (4), $a = 10$ and $b = e = 2.71828$. We find, from Table I, that

$$\log_{10} e = \log_{10} 2.71828 = 0.43429.$$

Hence,
$$\log_e N = \frac{\log_{10} N}{\log_{10} e} = \frac{\log_{10} N}{0.43429} = 2.3026 \log_{10} N,$$

that is,
$$\mathbf{\log_e N = 2.3026 \log_{10} N.} \tag{5}$$

Similarly, if we know the natural logarithm of N and wish to find its common logarithm, we have

$$\log_e N = \frac{\log_{10} N}{\log_{10} e}, \quad \text{or} \quad \log_{10} N = \log_{10} e \log_e N,$$

that is, since $\log_{10} e = 0.43429$,

$$\mathbf{\log_{10} N = 0.43429 \log_e N.} \tag{6}$$

EXAMPLE 1. Find $\log_e 5.2849$, using Table I.

From equation (5), we have

$$\log_e 5.2849 = 2.3026 \log_{10} 5.2849 = 2.3026 \times 0.72303 = 1.6649.$$

EXAMPLE 2. Find N, given $\log_e N = 5.4268$.

From equation (6), we have

$$\log_{10} N = 0.43429 \log_e N = 0.43429 \times 5.4268 = 2.3568.$$

Hence, from Table I, we find that $N = 227.4$.

Exercises

Find the natural logarithms of the following numbers.

1. 5. **2.** 7. **3.** 92. **4.** 872.
5. 6.83. **6.** 0.836. **7.** 0.09342. **8.** 4.1362

Find N to four significant figures, given that $\log_e N$ is:

9. 2.1643. **10.** 5.3426. **11.** 7.4688. **12.** 0.31485.
13. 6.5384. **14.** 12.738. **15.** 0.24862. **16.** 20.876.

Find the following logarithms.

17. $\log_2 91.476$. **18.** $\log_5 183.56$. **19.** $\log_7 3845.1$.
20. $\log_{1.5} 4.8372$. **21.** $\log_{3.14} 51.865$. **22.** $\log_{\sqrt{3}} 41.388$.

Chapter 11

Angles and Their Measurement

87. Directed Angles. If a half-line extending from a fixed point O, (Fig. 27) rotates around O from the position OA to OB, we shall say that it generates, by this rotation, an angle AOB having OA as its **initial side** and OB as its **terminal side.** The direction and magnitude of the rotation may be indicated by an arrow, as in the figure. If the direction of the rotation of the generating line is *counterclockwise*, the angle generated is **positive**; if *clockwise*, it is **negative.**

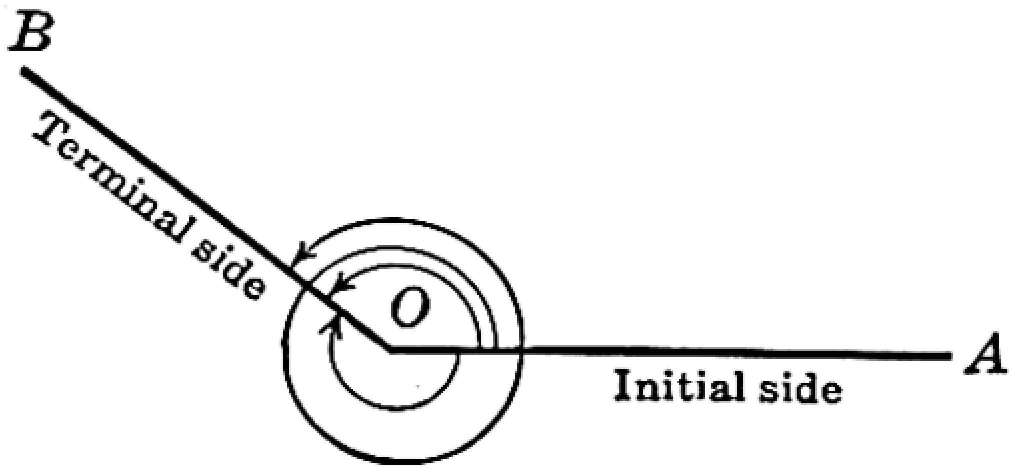


FIG. 27

There are indefinitely many angles, some positive and some negative, which have the same initial, and the same terminal, sides. Such angles are said to be **coterminal.** If, given any angle, we add to this angle, or subtract from it, any integral multiple of a complete revolution, we obtain an angle coterminal with the given one.

For example, in 15 minutes, the minute hand of a clock turns through an angle of $-90°$. In an hour and 15 minutes, it turns through an angle of $-450°$ and generates an angle coterminal with $-90°$.

88. Circular Measure. The Radian. The student is familiar with the measurement of angles in degrees, minutes, and seconds. While we shall, in what follows, use this system of measurement frequently, we shall also frequently use another system called **circular** (or **radian**) **measure.** In this system, the unit angle is called a **radian** and is defined as follows: *A radian is an angle which, if placed with its vertex at the center of a circle, will intercept on the circumference an arc equal in length to the radius of the circle.*

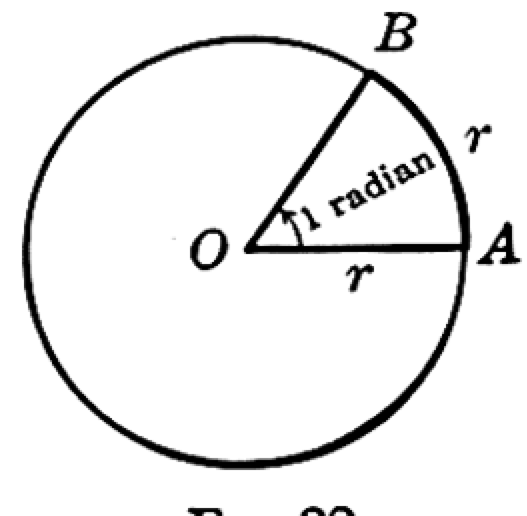


FIG. 28

Thus, in Figure 28, if the arc AB is equal to the radius OA, then, by definition, the angle AOB is one radian.

89. Degrees and Radians. To find the relation between the number of degrees and the number of radians in an angle, we observe that the circumference is 2π times the radius and that, every time we lay off the length of the radius on the circumference, the central angle subtended by it is one radian. It follows that, in a complete revolution, there are 2π radians; that is, $360° = 2\pi$ radians, or

$$\mathbf{180° = \pi \text{ radians},} \tag{1}$$

where $\pi = 3.1416$, approximately. It follows that

$$1^\circ = \frac{\pi}{180} \text{ radians} = .017453 \text{ radians, approximately,}$$

$$1 \text{ radian} = \left(\frac{180}{\pi}\right)^\circ = 57.296^\circ = 57^\circ\ 17'\ 45'', \text{ approximately.}$$

It should be observed that, customarily, when an angle is written in radians, no unit of angular measure is given. For example, an angle $\pi/4$ means an angle of $\pi/4$ radians and an angle 3 means an angle of 3 radians.

Exercises

Express the following angles in degrees.

1. $\frac{\pi}{6}$. **2.** $\frac{\pi}{4}$. **3.** $\frac{\pi}{2}$. **4.** $-\frac{2\pi}{3}$. **5.** $\frac{\pi}{12}$.

6. $\frac{3\pi}{4}$. **7.** $-\frac{13\pi}{6}$. **8.** $\frac{7\pi}{2}$. **9.** 2. **10.** 1.5.

Express the following angles in radians.

11. 60°. **12.** 150°. **13.** − 240°. **14.** 225°. **15.** 330°.

16. 450°. **17.** 630°. **18.** − 720°. **19.** 5°. **20.** 4.3°.

90. Central Angles and Their Intercepted Arcs. Let θ be the number of radians in an angle AOC at the center of a circle of radius r and let s be the length of its intercepted arc AC. Also, let AOB be an angle of one radian so that arc $AB = r$. Since, by elementary geometry, angles at the center of a circle are to each other as their intercepted arcs, we have

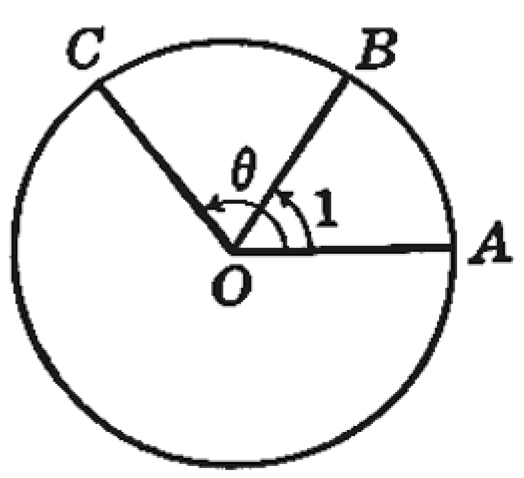


Fig. 29

$$\theta : 1 = s : r, \quad \text{or} \quad s = \theta r.$$

Hence,

$$s = \theta r; \tag{2}$$

that is, *the length of the arc intercepted by a central angle equals the number of radians in the angle times the radius.* If we are given two of the three numbers r, θ, and s, we are able, from this equation, to find the third.

In equation (2), r and s must be measured in the same units of length and θ must be measured in radians.

Example 1. An arc of 7 feet on a circle is intercepted by a central angle of 20°. Find the radius of the circle.

Since $20^\circ = 20\pi/180 = \pi/9$ radians, we have, from (2),

$$7 = \pi r/9, \quad \text{or} \quad r = 63/\pi = 20.053 \text{ feet.}$$

EXAMPLE 2. On a circle 10 feet in diameter, find in degrees the angle subtended by an arc of 35 inches.

We have $r = 60$ inches. Hence, $35 = 60\theta$, from which

$$\theta = \frac{7}{12}\text{radians, or } \theta = \left(\frac{7}{12}\,\frac{180}{\pi}\right)^\circ = \left(\frac{105}{\pi}\right)^\circ = 33.42^\circ.$$

Exercises

(In Ex. 1–18, take $\pi = 3.1416$ and compute the result to as many significant figures as are given in the statement of the exercise.)

Find s, given:

1. $r = 17$, $\theta = 3.1$.
2. $r = 4.83$, $\theta = 2.96$.
3. $r = 41.673$, $\theta = 5.2185$.
4. $r = 7.85$, $\theta = 36.2^\circ$.
5. $r = 978.32$, $\theta = 159.61^\circ$.
6. $r = 274.86$, $\theta = 321.47^\circ$.

Find r, given:

7. $s = 483$, $\theta = 2.65$.
8. $s = 483.68$, $\theta = 7.3421$.
9. $s = 0.13875$, $\theta = 1.1574$.
10. $s = 63.8$, $\theta = 27.6^\circ$.
11. $s = 59.368$, $\theta = 289.41^\circ$.
12. $s = 0.046359$, $\theta = 94.317^\circ$.

Find θ in degrees, given:

13. $r = 91.64$, $s = 27.85$.
14. $r = 71.638$, $s = 284.25$.
15. $r = 5.3684$, $s = 21.963$.
16. $r = 0.91616$, $s = 0.55765$.
17. $r = 4976.9$, $s = 3764.8$.
18. $r = 51.369$, $s = 473.57$.

19. How many radians in each angle of an equilateral triangle? in each of the equal angles of an isosceles right triangle?

20. How many radians in the angle between the hands of a clock at 2 o'clock? at 2:30 o'clock?

21. A railroad curve is laid out on an arc of a circle of radius 867 feet. Find its length, to the nearest foot, if it subtends an angle of $19^\circ 42'$ at the center of the circle.

22. An automobile tire is 32 inches in diameter. Find (*a*) how many radians and (*b*) how many revolutions it turns through in going one mile. (Take $\pi = \frac{22}{7}$)

23. A nautical mile is an arc of a great circle on the earth that subtends at the center an angle of one minute. Assuming that the earth is a sphere of radius 3959 statute miles, find, to the nearest foot, the length in feet of a nautical mile.

24. Using the data of Ex. 23, find, to four significant figures, the length in statute miles of an arc on a great circle of the earth that subtends a central angle of one degree.

91. Linear and Angular Velocities. If a point P moves at a uniform rate, the distance it moves in one unit of time is its **linear velocity** which

we shall denote by v. If a body, moving uniformly, goes a distance s in t units of time, then its linear velocity is

$$v = \frac{s}{t}. \tag{3}$$

Similarly, if a half-line rotates uniformly about its end point O, then the angle through which it rotates in one unit of time is its **angular velocity** which we shall denote by ω. If the half-line rotates through an angle θ in t units of time, its angular velocity is

$$\omega = \frac{\theta}{t}. \tag{4}$$

Suppose a point P moves uniformly along a circle of radius r and traverses an arc of length s in t units of time. Further, let θ be the number of radians in the angle which the half-line from the center through P turns in the time t. By equation (2), $s = \theta r$, hence $s/t = r\theta/t$ or, from equations (3) and (4),

$$v = r\omega. \tag{5}$$

In this equation, v is the linear velocity of P. The angular velocity ω of the half-line is also spoken of as the angular velocity of P.

Example. A flywheel 16 inches in diameter makes 13 revolutions per second. Find the linear velocity of a point on the rim in feet per minute.

The angular velocity of a point on the rim is $2\pi 13$ radians per second, or

$$\omega = 2\pi 13 \times 60 = 1560\pi \text{ radians per minute.}$$

The radius of the wheel is 8 inches, or $\frac{2}{3}$ feet. Hence, by (5),

$$v = 1560\pi\tfrac{2}{3} = 3267 \text{ feet per minute.}$$

Exercises

(In these exercises, find your answers to three significant figures.)

1. Find the angular velocity, in radians per second, of a point on the rim of a flywheel making 1000 revolutions per minute.

2. The minute hand of a clock is 5 inches long. Find the linear velocity, in inches per minute, of a point on the end of the hand.

3. It is required to construct a pulley that will make 15 revolutions per second when driven by a belt moving 1700 feet per minute. Find the radius of the pulley in inches.

4. An automobile tire is 31 inches in diameter. Find its angular velocity in radians per second if the car is going 40 miles an hour.

5. Taking the radius of the earth as 3959 miles, find the linear velocity, in feet per second, of a point on the equator.

6. Taking the radius of the earth's orbit as 92,900,000 miles, find the linear velocity of the earth in its orbit in miles per second.

Chapter 12

The Trigonometric Functions

92. Angles in Standard Position. An angle is in **standard position** with reference to a set of coördinate axes if its vertex is at the origin and its initial side (Art. 87) extends in the positive direction along the x-axis (Fig. 30).

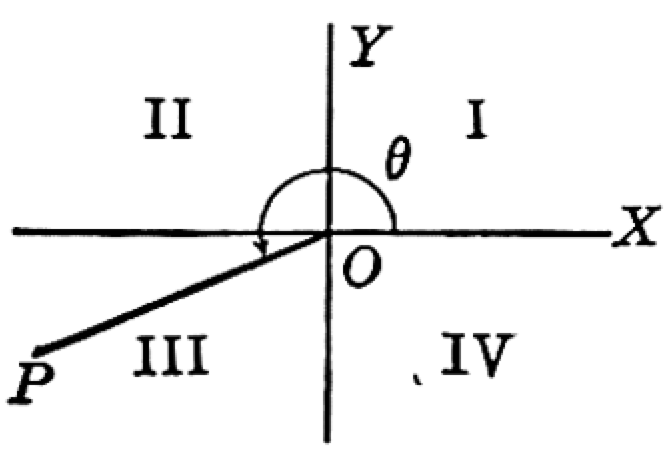


FIG. 30

We saw (Art. 38) that the coördinate axes divide the plane into four quadrants which are numbered as in Figure 30. We say that an angle, in standard position, lies in whatever quadrant its terminal side lies in. Thus, in Figure 30, the angle θ lies in the third quadrant because its terminal side OP lies in that quadrant.

Exercises

Draw each of the following angles in standard position with reference to a set of coördinate axes. Indicate by an arrow the amount and direction of rotation and state the quadrant in which the angle lies. Find two other angles, one positive and one negative, coterminal with the given angle.

1. $45°$.	**2.** $120°$.	**3.** $210°$.	**4.** $315°$.
5. $-30°$.	**6.** $-60°$.	**7.** $390°$.	**8.** $-750°$.
9. $\frac{\pi}{3}$.	**10.** $\frac{7\pi}{6}$.	**11.** $-\frac{5\pi}{8}$.	**12.** $-\frac{8\pi}{3}$.
13. $\frac{\pi}{4}$.	**14.** $\frac{11\pi}{6}$.	**15.** $-\frac{5\pi}{4}$.	**16.** $\frac{7\pi}{8}$.

17. Choose four points, one in each of the four quadrants. State the signs of the coördinates of each of these points.

93. Definitions of the Trigonometric Functions. Associated with a given angle θ, there are six quantities which are the values, for the angle θ, of the six trigonometric functions of θ. In this article, we shall set up the definitions of the values of these functions.

Place the given angle θ in standard position on a set of axes and choose any convenient point P on the terminal side of θ (Fig. 31). Let x and y be the coördinates of P and denote the length of the segment OP by r. We shall call this length, r, the **radius vector** of P and we shall assume, whenever we are dealing with the trigonometric functions, that r is *positive*.

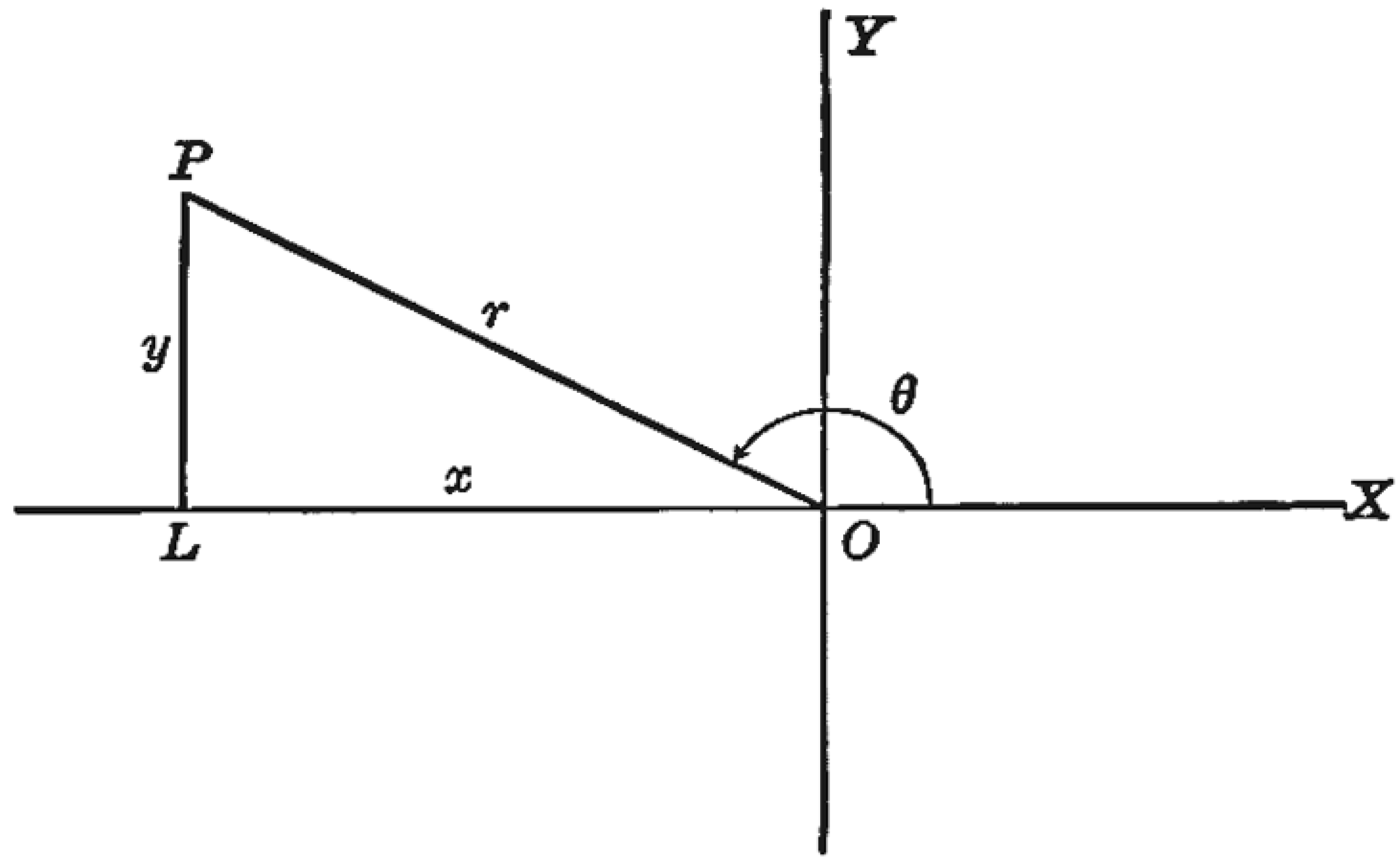


FIG. 31

Using the three numbers, x, y, and r, we can form six ratios. To each of these ratios, we attach a name, as follows:

$$\sin\theta = \frac{\text{ordinate}}{\text{radius vector}} = \frac{y}{r}, \qquad \text{(Read, "sine of } \theta\text{")}$$

$$\cos\theta = \frac{\text{abscissa}}{\text{radius vector}} = \frac{x}{r}, \qquad \text{(Read, "cosine of } \theta\text{")}$$

$$\tan\theta = \frac{\text{ordinate}}{\text{abscissa}} = \frac{y}{x}, \qquad \text{(Read, "tangent of } \theta\text{")}$$

$$\cot\theta = \frac{\text{abscissa}}{\text{ordinate}} = \frac{x}{y}, \qquad \text{(Read, "cotangent of } \theta\text{")}$$

$$\sec\theta = \frac{\text{radius vector}}{\text{abscissa}} = \frac{r}{x}, \qquad \text{(Read, "secant of } \theta\text{")}$$

$$\csc\theta = \frac{\text{radius vector}}{\text{ordinate}} = \frac{r}{y}. \qquad \text{(Read, "cosecant of } \theta\text{")}$$

These definitions will be used so frequently, from now on, that they should be memorized.

94. Some Properties of the Trigonometric Functions.

A. Each of these six trigonometric functions is the reciprocal of another one. We have, in fact,

$$\csc\theta = \frac{1}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}, \qquad \cot\theta = \frac{1}{\tan\theta}.$$

For, by definition,

$$\csc\theta = \frac{r}{y} = 1\Big/\frac{y}{r} = \frac{1}{\sin\theta},$$

and similarly for the other functions.

B. Since r is always positive, the signs of the trigonometric functions depend on the signs of x and y. The abscissa, x, is positive in the first and fourth quadrants and negative in the second and third; the ordinate, y, is positive in the first and second quadrants and negative in the third and fourth. It follows that:

In the first quadrant, all the functions are positive.
In the second quadrant, sin θ and csc θ are positive and the rest negative.
In the third quadrant, tan θ and cot θ are positive and the rest negative.
In the fourth quadrant, cos θ and sec θ are positive and the rest negative.

C. *The values of the functions are independent of the position of the point $P(x, y)$ on the terminal side of θ.* For if $P'(x', y')$ is a second point on the terminal side of θ (Fig. 32), then, since the triangles OLP and $OL'P'$ are similar,

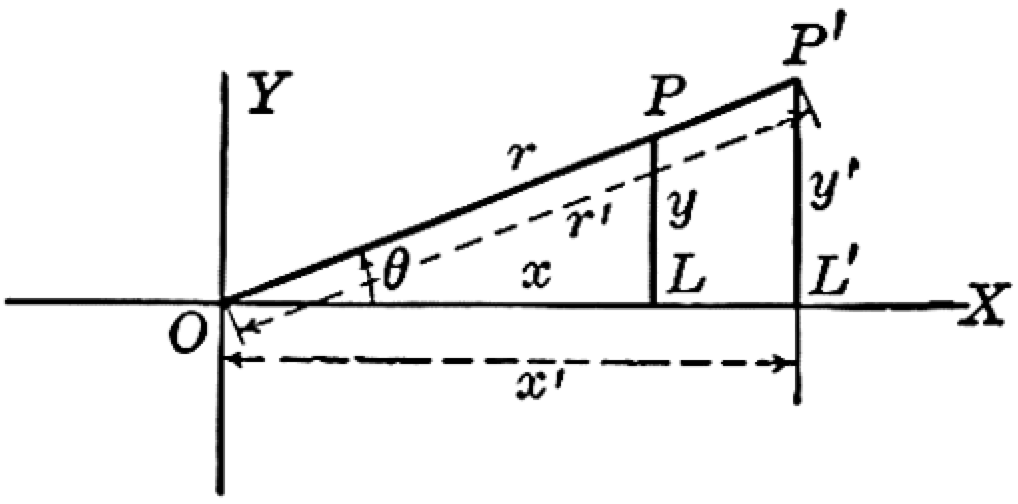


FIG. 32

$$\frac{x'}{x} = \frac{y'}{y} = \frac{r'}{r}.$$

It follows that $\dfrac{y'}{r'} = \dfrac{y}{r} = \sin\theta$, and similarly for the other functions.

Exercises

Using a protractor and coördinate paper, construct the given angle and find the value of each of its functions to two significant figures.

1. 17°. **2.** 53°. **3.** 72°. **4.** 114°.
5. 203°. **6.** 329°. **7.** − 34°. **8.** − 110°.
9. $\frac{\pi}{3}$. **10.** $\frac{3\pi}{4}$. **11.** $\frac{7\pi}{6}$. **12.** $\frac{19\pi}{10}$.

95. Determination of All the Functions from One of Them. Let there be given the value of one of the trigonometric functions of θ and, also, the quadrant in which θ lies. It is required to construct an angle θ satisfying these conditions and to find the values of its other five functions. The process is illustrated by the following examples.

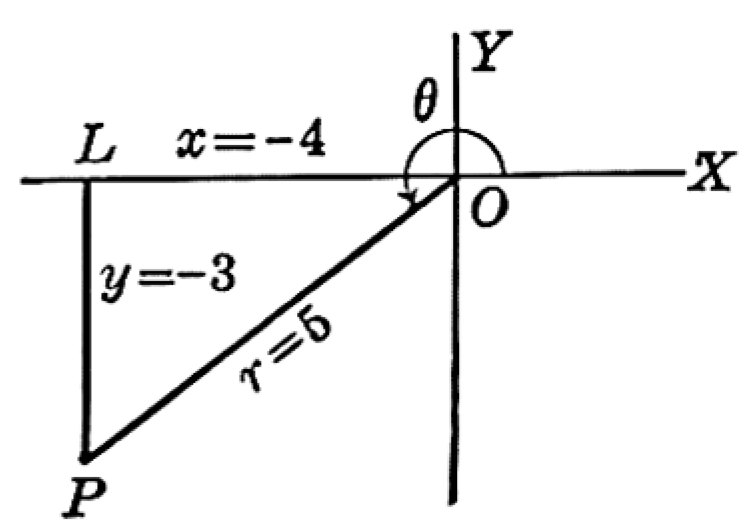


FIG. 33

EXAMPLE 1. Construct an angle θ in the third quadrant such that $\tan\theta = \frac{3}{4}$ and write the values of all of its functions.

Since, by definition, $\tan\theta = y/x$, we seek two numbers, x and y, both negative since the terminal side lies in the third quadrant such that $\frac{y}{x} = \frac{3}{4}$. The numbers $x = -4$, $y = -3$, constitute one such pair.

Plot the point $P(-4, -3)$ and draw OP. Then the angle θ (Fig. 33) is an angle in the third quadrant such that $\tan\theta = \frac{3}{4}$.

Further, by the Pythagorean theorem,

$$OP = r = \sqrt{(-4)^2 + (-3)^2} = 5.$$

If we substitute the values $x = -4$, $y = -3$, and $r = 5$ in the definitions of Art. 93, we get

$$\sin\theta = \frac{y}{r} = -\frac{3}{5}, \qquad \cos\theta = \frac{x}{r} = -\frac{4}{5}, \qquad \tan\theta = \frac{y}{x} = \frac{3}{4},$$

$$\csc\theta = \frac{r}{y} = -\frac{5}{3}, \qquad \sec\theta = \frac{r}{x} = -\frac{5}{4}, \qquad \cot\theta = \frac{x}{y} = \frac{4}{3}.$$

EXAMPLE 2. Construct an angle θ in the second quadrant such that $\sin\theta = \frac{3}{7}$ and write the values of all of its functions.

Since $\sin\theta = y/r = \frac{3}{7}$, we may take $y = 3$ and $r = 7$. Since $P(x, y)$ is in the second quadrant, x is negative and $x = -\sqrt{7^2 - 3^2} = -\sqrt{40} = -2\sqrt{10}$.

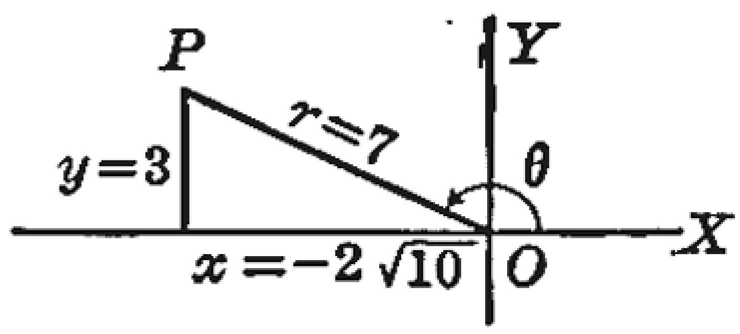


FIG. 34

Plot the point $P(-\sqrt{40}, 3)$ and draw OP. The angle $\theta = XOP$ (Fig. 34) is an angle in the second quadrant such that $\sin\theta = 3/7$. Put $x = -\sqrt{40} = -2\sqrt{10}$, $y = 3$, and $r = 7$ in the definitions of the functions of θ. We have

$$\sin\theta = \frac{3}{7}, \qquad \cos\theta = \frac{-2\sqrt{10}}{7}, \qquad \tan\theta = \frac{3}{-2\sqrt{10}} = -\frac{3\sqrt{10}}{20},$$

$$\csc\theta = \frac{7}{3}, \qquad \sec\theta = -\frac{7\sqrt{10}}{20}, \qquad \cot\theta = -\frac{2\sqrt{10}}{3}.$$

Exercises

Construct an angle θ satisfying the given conditions and write the values of all of its trigonometric functions.

1. $\tan\theta = \frac{5}{12}$, first quadrant.
2. $\cot\theta = -\frac{8}{15}$, second quadrant.
3. $\cos\theta = \frac{7}{25}$, fourth quadrant.
4. $\sin\theta = -\frac{9}{41}$, third quadrant.
5. $\csc\theta = -\frac{5}{2}$, third quadrant.
6. $\sec\theta = \frac{9}{7}$, first quadrant.
7. $\cot\theta = -2$, fourth quadrant.
8. $\csc\theta = 3$, second quadrant.
9. $\sin\theta = 0.2$, $\tan\theta$ negative.
10. $\tan\theta = -0.3$, $\sec\theta$ positive.

96. Values of the Functions of 45°, 135°, 225°, and 315°. There are quite a number of angles for which the values of the trigonometric functions can readily be determined by elementary geometry. In this and the following three articles, we shall consider the most important of these angles and we shall find the values of their functions. Since we shall assume, in the following chapters, that the student is able to write down the values of the functions of these angles, he should remember the figure, and the values of x, y, and r, for each of the angles discussed in these articles.

A diagonal of a square makes an angle of 45° with each of the sides of the square. Further, if the sides of the square are of length 1, the diagonal is of length $\sqrt{2}$.

Draw successively, in each quadrant, a square of side 1 with two of its sides extending along the coördinate axes (Fig. 35). Draw, also, OP, the diagonal of the square, through O.

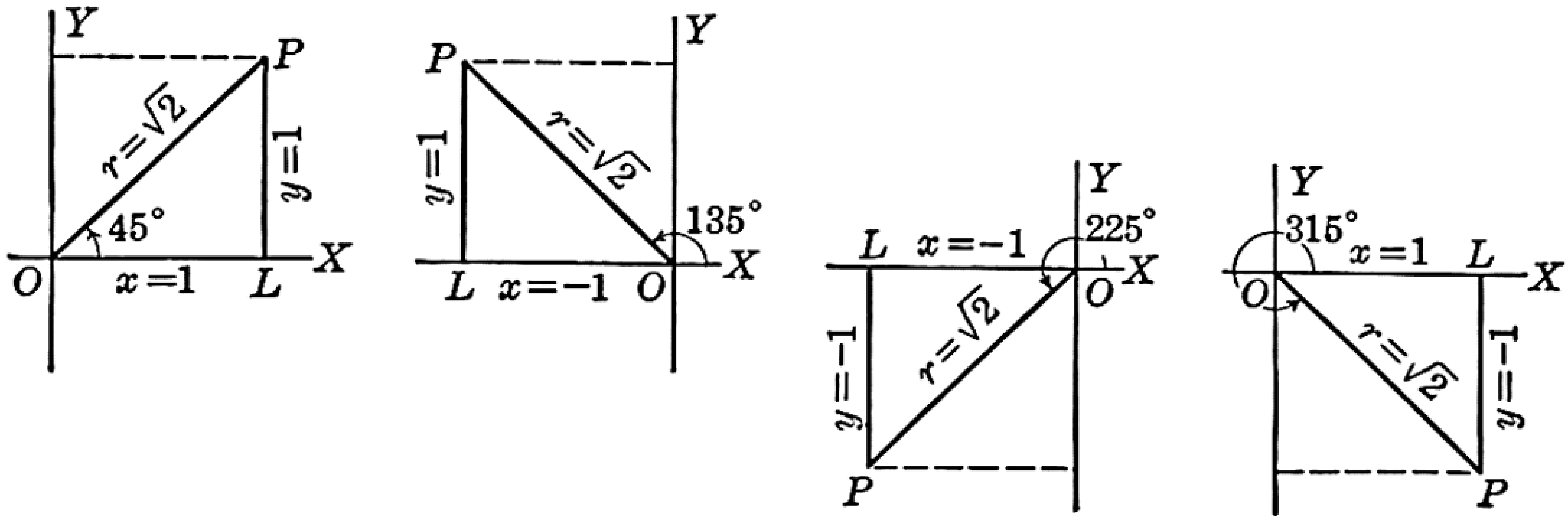


FIG. 35*a* FIG. 35*b* FIG. 35*c* FIG. 35*d*

Take OX as the initial side and OP as the terminal side of the angle under consideration and let P be the point whose coördinates and radius vector are used in defining the values of the functions of this angle.

In Fig. 35*a*, $\theta = 45°$, $x = 1$, $y = 1$, $r = \sqrt{2}$;
In Fig. 35*b*, $\theta = 180° - 45° = 135°$, $x = -1$, $y = 1$, $r = \sqrt{2}$;
In Fig. 35*c*, $\theta = 180° + 45° = 225°$, $x = -1$, $y = -1$, $r = \sqrt{2}$;
In Fig. 35*d*, $\theta = 360° - 45° = 315°$, $x = 1$, $y = -1$, $r = \sqrt{2}$.

If we substitute these values of the angle, and the corresponding values of x, y, and r, in the definitions of the trigonometric functions, we obtain the following table.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
45°	$\frac{\sqrt{2}}{2}$*	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
135°	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
225°	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
315°	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$

97. Values of the Functions of 60°, 120°, 240°, and 300°. The angles of an equilateral triangle OPQ (Fig. 36) are 60°. Further, the altitude PL bisects OQ and also bisects the angle at P. It follows that, if the

* Notice that $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

sides of the triangle are of length 2, then OL is of length 1 and LP, by the Pythagorean theorem, is of length $\sqrt{3}$.

Draw successively, in each quadrant, an equilateral triangle OPQ of side 2, with one vertex at O and one side OQ extending along the X-axis. Draw, also, PL perpendicular to OQ. Let OX be the initial side and OP the terminal side of the required angle and let P be the point used to find the values of the functions.

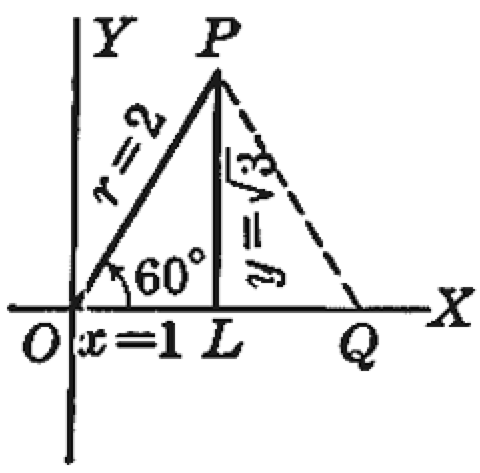


FIG. 36a

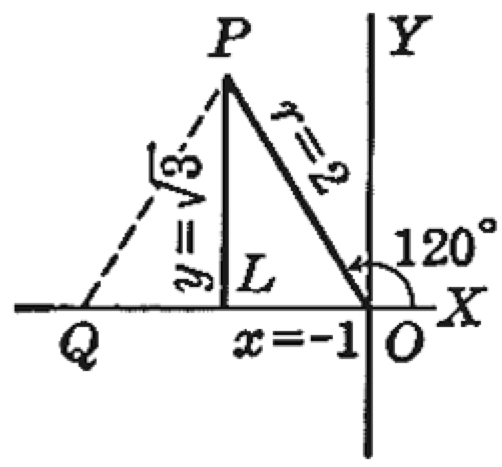


FIG. 36b

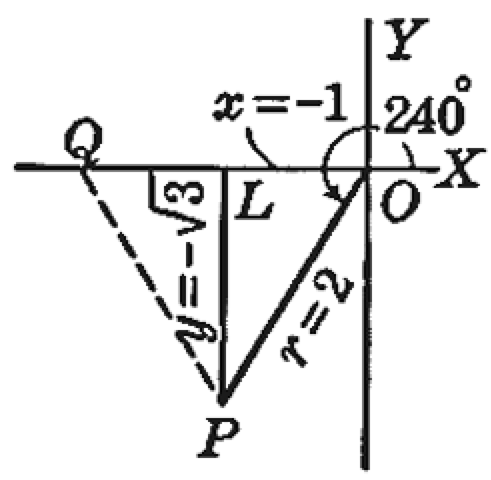


FIG. 36c

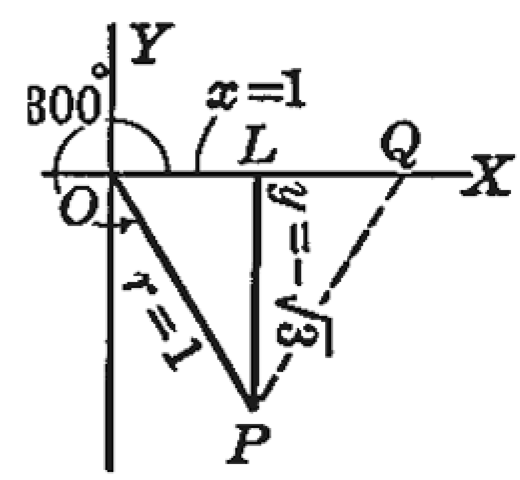


FIG. 36d

In Fig. 36a, $\theta = 60°$, $x = 1$, $y = \sqrt{3}$, $r = 2$;
In Fig. 36b, $\theta = 120°$, $x = -1$, $y = \sqrt{3}$, $r = 2$;
In Fig. 36c, $\theta = 240°$, $x = -1$, $y = -\sqrt{3}$, $r = 2$;
In Fig. 36d, $\theta = 300°$, $x = 1$, $y = -\sqrt{3}$, $r = 2$.

On substituting these values of θ, x, y, and r in the definitions, we have:

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$
120°	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{3}$	-2	$\frac{2\sqrt{3}}{3}$
240°	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	-2	$-\frac{2\sqrt{3}}{3}$
300°	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{3}$	2	$-\frac{2\sqrt{3}}{3}$

98. Values of the Functions of 30°, 150°, 210°, and 330°. To find the values of the functions of these angles, we again use an equilateral triangle of side 2 with one vertex at O but with the altitude OL of the triangle extending along the x-axis (Fig. 37). Since the perpendicular OL bisects the opposite side and the angle of the triangle at O, we have:

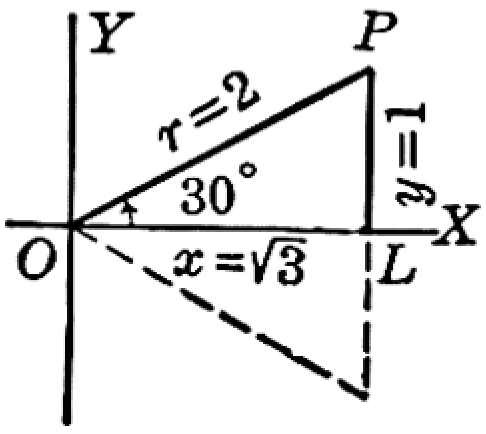


FIG. 37a

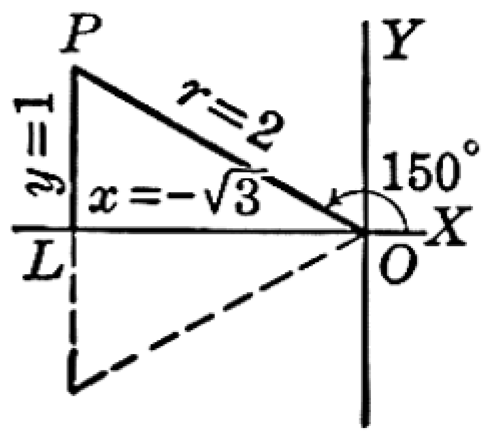


FIG. 37b

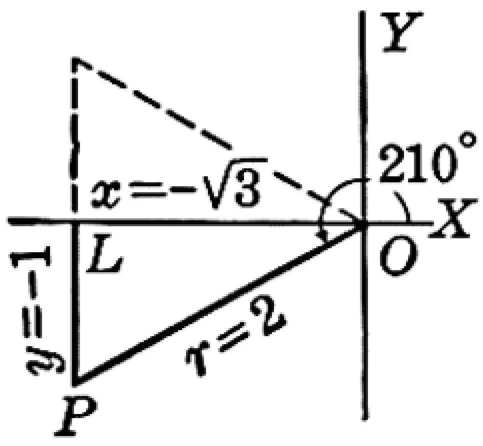


FIG. 37c

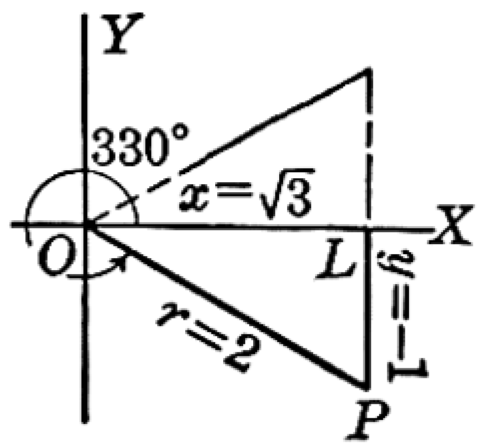


FIG. 37d

In Fig. 37a, $\theta = 30°$, $x = \sqrt{3}$, $y = 1$, $r = 2$;
In Fig. 37b, $\theta = 150°$, $x = -\sqrt{3}$, $y = 1$, $r = 2$;
In Fig. 37c, $\theta = 210°$, $x = -\sqrt{3}$, $y = -1$, $r = 2$;
In Fig. 37d, $\theta = 330°$, $x = \sqrt{3}$, $y = -1$, $r = 2$.

On inserting these values in the definitions, we have:

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
150°	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	$-\sqrt{3}$	$-\frac{2\sqrt{3}}{3}$	2
210°	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$-\frac{2\sqrt{3}}{3}$	-2
330°	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	$-\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	-2

99. Values of the Functions of 0°, 90°, 180°, and 270°. For each of these angles, the terminal side extends along one of the coördinate axes so that either the x or the y coördinate of P is zero (Fig. 38). If we take, along the terminal line, $OP = r = 1$, we have:

In Fig. 38a, $\theta = 0°$, $x = 1$, $y = 0$, $r = 1$;
In Fig. 38b, $\theta = 90°$, $x = 0$, $y = 1$, $r = 1$;
In Fig. 38c, $\theta = 180°$, $x = -1$, $y = 0$, $r = 1$;
In Fig. 38d, $\theta = 270°$, $x = 0$, $y = -1$, $r = 1$.

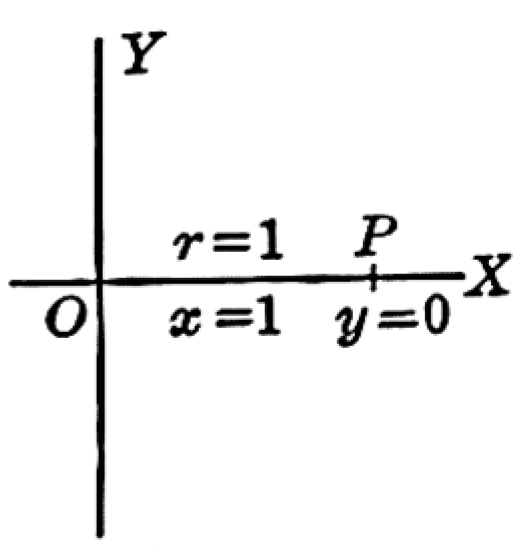


FIG. 38a

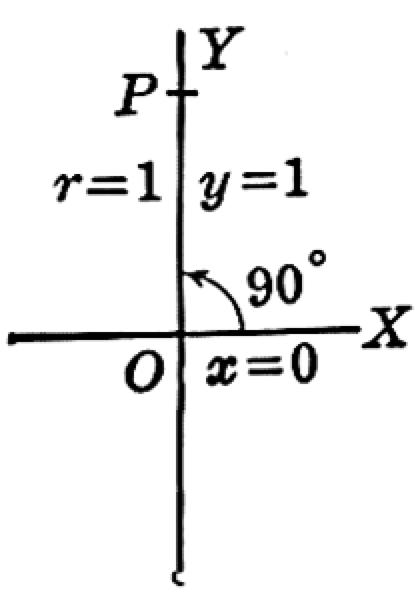


FIG. 38b

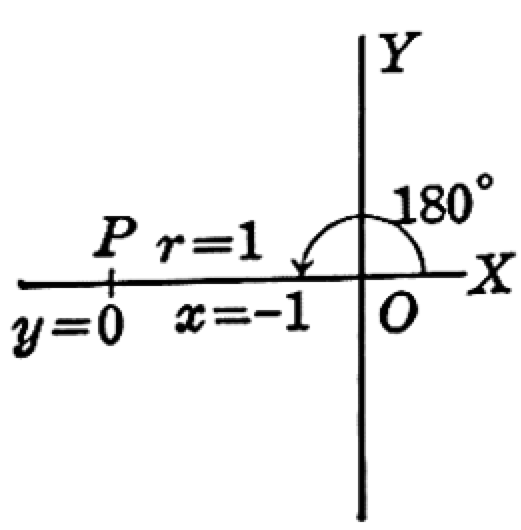


FIG. 38c

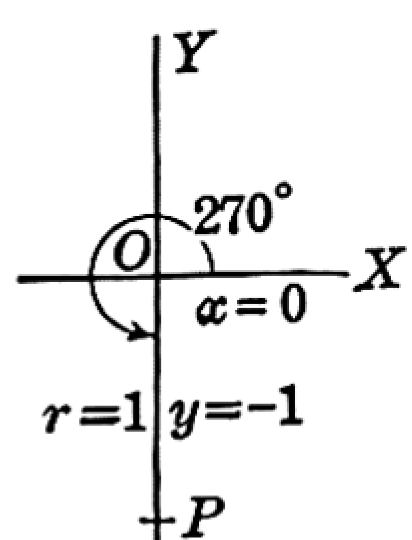


FIG. 38d

When we substitute these values in the definitions, some of the denominators will be zero. The values of the corresponding functions do not exist * since division by zero is excluded from our computations (Art. 4). The values of those functions of these angles that do exist are given in the following table.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
0°	0	1	0	——	1	——
90°	1	0	——	0	——	1
180°	0	− 1	0	——	− 1	——
270°	− 1	0	——	0	——	− 1

100. Tables of the Values of the Trigonometric Functions. For most angles, the determination of the values of the functions by elementary methods is impracticable and recourse is had to a table of the values of the functions. Table III, pages 498–520, gives to four decimal places the values of the sine, cosine, tangent, and cotangent of the angle for every minute from 0° to 90°. The method of using this table is explained in the Introduction to the Tables, pages 430–31.

Exercises

Find the required number, using Table III.

1. sin 41° 16′. **2.** sin 83° 11′. **3.** sin 16° 32′.
4. tan 9° 42′. **5.** tan 58° 15′. **6.** tan 64° 7′.
7. cos 52° 34′. **8.** cos 31° 56′. **9.** cos 17° 44′.
10. cot 13° 29′. **11.** cot 48° 23′. **12.** cot 61° 37′.

Find θ to the nearest minute, using Table III.

13. $\sin\theta = 0.8052$. **14.** $\sin\theta = 0.4572$. **15.** $\sin\theta = 0.6778$.
16. $\tan\theta = 0.4758$. **17.** $\tan\theta = 3.6269$. **18.** $\tan\theta = 2.7339$.
19. $\cos\theta = 0.9742$. **20.** $\cos\theta = 0.3647$. **21.** $\cos\theta = 0.1121$.
22. $\cot\theta = 1.6528$. **23.** $\cot\theta = 0.8020$. **24.** $\cot\theta = 0.5122$.

101. Logarithms of the Trigonometric Functions. In computations involving logarithms, instead of looking up the value of the function in Table III, then finding the logarithm of the resulting number in Table I, the required logarithm may be found directly from Table II. This table is arranged like Table III but the numbers given in the table are all 10 larger than the required logarithms. Tables of proportional parts are given to facilitate interpolation to tenths of a minute. The process of

* Not infrequently, one hears the statement that, for example, cot 0° is "infinity." This statement has significance if it is interpreted to mean that, if θ is nearly, but not exactly, zero then cot θ is numerically very large.

using the table is explained in the Introduction to the Tables, pages 428–430.

Exercises

Find the following logarithms, using Table II.

1. log sin 24° 51.2′.

2. log sin 74° 18.4′.

3. log sin 41° 53.8′.

4. log tan 23° 37.1′.

5. log tan 36° 49.3′.

6. log tan 62° 14.4′.

7. log cos 55° 21.4′.

8. log cos 79° 3.8′.

9. log cos 35° 18.4′.

10. log cot 38° 51.2′.

11. log cot 65° 21.7′.

12. log cot 84° 56.5′.

Find θ to the nearest tenth of a minute, using Table II.

13. $\log \sin \theta = 9.44491 - 10$.

14. $\log \sin \theta = 9.96821 - 10$.

15. $\log \sin \theta = 9.98434 - 10$.

16. $\log \tan \theta = 0.32420$.

17. $\log \tan \theta = 8.77121 - 10$.

18. $\log \tan \theta = 9.75694 - 10$.

19. $\log \cos \theta = 9.92247 - 10$.

20. $\log \cos \theta = 9.69423 - 10$.

21. $\log \cos \theta = 8.86780 - 10$.

22. $\log \cot \theta = 9.88687 - 10$.

23. $\log \cot \theta = 1.00762$.

24. $\log \cot \theta = 0.27103$.

Chapter 13

Solution of Right Triangles

102. Notation. Throughout the following discussion of the solution of right triangles, we shall denote the hypotenuse by c, the acute angles by α and β, and the sides opposite these acute angles by a and b, respectively (Fig. 39). These five quantities are called the **parts** of the right triangle.

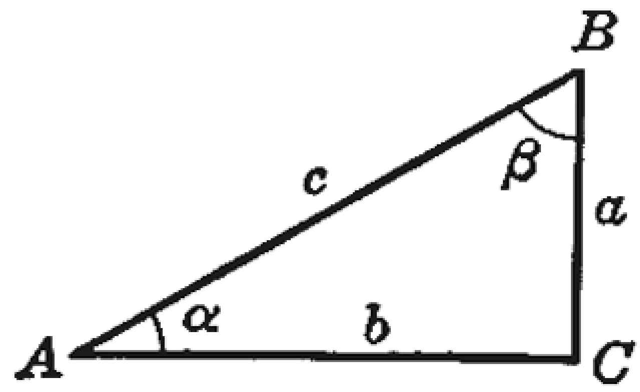


FIG. 39

If of these five quantities we know the values of any two (other than the two angles) we shall be able to find the values of the other three. The process of finding these three quantities when two are given, is called *solving the right triangle.*

103. Formulas. To obtain formulas for solving the right triangle, we first find values for the trigonometric functions of α. To obtain these from the definitions (Art. 93), we place the triangle in the first quadrant with α in standard position (Fig. 40). Then the coördinates of B are (b, a) and its radius vector is c. We have:

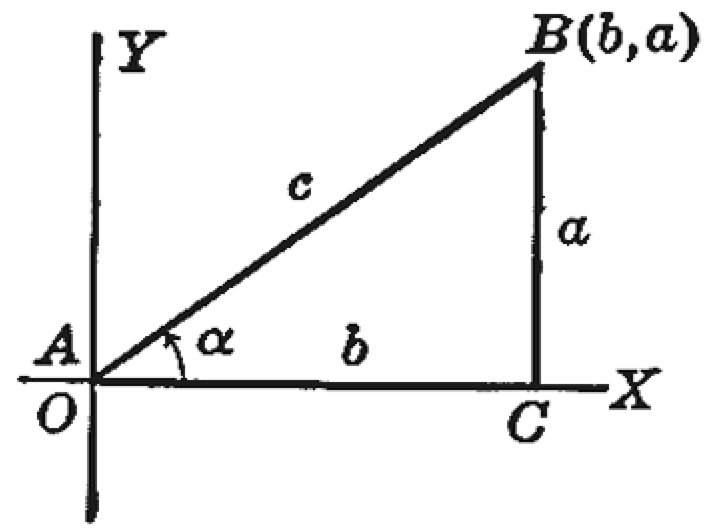


FIG. 40

$$\sin \alpha = \frac{a}{c}, \qquad \cos \alpha = \frac{b}{c}, \qquad \tan \alpha = \frac{a}{b}, \qquad \cot \alpha = \frac{b}{a}. \quad (1)$$

The formulas for sec α and csc α are omitted here because the values of these functions are not given in Tables II and III.

For the values of the functions of β, we find, in a similar way,

$$\sin \beta = \frac{b}{c}, \qquad \cos \beta = \frac{a}{c}, \qquad \tan \beta = \frac{b}{a}, \qquad \cot \beta = \frac{a}{b}. \quad (2)$$

Finally, we have, by elementary geometry,

$$\alpha + \beta = 90, \qquad a^2 + b^2 = c^2. \quad (3)$$

104. Solution by Natural Functions. To solve a right triangle, given two of its parts, we set up equations expressing each of the unknown quantities in terms of the given ones, using the formulas given in Art. 103. After solving these equations, as a check, we recompute one unknown, using a formula expressing this unknown in terms of the other two unknowns.

A figure, carefully drawn to scale, will not only frequently suggest the proper formulas to use in solving the triangle but will also reveal any gross errors that may have arisen in the solution.

The work may be arranged in the form shown in the following examples.

Example 1. Solve the triangle: $\alpha = 14° \, 23'$, $b = 31.72$.

Formulas: $\beta = 90° - \alpha$, $a = b \tan \alpha$, $c = b/\cos \alpha$. Check, $a = c \cos \beta$.
$\beta = 90° - 14° \, 23' = 75° \, 37'$.
$a = 31.72 \times 0.2564 = 8.133$.
$c = 31.72 \div 0.9687 = 32.74$.
$a = 32.74 \times 0.2484 = 8.133$ (Check).

Example 2. Solve the triangle: $a = 21.47$, $b = 19.84$.

Formulas: $c = \sqrt{a^2 + b^2}$, $\tan \alpha = a/b$, $\beta = 90° - \alpha$. Check, $b = c \sin \beta$.
$c = \sqrt{460.9 + 393.6} = \sqrt{854.5} = 29.23$.
$\tan \alpha = 21.47 \div 19.84 = 1.0822$, $\alpha = 47° \, 16'$.
$\beta = 90° - 47° \, 16' = 42° \, 44'$.
$b = 29.23 \times 0.6786 = 19.84$ (Check).

In this example, since the relation between the required angles does not involve the required side, we have checked by computing a given side in terms of the second computed angle and the required hypotenuse.

Exercises

Solve the following triangles, using Tables III and IV.* Find the lengths of the sides to four significant figures and the angles to the nearest minute.

1. $c = 41$, $\alpha = 62°$.
2. $a = 2.35$, $b = 3.52$.
3. $a = 5.483$, $\alpha = 62° \, 24'$.
4. $c = 5423$, $a = 3152$.
5. $b = 1.362$, $\alpha = 34° \, 17'$.
6. $a = 73.42$, $\beta = 21° \, 35'$.
7. $c = 8.137$, $b = 5.241$.
8. $c = 0.7643$, $\beta = 68° \, 11'$.
9. $b = 743.5$, $\beta = 59° \, 42'$.
10. $c = 28410$, $\alpha = 35° \, 51'$.
11. $a = 31.57$, $b = 17.63$.
12. $c = 72.15$, $a = 61.58$.
13. $c = 1.473$, $\beta = 28° \, 12'$.
14. $a = 3.587$, $\beta = 52° \, 37'$.

105. Angle of Elevation or Depression. The angle *HOP* which the line from an observer at *O* to an object at *P* makes with the horizontal

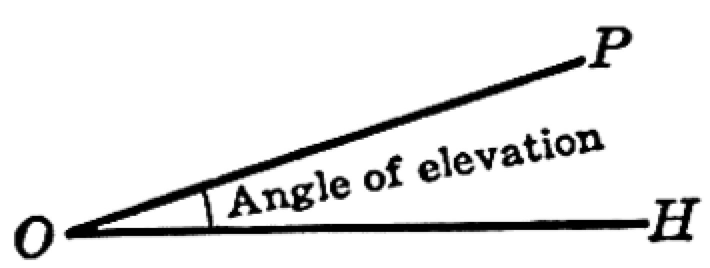


Fig. 41*a*

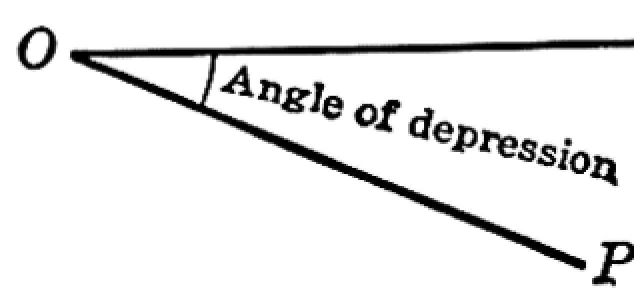


Fig. 41*b*

line *OH* through *O* is called the **angle of elevation** of *P* if *P* lies above *OH* (Fig. 41*a*); it is the **angle of depression** of *P* if *P* lies below *OH* (Fig. 41*b*).

* To find the square or the square root of a number of four significant figures from Table IV, the interpolation process must be used. This process is discussed, for this table, on pages 431–432 of the Introduction to the Tables.

106. Problems Involving the Solution of Right Triangles. Solve the following problems, using the natural functions. Find the required angles to the nearest 10 minutes and the required lengths to three significant figures.

1. The grade of a cog road up a mountain is 23% (23 feet rise to every 100 feet measured horizontally). What angle does it make with the horizontal?

2. The pitch of a roof is $\frac{1}{2}$ (6 inches rise to every foot measured horizontally). What angle does it make with the horizontal?

3. Solve Ex. 2 if the pitch is (*a*) $\frac{1}{3}$; (*b*) $\frac{1}{4}$.

4. What angle does the roof make with the horizontal if it rises 6 inches to every foot measured along the roof?

5. A ladder 21 feet long rests against a vertical wall. The bottom of the ladder is in the plane of the bottom of the wall and 8 feet from it. How high does the ladder reach up the wall?

6. A stairway is so constructed that the riser (vertical distance between the steps) is 7 inches and the tread (horizontal distance between the faces) is 11 inches. Find the angle of inclination of the hand rail.

7. In Ex. 6, if the vertical distance between the floors is 10.5 feet, find the length of the hand rail.

8. From a rowboat on a lake, the angle of elevation of the top of a cliff standing 113 feet above the water is $17° 20'$. How far is the boat from the foot of the cliff?

9. When the angle of elevation of the sun is 34°, the shadow of a building is 217 feet long. How high is the building?

10. The angle of elevation of a kite is 27°. The kite string is 360 feet long. Allow 10 feet for sag in the string and find the height of the kite.

11. A pole 24 feet long is used to brace a vertical wall. If the foot of the pole is 13 feet from the wall, what angle does the pole make with the wall?

12. A moving stairway carries shoppers in a store from one floor to the next, a vertical distance of 17 feet, in 8.4 seconds. If the inclination of the stairway is $26° 30'$, find its speed in feet per minute.

13. A building is surmounted by a flagstaff. From a point on the ground 137 feet from the building, the angle of elevation of the top of the building is $34° 20'$ and, of the top of the flagstaff, is $38° 50'$. Find the height of the building and of the flagstaff.

14. From two successive milestones on a straight, level road, a man observes the angle of elevation of the top of a hill directly ahead of him to be $6° 10'$ and $41° 40'$. How high is the top of the hill above the road?

15. The hour hand of a public clock is 11.2 inches long. At 15 minutes past 4 o'clock, the line joining the ends of the hour and minute hands is perpendicular to the hour hand. How long is the minute hand?

16. A regular octagon is circumscribed around a circle of radius 12 inches. Find the perimeter of the octagon.

17. The sides of an equilateral triangle are each 12 inches long. Find the radius (*a*) of the inscribed and (*b*) of the circumscribed circle.

107. Solution by Logarithms. The solution of triangles can usually be effected more easily by using logarithms. The formulas of Art. 103 should still be used except that the computation of c from the formula $c^2 = a^2 + b^2$ is no longer practical. If b is to be found from this formula, write it in the form $b = \sqrt{(c+a)(c-a)}$, giving $\log b = \frac{1}{2}[\log (c+a) + \log (c-a)]$. Similarly, $\log a = \frac{1}{2}[\log (c+b) + \log (c-b)]$.

The solution may be arranged as shown in the following example. Write out the entire form for the computation before looking up any logarithms.

EXAMPLE. Solve the right triangle: $c = 713.64$, $\alpha = 37° 28.3'$.

Given:	Find:
$c = 713.64$,	$\beta = 52° 31.7'$,
$\alpha = 37° 28.3'$.	$a = 434.16$,
	$b = 566.39$.

Formulas. $\beta = 90° - \alpha$, $a = c \sin \alpha$, $b = c \cos \alpha$. Check: $b = a \tan \beta$.

$$\begin{aligned} \log \sin \alpha &= 9.78417 - 10 \\ \log c &= 2.85348 \quad + \\ \log a &= 2.63765 \end{aligned} \qquad \begin{aligned} \log \cos \alpha &= 9.89963 - 10 \\ \log c &= 2.85348 \quad + \\ \log b &= 2.75311 \end{aligned}$$

$$\begin{aligned} \log \tan \beta &= 0.11546 \\ \log a &= 2.63765 + \\ \log b &= 2.75311 \end{aligned}$$

Exercises

Solve the following triangles, using Tables I and II. Find the required sides to five significant figures and the required angles to the nearest tenth of a minute.

1. $b = 91.35$, $\alpha = 62° 11'$.
2. $c = 86.32$, $a = 75.31$.
3. $a = 436.12$, $\beta = 53° 48.6'$.
4. $b = 41.037$, $\alpha = 62° 38.4'$.
5. $c = 434.36$, $b = 345.81$.
6. $b = 5.2839$, $\beta = 17° 53.6'$.
7. $c = 8421.5$, $\alpha = 31° 42.5'$.
8. $a = 13.428$, $b = 32.371$.
9. $a = 4728.9$, $\alpha = 54° 23.4'$.
10. $c = 0.024573$, $\beta = 28° 19.6'$.
11. $c = 88.916$, $a = 76.584$.
12. $a = 6.8385$, $\beta = 41° 52.3'$.
13. $c = 0.72952$, $\beta = 14° 26.2'$.
14. $c = 9312.4$, $b = 7184.8$.
15. $b = 8.6549$, $\beta = 72° 13.6'$.
16. $c = 2837.5$, $\alpha = 55° 41.8'$.
17. $a = 6.1394$, $b = 3.8762$.
18. $a = 853.46$, $\alpha = 82° 47.3'$.

108. Area of a Right Triangle. By elementary geometry, the area, S, of a right triangle, is

$$S = \tfrac{1}{2}ab.$$

With the aid of the formulas of Art. 103, this formula may be expressed in various other ways, corresponding to the ways in which the

triangle is determined. For example, since $a = c \sin \alpha$ and $b = c \cos \alpha$, we have

$$S = \tfrac{1}{2}c^2 \sin \alpha \cos \alpha.$$

Since $a = b \tan \alpha$, or $b = a \tan \beta$, we have

$$S = \tfrac{1}{2}b^2 \tan \alpha = \tfrac{1}{2}a^2 \tan \beta,$$

and so on.

Exercises

1–18. Find the areas of the triangles in Ex. 1–18, Art. 107, to five significant figures.

109. Applications.

PROJECTIONS. The **projection** of a line segment P_1P_2 on a line l (Fig. 42) is defined to be the segment L_1L_2 joining the feet of the perpendiculars from P_1 and P_2 on the line l.

If the segments are undirected, let α be the acute angle between l and the line through P_1 and P_2. Since angle $RP_1P_2 = \alpha$ (Fig. 42), it follows from the definition of $\cos \alpha$ that

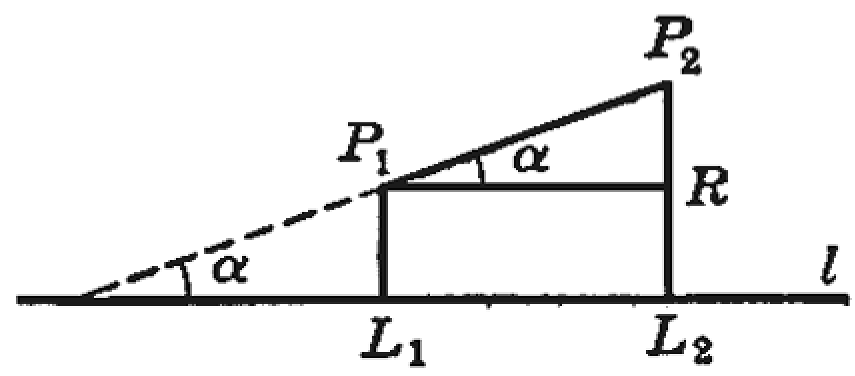


FIG. 42

$$L_1L_2 = P_1R = P_1P_2 \cos \alpha.$$

If the segments are directed, this formula still holds provided that α is taken to be the angle between the positive directions on the lines.

VECTORS. A **vector** *is a line segment having a fixed magnitude and a fixed direction.* Certain physical quantities are often represented by vectors. A velocity, for example, may be represented by a vector whose length represents the speed of the body and whose direction is the direction in which the body is moving. Similarly, a force may be represented by a vector whose length represents the magnitude of the force and whose direction is the direction in which the force acts.

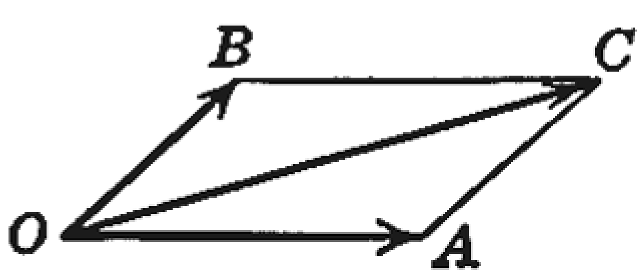


FIG. 43

The **resultant** of two vectors, OA and OB, extending from a point O, is the vector OC extending from O to the fourth vertex C of the parallelogram having OA and OB as two adjacent sides. The vectors OA and OB, in turn, are **components** of the vector OC.

Problems

In the following problems, find the required lengths to four significant figures and the required angles to the nearest minute. Logarithms may be used.

In Ex. 1–6, d is the length of a line segment and α is the angle it makes with the horizontal. It is required to find the lengths of its horizontal and vertical projections.

1. $d = 573.2$, $\alpha = 61° 14'$.
2. $d = 21.84$, $\alpha = 21° 42'$.
3. $d = 1736$, $\alpha = 38° 23'$.
4. $d = 537.2$, $\alpha = 57° 4'$.
5. $d = 92.85$, $\alpha = 40° 34'$.
6. $d = 0.3842$, $\alpha = 16° 24'$.

Find the length of a vector and the angle it makes with the horizontal, given that its horizontal and vertical components are, respectively:

7. 21.65, 17.94.
8. 3.482, 5.763.
9. 518.1, 204.8.
10. 8349, 3149.
11. 0.04281, 0.1317.
12. 29.34, 54.12.

Find the north or south and the east or west components of the velocity, v, of an airplane, given:

13. $v = 132.5$, $\alpha =$ N 21° 13′ E.
14. $v = 625.4$, $\alpha =$ S 51° 29′ E.

NOTE. The expression N 21° 13′ E means, "face north, turn 21° 13′ toward the east." You will then be facing in the direction in which the airplane is moving.

15. $v = 261.3$, $\alpha =$ S 37° 54′ W.
16. $v = 186.5$, $\alpha =$ N 47° 15′ E.

17. An airplane, headed north and traveling 136.4 miles an hour with reference to the air, is carried toward the east by a wind having a velocity of 34.2 miles an hour. Find the direction and speed of the plane with reference to the ground.

18. A surveyor, to avoid a lake, goes from A, 173 rods N 16° 12′ E to B, from B 128 rods N 71° 28′ E to C, and from C 213 rods S 51° 44′ E to D. Find the northerly and easterly projections of AD and the distance and direction of D from A.

19. To avoid an obstruction, a surveyor, surveying an east and west line, goes from A 312 yards S 25° 34′ W to B, from B 416 yards N 72° 31′ W to C, then from C N 28° 43′ W to a point D directly west of A. Find the distances AD and CD.

Chapter 14

Reduction Formulas

110. Introduction. Tables of the values of the trigonometric functions or their logarithms state the values of these functions for a limited interval only. In Tables II and III, for example, the angles for which the values of the functions are given lie in the interval 0° to 90°. It will frequently be necessary for us to find the value of a function of a negative angle or of one greater than 90°. In this chapter, we shall show how to express the value of a function of a positive or negative angle of any size in terms of the value of a function of an angle in the interval 0° to 90° and, thereby, to find its value with the aid of the tables.

111. Functions of $-\theta$. Place the angles θ and $-\theta$ in standard position (Fig. 44). Take a point P on the terminal line of θ and a point

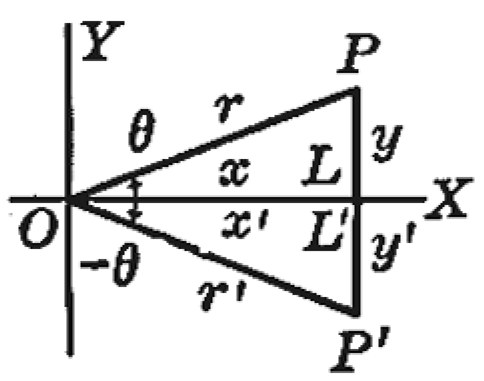


FIG. 44a

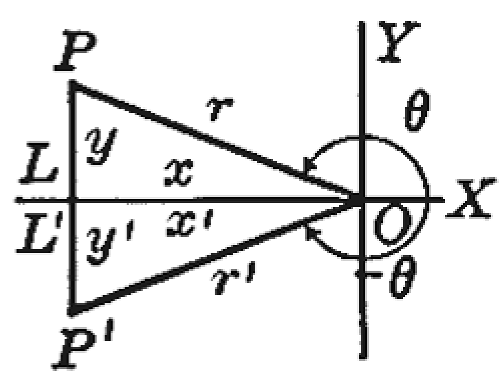


FIG. 44b

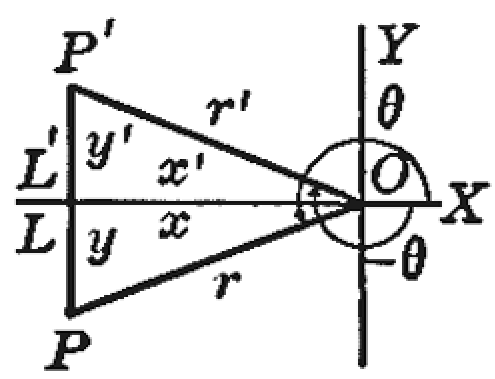


FIG. 44c

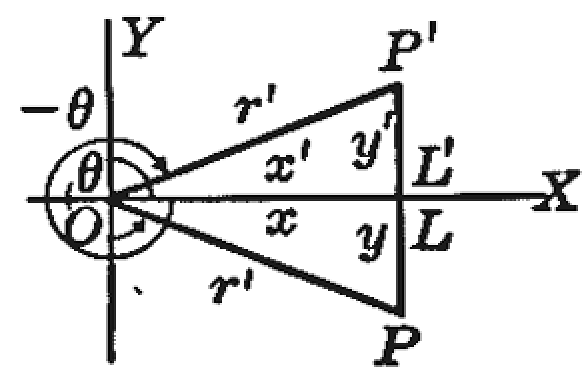


FIG. 44d

P' on the terminal line of $-\theta$ in such a way that $OP' = OP$. The right triangles OLP and $OL'P'$ are congruent since $OP' = OP$ and the acute angle $L'OP'$ equals the acute angle LOP (Why?). It follows by geometry that the corresponding sides of these two triangles are numerically equal. Giving due regard to the signs of the coördinates, we find, in each case, that

$$r' = r, \quad x' = x, \quad \text{and} \quad y' = -y.$$

Hence, by the definitions of the trigonometric functions,

$$\sin(-\theta) = \frac{y'}{r'} = -\frac{y}{r} = -\sin\theta, \qquad \csc(-\theta) = \frac{r'}{y'} = -\frac{r}{y} = -\csc\theta,$$

$$\cos(-\theta) = \frac{x'}{r'} = \frac{x}{r} = \cos\theta \qquad \sec(-\theta) = \frac{r'}{x'} = \frac{r}{x} = \sec\theta,$$

$$\tan(-\theta) = \frac{y'}{x'} = -\frac{y}{x} = -\tan\theta, \qquad \cot(-\theta) = \frac{x'}{y'} = -\frac{x}{y} = -\cot\theta.$$

The required relations are, therefore,

$$\begin{aligned} \sin(-\theta) &= -\sin\theta, & \csc(-\theta) &= -\csc\theta, \\ \cos(-\theta) &= \cos\theta, & \sec(-\theta) &= \sec\theta, \\ \tan(-\theta) &= -\tan\theta, & \cot(-\theta) &= -\cot\theta. \end{aligned} \tag{1}$$

Note particularly, since it is contrary to what one would expect, that $\cos(-\theta) = \cos\theta$, and $\sec(-\theta) = \sec\theta$.

112. Functions of $90° + \theta$. Place the angle θ in standard position and construct the angle $POP' = 90°$ so that the angle $XOP' = 90° + \theta$ (Fig. 45). Further, choose the points P and P' on the terminal lines of θ and $90° + \theta$ so that $OP = OP'$.

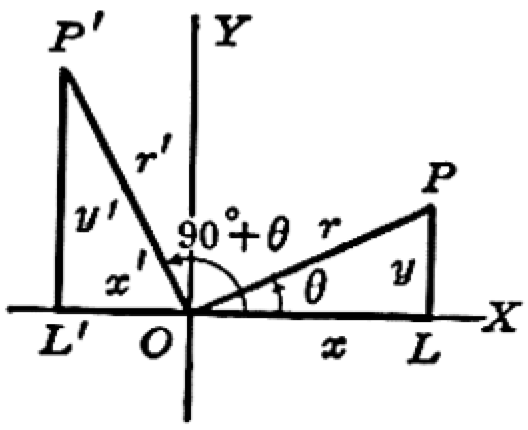


FIG. 45*a*

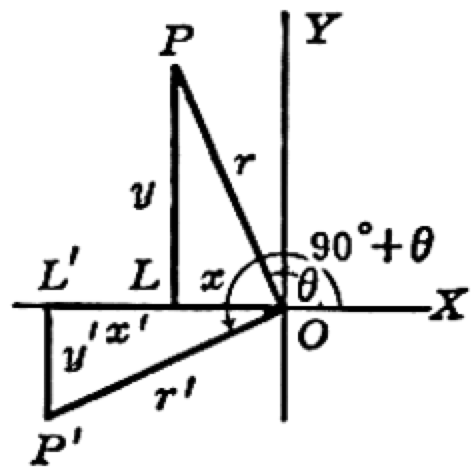


FIG. 45*b*

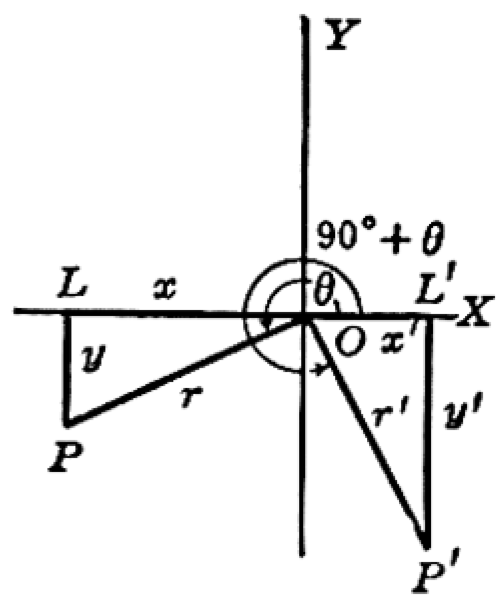


FIG. 45*c*

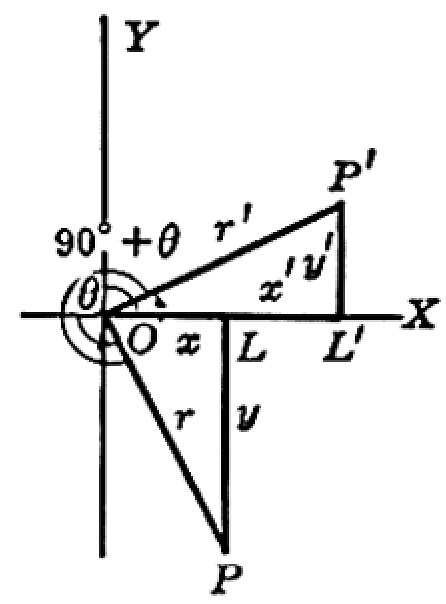


FIG. 45*d*

The right triangles OLP and $OL'P'$ are congruent since $OP' = OP$ and the corresponding sides of the two triangles are respectively perpendicular. It follows that the corresponding sides are numerically equal. Giving due regard to the signs of the coördinates, we find, in each case, that

$$r' = r, \quad x' = -y, \quad \text{and} \quad y' = x.$$

Hence,

$$\sin(90° + \theta) = \frac{y'}{r'} = \frac{x}{r} = \cos\theta, \qquad \csc(90° + \theta) = \frac{r'}{y'} = \frac{r}{x} = \sec\theta,$$

$$\cos(90° + \theta) = \frac{x'}{r'} = -\frac{y}{r} = -\sin\theta, \qquad \sec(90° + \theta) = \frac{r'}{x'} = -\frac{r}{y} = -\csc\theta,$$

$$\tan(90° + \theta) = \frac{y'}{x'} = -\frac{x}{y} = -\cot\theta, \qquad \cot(90° + \theta) = \frac{x'}{y'} = -\frac{y}{x} = -\tan\theta.$$

The required relations are, therefore,

$$\begin{aligned} \sin(90° + \theta) &= \cos\theta, & \csc(90° + \theta) &= \sec\theta, \\ \cos(90° + \theta) &= -\sin\theta, & \sec(90° + \theta) &= -\csc\theta, \\ \tan(90° + \theta) &= -\cot\theta, & \cot(90° + \theta) &= -\tan\theta. \end{aligned} \tag{2}$$

In Figures 44 and 45, we have, for definiteness, indicated the angles θ as positive and less than 360°. The proofs, however, and the resulting formulas, hold equally well if θ is negative, or if it is positive and greater than 360°.

113. Functions of $90° - \theta$. Since $90° - \theta = 90° + (-\theta)$, the formulas for functions of $90° - \theta$ may be derived from those of the preceding two articles in the following way.

$$\sin(90° - \theta) = \sin[90° + (-\theta)] = \cos(-\theta) = \cos\theta.$$

Proceeding in this way for each function, we find that

$$\begin{aligned} \sin(90^\circ - \theta) &= \cos\theta, & \csc(90^\circ - \theta) &= \sec\theta, \\ \cos(90^\circ - \theta) &= \sin\theta, & \sec(90^\circ - \theta) &= \csc\theta, \\ \tan(90^\circ - \theta) &= \cot\theta, & \cot(90^\circ - \theta) &= \tan\theta. \end{aligned} \quad (3)$$

114. Reduction of the Functions of a Given Angle. If θ is a positive angle greater than 90°, we can replace θ by $90^\circ + \theta_1$, where θ_1 is an angle less by 90° than θ, and express any given function of θ in terms of a function of θ_1 by means of equations (2). If θ_1 still is greater than 90°, we can replace θ_1 by $90^\circ + \theta_2$ and express the function of θ_1 in terms of a function of θ_2, and so on.

Thus, $\sin 349^\circ = \sin(90^\circ + 259^\circ) = \cos 259^\circ = \cos(90^\circ + 169^\circ)$
$= -\sin 169^\circ = -\sin(90^\circ + 79^\circ) = -\cos 79^\circ.$

If θ is a negative angle, we can, by using equations (1), replace the given function of θ by a function of a positive angle, then reduce this positive angle as in the preceding case.

Thus, $\tan(-217^\circ) = -\tan 217^\circ = -\tan(90^\circ + 127^\circ) = \cot 127^\circ$
$= \cot(90^\circ + 37^\circ) = -\tan 37^\circ.$

The foregoing computations can be abbreviated. Every time we decrease the angle by 90°, we replace the given function by its cofunction (sine by cosine, cosine by sine, tangent by cotangent, etc.). Hence, if we decrease the angle by 90° an *even* number of times, we end with the same function as that with which we started; if we decrease it by 90° an *odd* number of times, we end with its cofunction. Hence, we should *replace the given angle by $n\,90^\circ + \theta$, where θ is an angle in the first quadrant. If n is an even number, replace the given function of $n\,90^\circ + \theta$ by plus or minus the same function of θ; if n is odd, replace it by plus or minus the cofunction of θ.*

To determine which algebraic sign to use, since all the functions of an angle in the first quadrant are positive, *determine the sign of the originally given function of the given angle and place that sign in front of the final function of θ.*

EXAMPLE 1. Find the value of $\sin 247^\circ 21'$.

$247^\circ 21' = 2 \cdot 90^\circ + 67^\circ 21'$. Since 2 is an even number,

$$\sin 247^\circ 21' = \sin(2 \cdot 90^\circ + 67^\circ 21') = \pm \sin 67^\circ 21'.$$

Since $247^\circ 21'$ is an angle in the third quadrant, its sine is negative.

$$\sin 247^\circ 21' = -\sin 67^\circ 21' = -0.9229.$$

EXAMPLE 2. Find the value of $\tan(-283^\circ 34')$.

Since the given *negative* angle lies in the first quadrant, its tangent is positive.

$$\begin{aligned} \tan(-283^\circ 34') &= -\tan 283^\circ 34' = -\tan(3 \cdot 90^\circ + 13^\circ 34') \\ &= \cot 13^\circ 34' = 4.1441. \end{aligned}$$

Sometimes it is required to express a function of an angle of the form $n \cdot 90^\circ \pm \theta$ in terms of a function of θ, when the value of θ is not specified. Since the required sign is independent of the value of θ, *determine the sign by supposing that θ is in the first quadrant.*

EXAMPLE 3. Express in terms of a function of θ: $\sec(540^\circ + \theta)$.

If θ is in the first quadrant, $540^\circ + \theta$ is in the third quadrant and its secant is negative.

$$\sec(540^\circ + \theta) = \sec(6 \cdot 90^\circ + \theta) = -\sec\theta.$$

EXAMPLE 4. Express in terms of a function of θ: $\tan(270^\circ - \theta)$.

If θ is in the first quadrant, $270^\circ - \theta$ is in the third quadrant and its tangent is positive.

$$\tan(270^\circ - \theta) = \tan(3 \cdot 90^\circ - \theta) = \cot\theta.$$

Exercises

Find the values of the following functions, using Table III.

1. $\sin 234^\circ 15'$. **2.** $\cot 291^\circ 12'$. **3.** $\tan 347^\circ 41'$.
4. $\cos 471^\circ 24'$. **5.** $\cot 242^\circ 38'$. **6.** $\sin 417^\circ 26'$.
7. $\cos 589^\circ 46'$. **8.** $\sin 647^\circ 14'$. **9.** $\tan 873^\circ 43'$.
10. $\sin(-194^\circ 31')$. **11.** $\tan(-237^\circ 6')$. **12.** $\cos(-483^\circ 51')$.

Find the logarithms of the values of the following functions. If the value of the given function is negative, write the letter (n), in parentheses, after the logarithm.

13. $\cot 251^\circ 16.4'$. **14.** $\sin 295^\circ 39.2'$. **15.** $\tan 508^\circ 52.6'$.

Express each of the following in terms of a function of θ.

16. $\cos(180^\circ + \theta)$. **17.** $\sin(180^\circ - \theta)$. **18.** $\cot(270^\circ - \theta)$.
19. $\cot(270^\circ + \theta)$. **20.** $\csc(450^\circ + \theta)$. **21.** $\cos(630^\circ - \theta)$.
22. $\sin\left(\frac{3\pi}{2} + \theta\right)$. **23.** $\sec(\pi + \theta)$. **24.** $\tan\left(\frac{9\pi}{2} - \theta\right)$.

115. Periodicity. Since $360^\circ = 4 \cdot 90^\circ$, we have for all values of θ,

$$\sin(360^\circ + \theta) = \sin\theta, \qquad \csc(360^\circ + \theta) = \csc\theta,$$
$$\cos(360^\circ + \theta) = \cos\theta, \qquad \sec(360^\circ + \theta) = \sec\theta;$$

that is, *if we add* 360° *to any angle, or subtract* 360° *from it, we do not change the value of any one of these four functions.* This fact is expressed by the statement: *The functions $\sin\theta$, $\cos\theta$, $\sec\theta$, and $\csc\theta$ are periodic with period* 360°.

We find in a similar way that

$$\tan(180^\circ + \theta) = \tan\theta, \qquad \cot(180^\circ + \theta) = \cot\theta;$$

that is, *$\tan\theta$ and $\cot\theta$ are periodic with period* 180°.

If the angles are measured in radians, since 360° = 2π radians and 180° = π radians, the above equations become

$$\begin{aligned} \sin(2\pi + \theta) &= \sin\theta, & \csc(2\pi + \theta) &= \csc\theta, \\ \cos(2\pi + \theta) &= \cos\theta, & \sec(2\pi + \theta) &= \sec\theta, \\ \tan(\pi + \theta) &= \tan\theta, & \cot(\pi + \theta) &= \cot\theta; \end{aligned}$$

that is, *$\sin\theta$, $\cos\theta$, $\sec\theta$, and $\csc\theta$ are periodic with period 2π, and $\tan\theta$ and $\cot\theta$ are periodic with period π.*

116. The Graphs of the Trigonometric Functions. The graphs of the trigonometric functions can be drawn by plotting points on the graph and drawing the curve through the plotted points. We shall suppose that the angle is measured in radians; that is, we shall measure on the x-axis as many units of length as the number of radians in the angle. Since the functions are periodic, we need to compile a table of coördinates of points for a single period only. By drawing the graph for this period and repeating the part so drawn, indefinitely far, in both directions, we have the required curve.

The sine curve. To draw the sine curve,

$$y = \sin x,$$

we first make a table of pairs of values of x and y.

x	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/3$	π	$7\pi/6$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$11\pi/6$	2π
y	0	0.5	.87	1.0	.87	0.5	0	−0.5	−.87	−1.0	−.87	−0.5	0

This table has been computed for intervals of $\pi/6$ on the x-axis. The values of y may be found with the aid of Table III and the reduction formulas or from the values given in Arts. 96 to 99.

By plotting the points whose coördinates are given, drawing a smooth curve through them, and repeating the part so drawn in both directions, we obtain Figure 46.

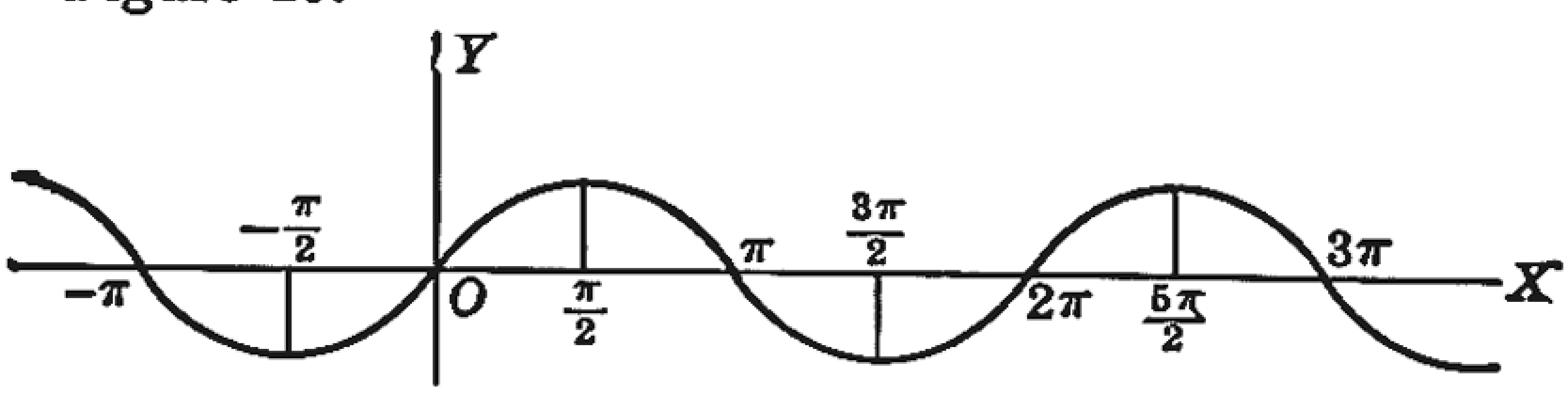


FIG. 46

The cosine curve. The graph of the equation,

$$y = \cos x,$$

is obtained in a similar way (Fig. 47).

The cosine curve is congruent to the sine curve but it is placed in a slightly different position with reference to the y-axis. It can be shown,

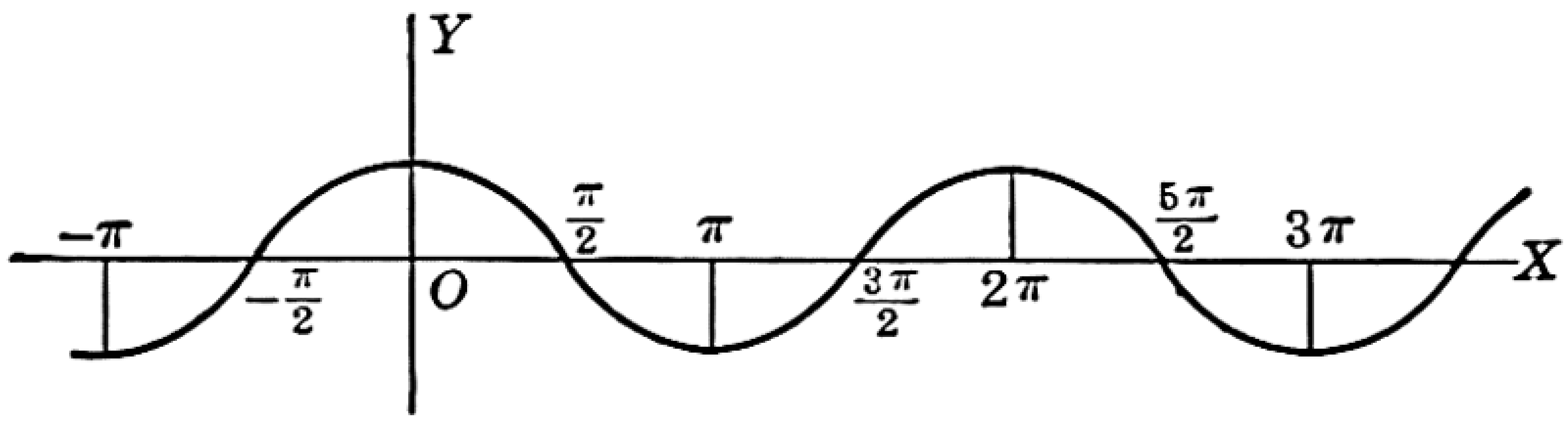


FIG. 47

in fact, since $\sin(\pi/2 + x) = \cos x$, that, if we draw a sine curve, then move the y-axis $\pi/2$ units to the right, we have a cosine curve.

The student should show, using the identity $\cos(-x) = \cos x$, that the cosine curve is symmetric (Art. 41) with respect to the y-axis.

The tangent curve. The graph of the equation,

$$y = \tan x,$$

is obtained in the same way that the preceding curves were obtained. The curve is shown in Figure 48.

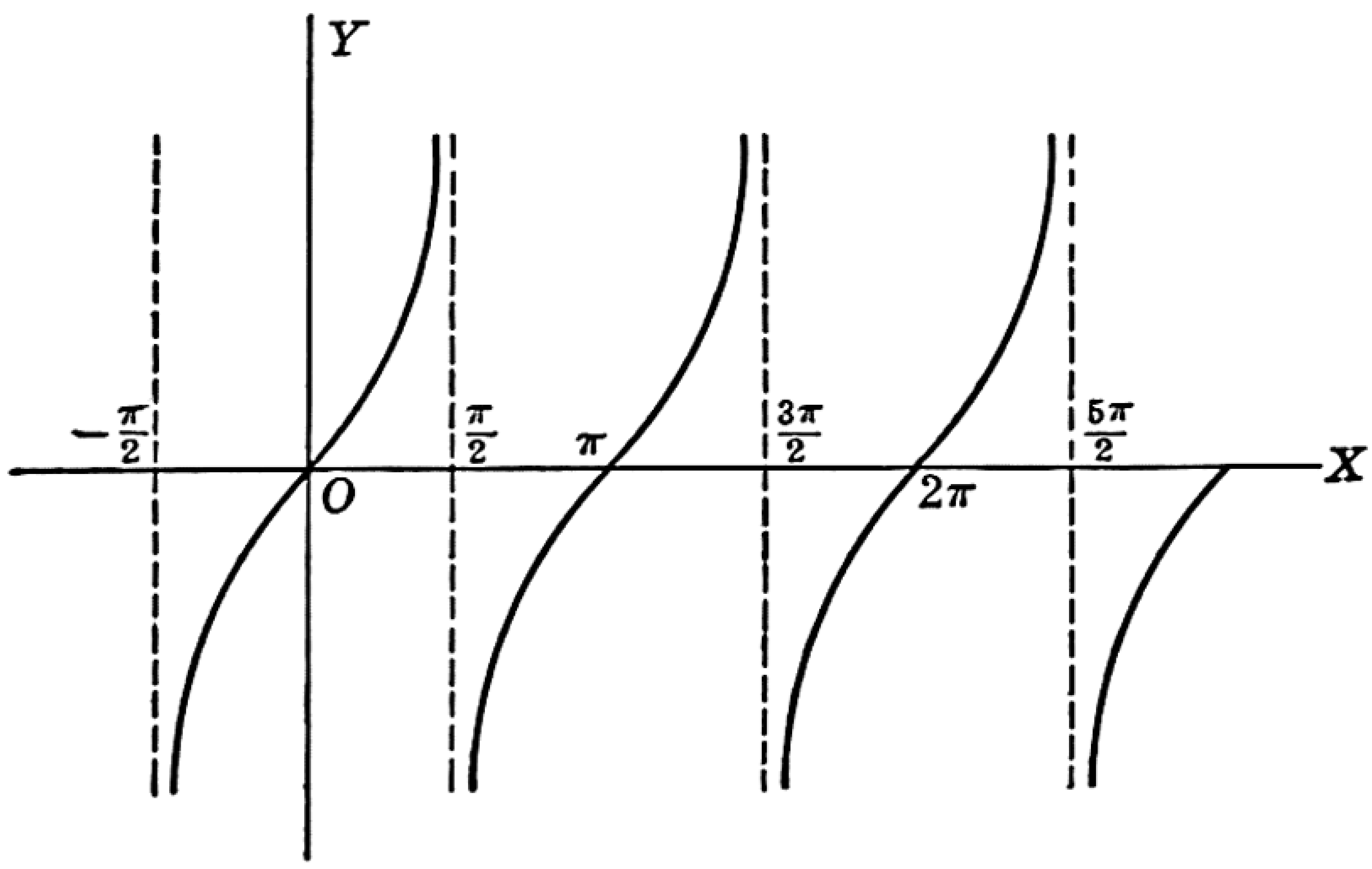


FIG. 48

Since the numerical value of $\tan x$ increases indefinitely as x approaches $-\pi/2$, $\pi/2$, $3\pi/2$, etc., the graph recedes indefinitely far from the x-axis in such a way that it approaches the vertical lines $x = -\pi/2$, $x = \pi/2$, $x = 3\pi/2$, etc., as shown in the figure. Because of this property, these lines are called **asymptotes** to the curve.

Since $\tan(-x) = -\tan x$, if the point (x, y) lies on the tangent curve, so also does the point $(-x, -y)$. Because of this property, the tangent curve is said to be **symmetric** with respect to the origin.

The problem of drawing the graph of the cotangent curve, $y = \cot x$, is left as an exercise for the student.

The secant curve. The graph of the equation

$$y = \sec x,$$

is the secant curve (Fig. 49).

What are the equations of the asymptotes to the secant curve? Are there any symmetries? According to the graph, there are no values of x for which $\sec x$ is numerically less than unity. Show from the definition that this is so. Draw the cosecant curve, $y = \csc x$.

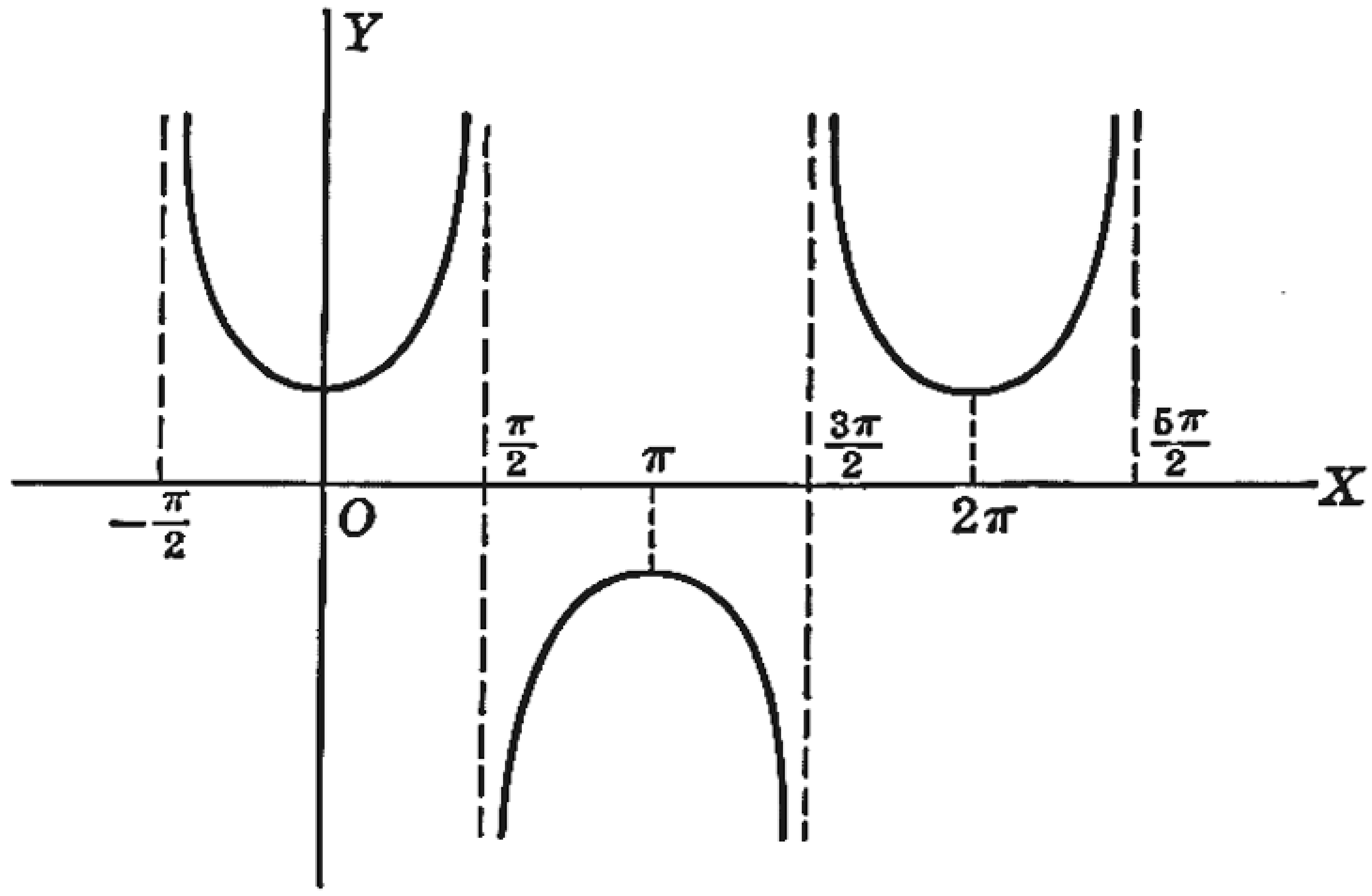


FIG. 49

117. The Line Values of the Functions. Draw a circle of radius unity having its center at the origin of coördinates and place the given angle θ in standard position with reference to the axes.

Let P be the point where the terminal side of angle θ meets the circle, let T be the point where the terminal line meets the tangent line to the circle at A and let S be the point where it meets the tangent at B (Fig. 50).

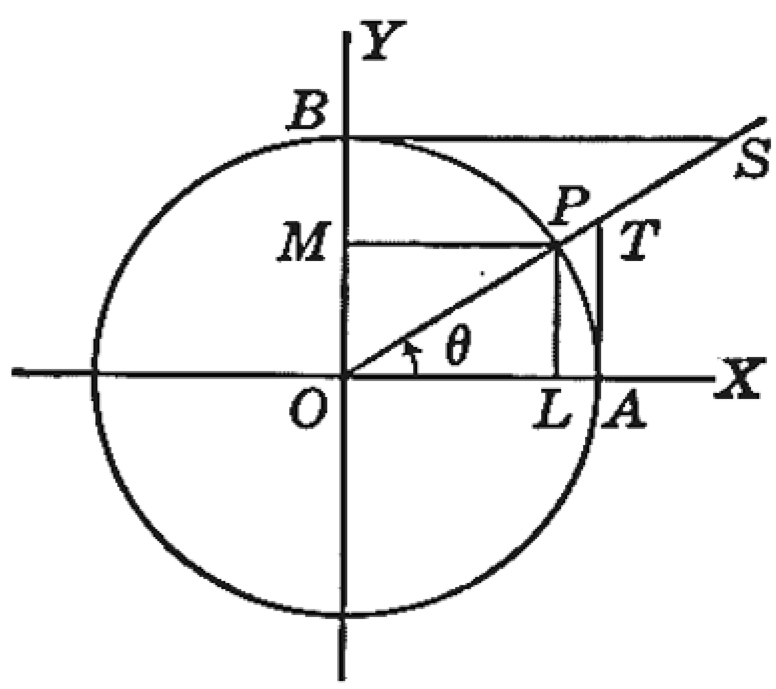


FIG. 50a

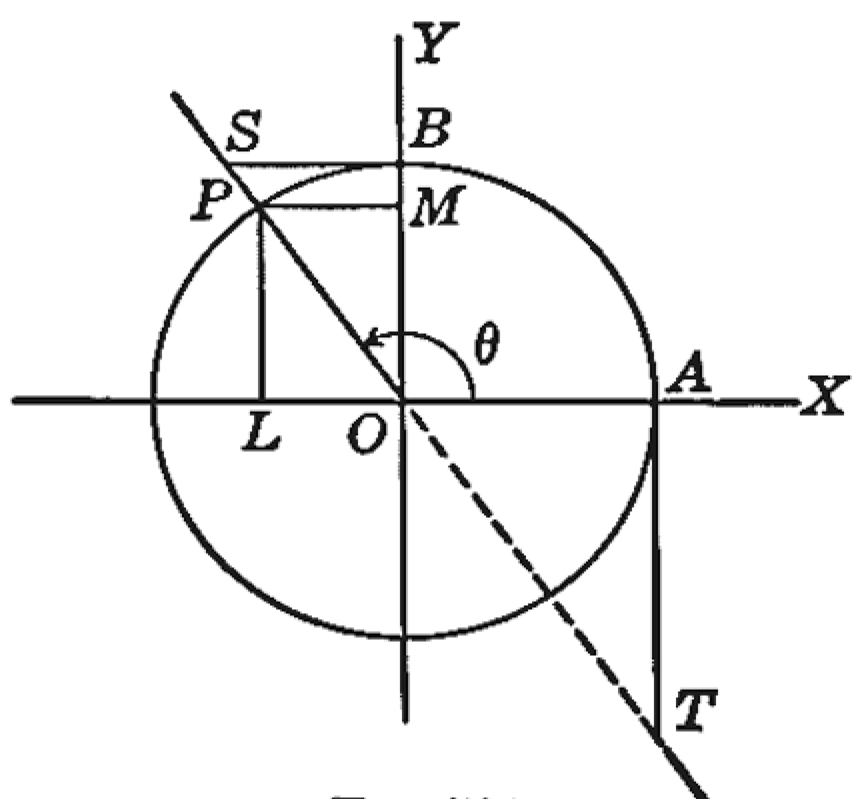


FIG. 50b

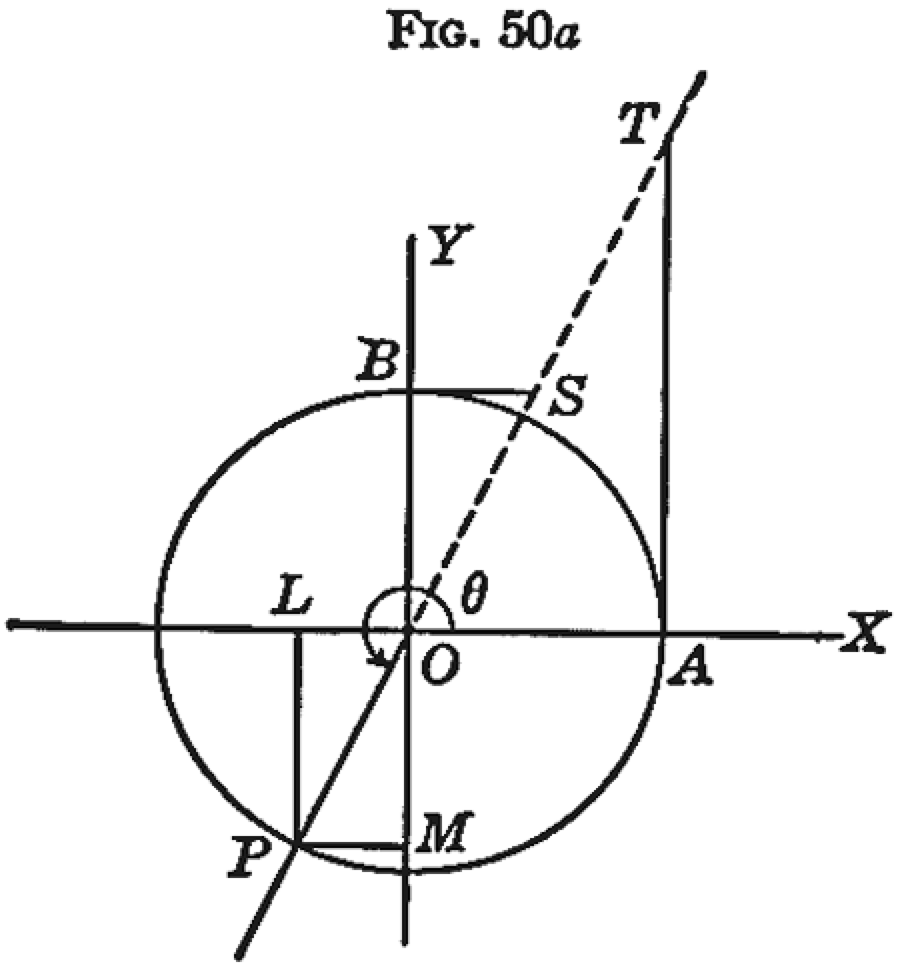


FIG. 50c

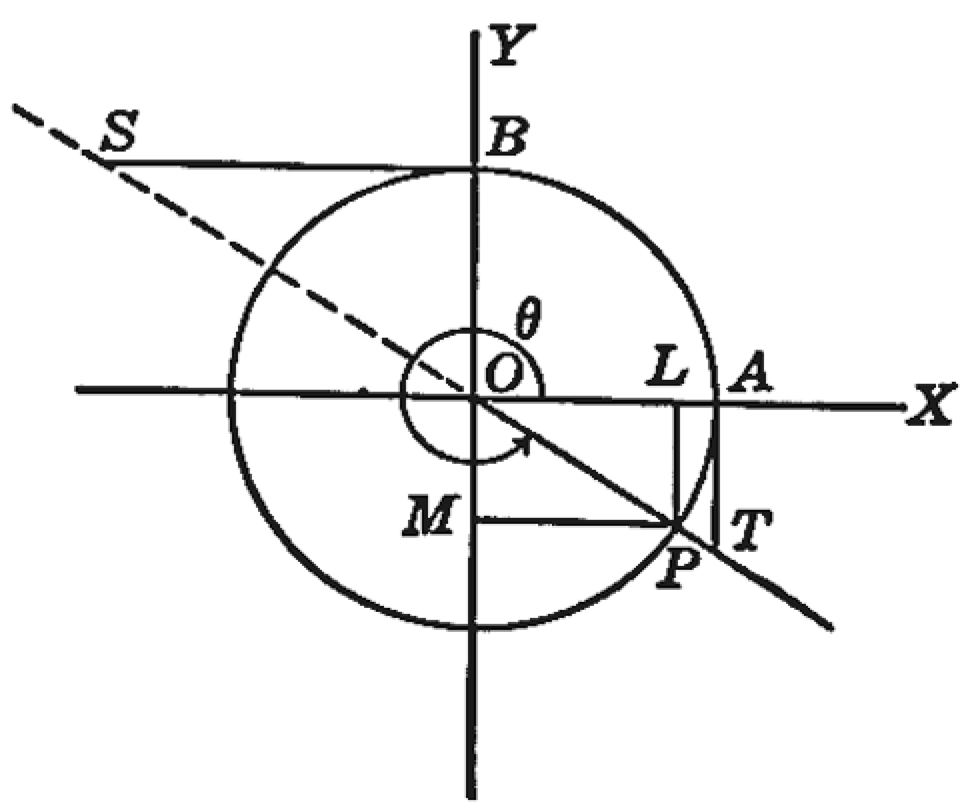


FIG. 50d

Let us agree, not only to measure horizontal and vertical segments as positive and negative in the customary way, but also to consider segments measured in the direction of the terminal side of θ as positive and segments measured in the direction opposite to this side as negative.

Using the definitions of the trigonometric functions, together with similar triangles when necessary, we find, in each quadrant, that

$$\sin\ \theta = \frac{LP}{OP}, \qquad \csc\ \theta = \frac{OS}{OB},$$

$$\cos\ \theta = \frac{OL}{OP}, \qquad \sec\ \theta = \frac{OT}{OA},$$

$$\tan\ \theta = \frac{AT}{OA}, \qquad \cot\ \theta = \frac{BS}{OB}.$$

But $OP = OA = OB = 1$. Hence the above equations reduce to:

$$\begin{aligned} \sin\ \theta &= LP, & \csc\ \theta &= OS, \\ \cos\ \theta &= OL, & \sec\ \theta &= OT, \\ \tan\ \theta &= AT, & \cot\ \theta &= BS. \end{aligned} \tag{4}$$

Equations (4) express the value of each of the trigonometric functions as the length of a directed line segment. Because of these relations, the trigonometric functions are sometimes spoken of as *trigonometric lines*.

Chapter 15

Trigonometric Identities and Equations

118. The Fundamental Identities. The following eight equations connecting the six trigonometric functions of any angle are the **fundamental identities** connecting these functions.

$$\csc\theta = \frac{1}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}, \qquad \cot\theta = \frac{1}{\tan\theta},$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}, \tag{1}$$

$$\sin^2\theta + \cos^2\theta = 1, \quad \tan^2\theta + 1 = \sec^2\theta, \quad \cot^2\theta + 1 = \csc^2\theta.$$

The first three of these equations were proved in Art. 94. To prove the next two, write down the definitions of $\tan\theta$ and $\cot\theta$, divide each numerator and denominator by r, and write the name of each of the resulting fractions, thus:

$$\tan\theta = \frac{y}{x} = \frac{y/r}{x/r} = \frac{\sin\theta}{\cos\theta}; \qquad \cot\theta = \frac{x}{y} = \frac{x/r}{y/r} = \frac{\cos\theta}{\sin\theta}.$$

To derive the last three equations, we start from the equation $x^2 + y^2 = r^2$, which is true by the Pythagorean theorem. Divide this equation successively by r^2, by x^2, and by y^2, then write down the names of the resulting fractions, thus:

$$\frac{y^2}{r^2} + \frac{x^2}{r^2} = 1, \quad \text{or} \quad \sin^2\theta + \cos^2\theta = 1,$$

$$\frac{y^2}{x^2} + 1 = \frac{r^2}{x^2}, \quad \text{or} \quad \tan^2\theta + 1 = \sec^2\theta,$$

$$1 + \frac{x^2}{y^2} = \frac{r^2}{y^2}, \quad \text{or} \quad 1 + \cot^2\theta = \csc^2\theta.$$

Exercises

Verify that the eight equations (1) are true for the given angle by inserting the values of the functions of the angle from Arts. 96 to 98.

1. $30°$. **2.** $120°$. **3.** $225°$. **4.** $\frac{5\pi}{6}$. **5.** $\frac{5\pi}{3}$. **6.** $\frac{7\pi}{4}$.

By inserting the numerical values, verify that each of the equations $\cos\theta = \pm\sqrt{1 - \sin^2\theta}$, $\sec\theta = \pm\sqrt{1 + \tan^2\theta}$, and $\csc\theta = \pm\sqrt{1 + \cot^2\theta}$,

is true for each given angle. In each case, insert the proper sign before the radical.

7. $45°$. **8.** $210°$. **9.** $330°$. **10.** $\frac{\pi}{3}$. **11.** $\frac{3\pi}{4}$. **12.** $\frac{4\pi}{3}$.

Using equations (1), write the values of all of the functions of α, when α is in the quadrant indicated.

13. $\sin \alpha = -\frac{4}{5}$; fourth.
14. $\tan \alpha = -\frac{12}{5}$; second.
15. $\sec \alpha = -\frac{17}{8}$; third.
16. $\csc \alpha = \frac{25}{24}$; first.
17. $\cot \alpha = \frac{5}{2}$; third.
18. $\cos \alpha = \frac{4}{7}$; fourth.

19. Find the value of $\sin \beta \sec^2 \beta$, given $\csc \beta = -3$.

20. Find the value of $(2 \cos^2 \beta - 1)(1 - \cot \beta)$, given $\tan \beta = 2$.

21. Find the value of $\cos \beta (1 + \tan \beta)$, given $\sin \beta = \frac{3}{4}$, with β in the second quadrant.

22. Find the value of $\dfrac{\sec \beta \csc \beta}{\sec \beta + \csc \beta}$, given $\cos \beta = -\dfrac{1}{4}$, with β in the third quadrant.

Express all of the other functions of the angle x in terms of the given function.

23. $\sin x$. **24.** $\cos x$. **25.** $\tan x$. **26.** $\cot x$.

119. Identities and Equations of Condition. An equation is an **identity** if it is true for all values of the quantities involved in it for which both of its members have a meaning.

Thus, the equations,

$$(x - y)^2 = x^2 - 2xy + y^2,$$

and

$$\frac{x^2 - 7x + 10}{x - 5} = x - 2,$$

are identities. The first is true for all values of x and y; the second is true for all values of x except $x = 5$ for which the first member has no meaning.

All of equations (1) of the preceding article are identities because they are true for all values of θ for which both members have a meaning.

An **equation of condition** is one that is true only for certain values of the quantities contained in it.

Thus, the equation

$$\cos \theta = \tfrac{1}{2}$$

is an equation of condition. It is true if $\theta = \pm 60°$, $\pm 420°$, and so on, but it is not true for many other values of θ.

120. Trigonometric Identities. In the applications of trigonometry, it is often necessary to transform one trigonometric expression into another one which is more convenient for the purpose for which it is to be used. The following exercises will afford practice in carrying through such transformations.

To prove an identity, one should, with the aid of equations (1), either (*a*) transform one member of the equation into the other or (*b*) transform each member, separately, into a single expression which is identically equal to each member of the given equation.

In the process of proving an identity, one should not remove factors common to the two members or otherwise modify the values of the separate members. The proof should consist in showing that each member of the given equation is identically equal to the final result. This will not be true if the values are altered during the course of the proof.

To carry through the proof of an identity, one should, in most cases, express both members in terms of sines and cosines, then simplify these results, on both sides of the equation, to the same form. In some cases, it is easier to express both members in terms of some one of the other functions and then to simplify these results.

The operations to be performed will consist largely of transformations of fractions and of factoring. The student should refer to Chapter 2 for the methods and formulas involved in these operations.

EXAMPLE 1. Prove the identity: $\dfrac{\cot\alpha + 1}{\cot\alpha - 1} = \dfrac{1 + \tan\alpha}{1 - \tan\alpha}.$

Since the second member contains only the function $\tan\alpha$, we shall try to express the first member in terms of $\tan\alpha$ and then to reduce the resulting expression to the form of the second member.

$$\frac{\cot\alpha + 1}{\cot\alpha - 1} = \frac{\dfrac{1}{\tan\alpha} + 1}{\dfrac{1}{\tan\alpha} - 1} = \frac{\dfrac{1 + \tan\alpha}{\tan\alpha}}{\dfrac{1 - \tan\alpha}{\tan\alpha}} = \frac{1 + \tan\alpha}{1 - \tan\alpha}.$$

In this example, we have proved the identity by transforming the first member into the second.

EXAMPLE 2. Prove the identity: $\tan x \csc x = \tan x \sin x + \cos x$.

Replace each function by its value in terms of $\sin x$ and $\cos x$ and simplify each member separately, using the fundamental identities (1).

$$\frac{\sin x}{\cos x}\,\frac{1}{\sin x} = \frac{\sin^2 x}{\cos x} + \cos x$$

$$\frac{1}{\cos x} = \frac{\sin^2 x + \cos^2 x}{\cos x}$$

$$\frac{1}{\cos x} = \frac{1}{\cos x}.$$

Since each member of the given equation is identically equal to $1/\cos x$, it is identically equal to the other member.

Exercises

Prove the following identities.

1. $\sin\theta = \cos\theta\tan\theta.$

2. $\cos\theta\csc\theta = \cot\theta.$

3. $\sin^2\theta = (1-\cos\theta)(1+\cos\theta).$

4. $(\sin\theta+\cos\theta)^2 = 1+2\sin\theta\cos\theta.$

5. $\tan\theta = \dfrac{\sec\theta}{\csc\theta}.$

6. $\dfrac{1-\cos\theta}{1+\cos\theta} = \dfrac{\sec\theta-1}{\sec\theta+1}.$

7. $\dfrac{1+\sin\alpha}{\cos\alpha} = \sec\alpha+\tan\alpha.$

8. $\tan^2\alpha = \dfrac{1-\cos^2\alpha}{\cos^2\alpha}.$

9. $(\sec\alpha-1)(\sec\alpha+1) = \tan^2\alpha.$

10. $(1-\tan^2\alpha)\cot\alpha = \cot\alpha-\tan\alpha.$

11. $\cos^2\alpha(1+\cot^2\alpha) = \cot^2\alpha.$

12. $\csc\alpha-\sin\alpha = \sin\alpha\cot^2\alpha.$

13. $\dfrac{1+\cot^2\beta}{1+\tan^2\beta} = \cot^2\beta.$

14. $\sec\beta\csc\beta = \tan\beta+\cot\beta.$

15. $(\sec\beta+\tan\beta)^2 = \dfrac{1+\sin\beta}{1-\sin\beta}.$

16. $\sec^2\beta\csc^2\beta = \sec^2\beta+\csc^2\beta.$

17. $\cos^2\beta-\sin^2\beta = \dfrac{\cot^2\beta-1}{\cot^2\beta+1}.$

18. $\dfrac{\tan\beta+\sec\beta}{\tan\beta\sec\beta} = \cot\beta+\cos\beta.$

19. $\tan^2\beta-\sin^2\beta = \tan^2\beta\sin^2\beta.$

20. $\dfrac{\tan\beta+\cot\beta}{\tan\beta-\cot\beta} = \dfrac{\sec^2\beta}{\tan^2\beta-1}.$

21. $\dfrac{1-2\cos^2 x}{\sin x\cos x} = \tan x-\cot x.$

22. $\dfrac{\tan^2 x}{\sin^2 x} = 1+\tan^2 x.$

23. $\dfrac{\tan^2 x+1}{\sec^2 x-1} = \csc^2 x.$

24. $\dfrac{\cot x+\csc x}{1+\cos x} = \csc x.$

25. $\dfrac{\sec^2 x+\csc^2 x}{\sec x\csc x} = \tan x+\cot x.$

26. $\dfrac{\cos^2 x}{(1+\sin x)^2} = \dfrac{(1-\sin x)^2}{\cos^2 x}.$

27. $\sin^4 x-\cos^4 x = 1-2\cos^2 x.$

28. $\sec^4 x-\sec^2 x = \tan^4 x+\tan^2 x.$

29. $1+\cot^2 x = \dfrac{\sec^2 x}{\sec^2 x-1}.$

30. $\dfrac{\cot x-\tan x}{\cot x+\tan x} = 1-2\sin^2 x.$

31. $(2\cos^2 x-1)^2 = 1-4\sin^2 x\cos^2 x.$

32. $\dfrac{\sin x}{1+\cos x}+\dfrac{1+\cos x}{\sin x} = 2\csc x.$

33. $(\sin^2 x-\cos^2 x)\tan x\sec x\csc x = \tan^2 x-1.$

34. $(\sec x+\csc x)^2 = \sec^2 x\csc^2 x+2\sec x\csc x.$

35. $(\sin y-\cos y)^2 = 2-(\sin y+\cos y)^2.$

36. $\dfrac{2}{\tan y} = \dfrac{1}{\csc y-\cot y}-\dfrac{1}{\csc y+\cot y}.$

37. $\dfrac{\sin y}{\cos y}+\dfrac{\tan y}{\cot y}+\dfrac{\sec y}{\csc y} = \dfrac{2\cot y+1}{\cot^2 y}.$

38. $\sec y+\cos y+\sec y\sin^2 y = 2\sin y\sec^2 y\cot y.$

39. $\dfrac{\tan x+\cot y}{\cot x+\tan y} = \dfrac{\tan x}{\tan y}.$

40. $\dfrac{\tan x+\tan y}{\cot x+\cot y} = \tan x\tan y.$

41. $\dfrac{\tan x+\tan y}{\sec x-\sec y} = \dfrac{\sec x+\sec y}{\tan x-\tan y}.$

42. $\dfrac{\tan x - \tan y}{1 + \tan x \tan y} = \dfrac{\cot y - \cot x}{1 + \cot x \cot y}$.

43. $(\sin \alpha \cos \beta + \cos \alpha \sin \beta)^2 + (\cos \alpha \cos \beta - \sin \alpha \sin \beta)^2 = 1$.

121. Trigonometric Equations of Condition. If the given equation can readily be expressed in terms of a single function of the angle, express it in terms of that function and solve the resulting equation considering this function as the unknown. When the values of the function that satisfy the equation have been found, find all the angles (positive or zero but less than 360°) for which this function has the prescribed values. For this purpose, the values given in Arts. 96 to 99 should be used whenever possible. If these are not applicable, use Table III and the reduction formulas.

EXAMPLE 1. Solve the equation: $2 \cos^2 x + \cos x - 1 = 0$.

Factor the first member: $(\cos x + 1)(2 \cos x - 1) = 0$.

Equate the factors to zero and solve, first for $\cos x$, then for x.

If $\cos x = -1$, $x = 180°$. If $\cos x = \frac{1}{2}$, $x = 60°$ or $300°$. The required values of x are 60°, 180°, and 300°.

EXAMPLE 2. Solve the equation: $2 \sec^2 x - 5 \tan x + 1 = 0$.

Replace $\sec^2 x$ by $1 + \tan^2 x$ and write the equation in the form

$$2 \tan^2 x - 5 \tan x + 3 = 0.$$

Solve this quadratic equation in $\tan x$, giving either:

$\tan x = 1$, $x = 45°$ or $225°$; or $\tan x = 1.5$, $x = 56° 19'$ or $236° 19'$.

Sometimes the equation can be written as the product of two factors equated to zero in such a way that each factor contains only one trigonometric function.

EXAMPLE 3. Solve: $2 \sin x \sec x - 4 \sin x + \sec x = 2$.

Transpose the 2 and factor: $(2 \sin x + 1)(\sec x - 2) = 0$.

Equate each factor to zero and solve. We have, either:

$\sin x = -\frac{1}{2}$, $x = 210°$ or $330°$; $\sec x = 2$, $x = 60°$ or $300°$.

Occasionally it is necessary to square both sides in order to express the equation rationally in terms of one function. When this is done, it must be remembered that extraneous solutions may be introduced and that the results must be checked in the given equation to see whether or not they satisfy it.

EXAMPLE 4. Solve the equation: $\cos x + 2 \sin x = 1$.

Solve for $\cos x$: $\cos x = 1 - 2 \sin x$.

Square both sides: $\cos^2 x = (1 - 2 \sin x)^2 = 1 - 4 \sin x + 4 \sin^2 x$.

Replace $\cos^2 x$ by $1 - \sin^2 x$ and simplify. We have $5 \sin^2 x - 4 \sin x = 0$.

If $\sin x = 0$, $x = 0°$ or $180°$; if $\sin x = \frac{4}{5} = 0.8000$, $x = 53° 8'$ or $126° 52'$. By checking in the given equation, we find that $180°$ and $53° 8'$ are extraneous. The required solutions are $x = 0$ and $126° 52'$.

Exercises

Find all the positive or zero angles less than 360° that satisfy the given equation.

1. $\tan^2 x = 3$.
2. $2 \sin^2 x = 1$.
3. $\sec^2 x = 1$.
4. $4 \cos^2 x = 3$.
5. $\sin^2 x = 1$.
6. $\sec^2 x = 4$.
7. $5 \cos x + 2 = 3(2 - \cos x)$.
8. $2 \sin^2 x + 1 = 3 \sin x$.
9. $\sqrt{2} \tan x \sin x + \tan x = 0$.
10. $2 \csc x \cos x + \sqrt{3} \csc x = 0$.
11. $2 \sin^2 x + 3 \cos x = 0$.
12. $\sec x = \csc x$.
13. $(\sqrt{2} \cos x - 1)(\tan x - \sqrt{3}) = 0$.
14. $(\cos x - 1)(\csc x + 2) = 0$.
15. $\sin x + \cos x = 0$.
16. $3 \csc^2 x + \cot^2 x = 15$.
17. $3 \sin \theta \sec^2 \theta = 4 \sin \theta$.
18. $4 \cot^2 \theta = 3 \csc^2 \theta$.
19. $1 + \sin \theta = 3 \cos^2 \theta$.
20. $3 \sin \theta \tan \theta = \cos \theta$.
21. $1 - \cos \theta = 2 \cos^2 \theta$.
22. $1 + \sin \theta + \cos \theta = 0$.
23. $\tan \alpha - 1 = \sqrt{3}(\cot \alpha - 1)$.
24. $2 \sin \alpha - \csc \alpha = 1$.
25. $\sec \alpha = 1 + \tan \alpha$.
26. $\sin \alpha - \cos \alpha = 1$.
27. $6 \sin^2 \beta - 5 \sin \beta + 1 = 0$.
28. $\cot^2 \beta - 3 \cot \beta = 4$.
29. $3 \sec^2 \beta + \cot^2 \beta = 7$.
30. $3 \sec^2 \beta - \cot^2 \beta = 5$.

Chapter 16

Functions of Two Angles

122. Introduction. We learned in algebra that, if a, b, and c are any three numbers, $a(b + c) = ab + ac$. By analogy, one would expect $\sin(\alpha + \beta)$ to equal $\sin\alpha + \sin\beta$. That this is not so can be shown by actual trial, as in the following example.

EXAMPLE. Given $\alpha = 30°$, $\beta = 60°$, find $\sin(\alpha + \beta)$ and $\sin\alpha + \sin\beta$.

$$\sin(\alpha + \beta) = \sin(30° + 60°) = \sin 90° = 1.$$

$$\sin\alpha + \sin\beta = \sin 30° + \sin 60° = \frac{1}{2} + \frac{\sqrt{3}}{2} = \frac{1 + 1.732}{2} = 1.366.$$

Since $1 \neq 1.366$, in this case, at least, $\sin(\alpha + \beta) \neq \sin\alpha + \sin\beta$.

In the next article, we shall take up the problem of finding correct expressions for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$ in terms of the sines and cosines of α and β.

123. The Sine and Cosine of the Sum of Two Angles. We shall denote the two angles by α and β. In the present article, we suppose both of these angles to be acute. Their sum may be acute, as in Figure 51*a*, or obtuse, as in Figure 51*b*.

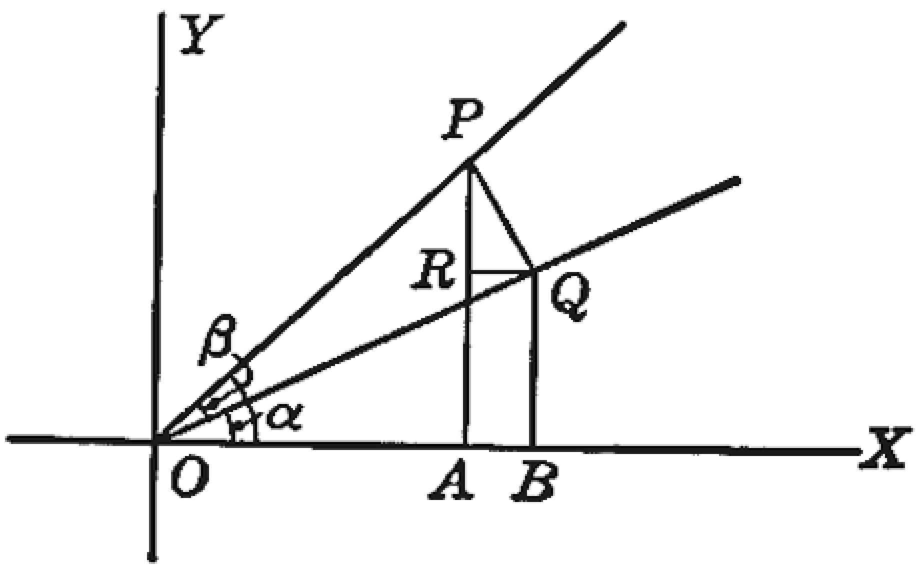


FIG. 51*a*

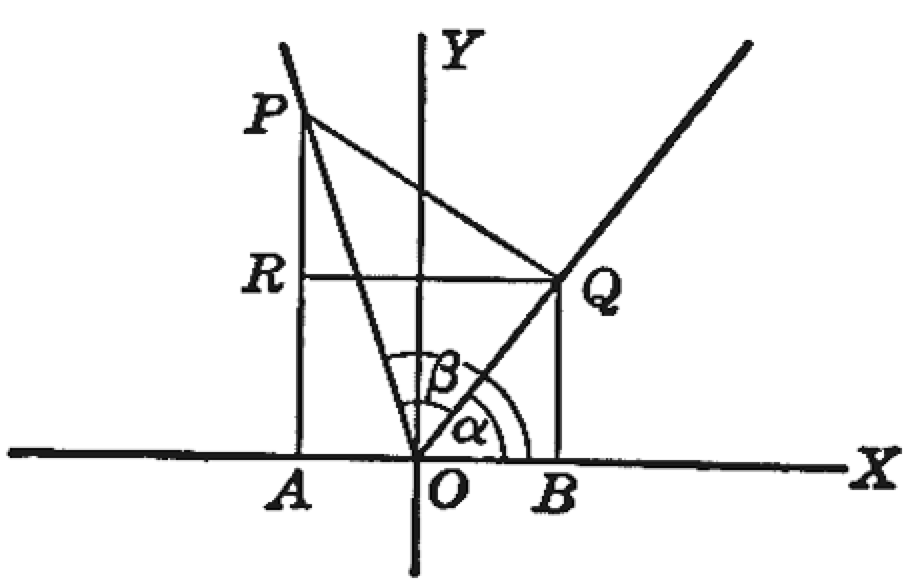


FIG. 51*b*

Place the angle α in standard position and place β with its initial side on the terminal side of α. Let P be any point on the terminal side of β. Then the angle $XOP = \alpha + \beta$.

From P draw PA perpendicular to OX and PQ perpendicular to the terminal side of α. From Q, draw QB perpendicular to OX and QR perpendicular to AP.

Then $AB = RQ$ and $AR = BQ$, since they are pairs of opposite sides of a rectangle. Also, angle $RPQ = \alpha$, since both angles are acute and their sides are respectively perpendicular.

By the definition of the sine of an angle,

$$\sin(\alpha+\beta) = \frac{AP}{OP} = \frac{AR+RP}{OP} = \frac{BQ}{OP} + \frac{RP}{OP} = \frac{BQ}{OP}\cdot\frac{OQ}{OQ} + \frac{RP}{OP}\cdot\frac{QP}{QP}$$
$$= \frac{BQ}{OQ}\cdot\frac{OQ}{OP} + \frac{RP}{QP}\cdot\frac{QP}{OP}.$$

But $\frac{BQ}{OQ} = \sin\alpha$, $\frac{OQ}{OP} = \cos\beta$, $\frac{RP}{QP} = \cos\alpha$, and $\frac{QP}{OP} = \sin\beta$. On substituting these values in the last member of the preceding series of equations, we have

$$\mathbf{\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.} \qquad \text{I}$$

In a similar way, starting from the definition of the cosine, we have

$$\cos(\alpha+\beta) = \frac{OA}{OP} = \frac{OB-AB}{OP} = \frac{OB}{OP} - \frac{RQ}{OP} = \frac{OB}{OP}\cdot\frac{OQ}{OQ} - \frac{RQ}{OP}\cdot\frac{QP}{QP}$$
$$= \frac{OB}{OQ}\cdot\frac{OQ}{OP} - \frac{RQ}{QP}\cdot\frac{QP}{OP}.$$

But $\frac{OB}{OQ} = \cos\alpha$, $\frac{OQ}{OP} = \cos\beta$, $\frac{RQ}{QP} = \sin\alpha$, and $\frac{QP}{OP} = \sin\beta$. On making these substitutions, we have

$$\mathbf{\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.} \qquad \text{II}$$

Formulas I and II should be memorized in words, thus:

The sine of the sum of two angles equals the sine of the first times the cosine of the second plus the cosine of the first times the sine of the second.

The cosine of the sum of two angles equals the cosine of the first times the cosine of the second minus the sine of the first times the sine of the second.

The verification that formulas I and II hold when either α or β, or their sum, is 0° or 90° is left as an exercise for the student (see Art. 112).

EXAMPLE. Find the values of sin 75° and cos 75°.

$$\sin 75° = \sin(45° + 30°) = \sin 45°\cos 30° + \cos 45°\sin 30°$$
$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}.$$
$$\cos 75° = \cos(45° + 30°) = \cos 45°\cos 30° - \sin 45°\sin 30°$$
$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}.$$

124. Extension of the Formulas to Angles of Any Size. We know that formulas I and II hold if α and β lie in the interval 0° to 90°. We now wish to show that they still hold if α and β are positive or negative angles of any size.

Let α and β be any two angles for which formulas I and II are known to hold. We shall first show that these formulas are still true if either α or β is increased by 90°.

Let $\alpha' = 90° + \alpha$. By formulas (2) of Art. 112,

$$\sin \alpha' = \sin (90° + \alpha) = \cos \alpha; \cos \alpha' = \cos (90° + \alpha) = -\sin \alpha.$$

Further,

$$\begin{aligned} \sin (\alpha' + \beta) &= \sin (90° + \alpha + \beta) = \cos (\alpha + \beta) \\ &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \qquad (1) \\ &= \sin \alpha' \cos \beta + \cos \alpha' \sin \beta. \\ \cos (\alpha' + \beta) &= \cos (90° + \alpha + \beta) = -\sin (\alpha + \beta) \\ &= -\sin \alpha \cos \beta - \cos \alpha \sin \beta \qquad (2) \\ &= \cos \alpha' \cos \beta - \sin \alpha' \sin \beta. \end{aligned}$$

It follows that, if formulas I and II hold for the angles α and β, they hold also for the angles $\alpha' = 90° + \alpha$ and β. A precisely similar proof will hold when β is increased by 90°. We already know that these formulas hold for all angles α and β in the interval 0° to 90°. It now follows that they hold for all angles in the interval 0° to 180°. But, if they hold from 0° to 180°, they hold to 270°, and so on to any positive angle whatever.

Further, equations (1) and (2) show that, if α' and β are the angles for which formulas I and II are known to be true, then $\alpha = \alpha' - 90°$ and β are also angles for which these formulas are true and a similar proof holds for α and $\beta = \beta' - 90°$. Hence the formulas are true for all negative angles. It now follows that these formulas hold for all angles, positive, negative, and zero.

125. The Tangent and Cotangent of the Sum. We have

$$\tan (\alpha + \beta) = \frac{\sin (\alpha + \beta)}{\cos (\alpha + \beta)} = \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta - \sin \alpha \sin \beta}.$$

Divide each term of the numerator and denominator of the last member by $\cos \alpha \cos \beta$. The result is

$$\tan (\alpha + \beta) = \frac{\dfrac{\sin \alpha}{\cos \alpha} + \dfrac{\sin \beta}{\cos \beta}}{1 - \dfrac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta}},$$

or

$$\tan (\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}. \qquad \text{III}$$

Similarly, we have

$$\cot (\alpha + \beta) = \frac{\cos (\alpha + \beta)}{\sin (\alpha + \beta)} = \frac{\cos \alpha \cos \beta - \sin \alpha \sin \beta}{\sin \alpha \cos \beta + \cos \alpha \sin \beta}.$$

Divide each term in the numerator and denominator by $\sin \alpha \sin \beta$ and simplify. We obtain

$$\cot (\alpha + \beta) = \frac{\cot \alpha \cot \beta - 1}{\cot \beta + \cot \alpha}. \qquad \text{IV}$$

EXAMPLE. Find the value of cot 75°.

$$\cot 75^\circ = \cot (45^\circ + 30^\circ) = \frac{\cot 45^\circ \cot 30^\circ - 1}{\cot 30^\circ + \cot 45^\circ}$$
$$= \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = \frac{(\sqrt{3} - 1)(\sqrt{3} - 1)}{(\sqrt{3} + 1)(\sqrt{3} - 1)} = 2 - \sqrt{3}.$$

126. Functions of the Difference of Two Angles. Since $\alpha - \beta = \alpha + (-\beta)$, we can, by replacing β by $-\beta$ in formulas I to IV, express the sine, cosine, tangent, and cotangent of $\alpha - \beta$ in terms of functions of α and $-\beta$. Further, by using equations (1) of Art. 111, we can express the functions of $-\beta$ in terms of those of β. Thus,

$$\sin (\alpha - \beta) = \sin [\alpha + (-\beta)] = \sin \alpha \cos (-\beta) + \cos \alpha \sin (-\beta)$$
$$= \sin \alpha \cos \beta - \cos \alpha \sin \beta;$$

that is,

$$\boldsymbol{\sin (\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta.} \qquad \text{V}$$

In a similar way, we find that

$$\boldsymbol{\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta.} \qquad \text{VI}$$

$$\boldsymbol{\tan (\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}.} \qquad \text{VII}$$

$$\boldsymbol{\cot (\alpha - \beta) = \frac{\cot \alpha \cot \beta + 1}{\cot \beta - \cot \alpha}.} \qquad \text{VIII}$$

Exercises

1. Find the values of sin 45° + sin 30°, cos 45° + cos 30°, and cot 45° + cot 30°. Compare your results with the values of sin (45° + 30°), cos (45° + 30°), and cot (45° + 30°) found in the examples of Arts. 123 and 125.

2. Using the equation 105° = 60° + 45°, find the values of sin 105°, cos 105°, tan 105°, and cot 105°.

3. Using the equation 15° = 45° − 30°, find the values of sin 15°, cos 15°, tan 15°, and cot 15°.

Find the value of $\sin (\alpha + \beta)$, $\cos (\alpha + \beta)$, $\sin (\alpha - \beta)$, and $\cos (\alpha - \beta)$, given:

4. $\sin \alpha = \frac{3}{5}$, $\cos \beta = \frac{5}{13}$; α and β in the first quadrant.

5. $\sin \alpha = \frac{7}{25}$, $\tan \beta = -\frac{8}{15}$; α and β in the second quadrant.

6. $\tan \alpha = -\frac{4}{3}$, $\tan \beta = \frac{3}{4}$; α in the fourth quadrant, β in the third.

7. $\tan \alpha = -\frac{5}{12}$, $\tan \beta = -\frac{15}{8}$; α in the second quadrant, β in the fourth.

8. $\cos \alpha = \frac{1}{4}$, $\cot \beta = \frac{1}{2}$; α in the fourth quadrant, β in the third.

9. $\csc \alpha = -3$, $\tan \beta = 5$; α and β in the same quadrant.

10. Given $\tan \alpha = 0.6959$ and $\tan \beta = 0.4108$, α and β both positive and less than 90°. Find $\tan (\alpha + \beta)$, using formula III. Verify your result, using Table III.

Express the following functions in terms of functions of θ.

11. $\sin (45° + \theta)$. **12.** $\cos (30° - \theta)$. **13.** $\tan (135° + \theta)$.

14. $\cos \left(\frac{\pi}{3} - \theta\right)$. **15.** $\sin \left(\frac{2\pi}{3} + \theta\right)$. **16.** $\cot \left(\frac{3\pi}{4} - \theta\right)$.

Prove the following identities.

17. $\sin (60° + x) - \cos (30 + x) = \sin x$.

18. $\tan \left(\frac{\pi}{4} + x\right) = \cot \left(\frac{\pi}{4} - x\right) = \frac{1 + \tan x}{1 - \tan x}$.

19. $\cos \left(x + \frac{\pi}{6}\right) - \cos \left(x - \frac{\pi}{6}\right) = - \sin x$.

20. $\sin (x + 60°) + \sin (x - 60°) = \sin x$.

21. $\sin (x + y) \sin (x - y) = \sin^2 x - \sin^2 y$.

22. $\cos (x + y) \cos (x - y) = \cos^2 x - \sin^2 y$.

23. $\sin 3\theta \cos \theta - \cos 3\theta \sin \theta = \sin 2\theta$.

HINT. Use formula V with $\alpha = 3\theta$ and $\beta = \theta$.

24. $\cos 3\theta \cos \theta + \sin 3\theta \sin \theta = \cos 2\theta$.

25. Express $\sin (\alpha + \beta + \gamma)$ in terms of functions of α, β, and γ.

HINT. $\sin (\alpha + \beta + \gamma) = \sin [(\alpha + \beta) + \gamma]$.

26. Express $\cos (\alpha + \beta + \gamma)$ in terms of functions of α, β, and γ.

127. Functions of Twice an Angle. If, in formulas I to IV, we put $\beta = \alpha$, we obtain formulas for the functions of 2α. Thus, from I, we have

$$\sin (\alpha + \alpha) = \sin \alpha \cos \alpha + \cos \alpha \sin \alpha;$$

that is,

$$\mathbf{\sin 2\alpha = 2 \sin \alpha \cos \alpha.} \qquad \text{IX}$$

From formulas II, III, and IV, we obtain, in the same way,

$$\begin{aligned} \cos 2\alpha &= \cos^2 \alpha - \sin^2 \alpha \\ &= 2 \cos^2 \alpha - 1 \qquad \text{X} \\ &= 1 - 2 \sin^2 \alpha. \end{aligned}$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}. \qquad \text{XI}$$

$$\cot 2\alpha = \frac{\cot^2 \alpha - 1}{2 \cot \alpha}. \qquad \text{XII}$$

Exercises

Using the known values of the functions of 30°, 45°, 120°, and 150° and the double angle formulas, find expressions for the sine, cosine, tangent, and cotangent of each of the following angles.

1. 60°. **2.** 90°. **3.** 240°. **4.** 300°.

Find the values of $\sin 2x$, $\cos 2x$, $\tan 2x$, and $\cot 2x$, given that x is in the quadrant indicated and that:

5. $\sin x = \frac{4}{5}$; second.
6. $\tan x = \frac{8}{15}$; third.
7. $\cos x = -\frac{12}{13}$; second.
8. $\sec x = \frac{25}{24}$; fourth.
9. $\cot x = -3$; fourth.
10. $\csc x = 4$; second.

Prove the following identities.

11. $2 \csc 2\theta = \sec \theta \csc \theta$.
12. $\sin 2\theta (\tan \theta + \cot \theta) = 2$.
13. $\dfrac{\sin 2\theta}{2} = \dfrac{\tan \theta}{1 + \tan^2 \theta}$.
14. $\cos 2\theta = \dfrac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$.
15. $\dfrac{\sin 2x}{1 - \cos 2x} = \cot x$.
16. $\sec 2x = \dfrac{\sec^2 x}{2 - \sec^2 x}$.
17. $\sec 2x + \tan 2x = \dfrac{1 + \tan x}{1 - \tan x}$.
18. $2 \cot 2x = \cot x - \tan x$.
19. $\sin 3x = 3 \sin x - 4 \sin^3 x$.
20. $\cos 3x = 4 \cos^3 x - 3 \cos x$.
21. $\cos 4x = 1 - 8 \sin^2 x \cos^2 x$.
22. $(\sin x - \cos x)^2 = 1 - \sin 2x$.
23. $8 \cos^4 x = 3 + 4 \cos 2x + \cos 4x$.
24. $\sin 4x = 4 (\sin x \cos^3 x - \sin^3 x \cos x)$.

128. Functions of Half an Angle. From formulas X, we have the following two equations,

$$\cos 2\alpha = 1 - 2 \sin^2 \alpha, \quad \cos 2\alpha = 2 \cos^2 \alpha - 1.$$

If we solve the first of these equations for $\sin \alpha$ and the second for $\cos \alpha$, we get

$$\sin \alpha = \pm \sqrt{\frac{1 - \cos 2\alpha}{2}}, \quad \text{and} \quad \cos \alpha = \pm \sqrt{\frac{1 + \cos 2\alpha}{2}}.$$

Let us denote the angle 2α by θ. Then $\alpha = \theta/2$ and the preceding two equations become

$$\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}}, \qquad \text{XIII}$$

and

$$\cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}}. \qquad \text{XIV}$$

The sign before the radical, in each of these equations, is to be determined by the quadrant in which $\theta/2$ lies.

Since $\tan \theta/2 = \dfrac{\sin (\theta/2)}{\cos (\theta/2)}$, we have by substituting the values of $\sin (\theta/2)$ and $\cos (\theta/2)$ from XIII and XIV and simplifying,

$$\tan \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}}. \qquad \text{XV}$$

Similarly,

$$\cot \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{1 - \cos \theta}}. \qquad \text{XVI}$$

These equations can be written in a form free from radicals. Thus,

$$\tan\frac{\theta}{2}=\frac{\sin(\theta/2)}{\cos(\theta/2)}\frac{2\cos(\theta/2)}{2\cos(\theta/2)}=\frac{2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)}=\frac{\sin\theta}{1+\cos\theta}=\frac{1-\cos\theta}{\sin\theta}.$$

$$\cot\frac{\theta}{2}=\frac{1+\cos\theta}{\sin\theta}=\frac{\sin\theta}{1-\cos\theta}.$$

Exercises

Using the known values of the functions of 30°, 45°, 135°, and 210° and the half-angle formulas, find the values of the sine, cosine, tangent, and cotangent of each of the following angles.

1. 15°. **2.** 22° 30′. **3.** 67° 30′. **4.** 105°.

Find the values of $\sin(x/2)$, $\cos(x/2)$, $\tan(x/2)$, and $\cot(x/2)$, given that x is positive, less than 360°, lies in the quadrant indicated, and that:

5. $\cos x=\frac{3}{5}$; first. **6.** $\sin x=\frac{24}{25}$; second.

7. $\tan x=-\frac{40}{9}$; fourth. **8.** $\cot x=\frac{119}{120}$; third.

9. $\sec x=-5$; third. **10.** $\cos x=-\frac{2}{7}$; second.

Prove the following identities.

11. $2\csc x=\tan\frac{x}{2}+\cot\frac{x}{2}.$ **12.** $\sec^2\frac{x}{2}=\frac{2\tan x}{\tan x+\sin x}.$

13. $\csc x-\cot x=\tan\frac{x}{2}.$ **14.** $\tan^2\frac{x}{2}=\frac{\sec x-1}{\sec x+1}.$

15. $\frac{1+\tan(x/2)}{1-\tan(x/2)}=\sec x+\tan x.$ **16.** $\cot x=\frac{\cot^2(x/2)-1}{2\cot(x/2)}.$

17. $\cos x=\frac{1-\tan^2(x/2)}{\sec^2(x/2)}.$ **18.** $\tan\left(45°+\frac{x}{2}\right)=\cot\left(45°-\frac{x}{2}\right).$

19. $\cot\frac{x}{2}-\tan\frac{x}{2}=2\cot x.$ **20.** $\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2=1+\sin x.$

129. The Product Formulas. In Arts. 123 and 126, we derived the following four formulas.

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta, \tag{3}$$
$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta, \tag{4}$$
$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta, \tag{5}$$
$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta. \tag{6}$$

Let us (*a*) add equations (3) and (4), (*b*) subtract (4) from (3), (*c*) add (5) and (6), and (*d*) subtract (6) from (5). The results are:

$$\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta, \tag{7}$$
$$\sin(\alpha+\beta)-\sin(\alpha-\beta)=2\cos\alpha\sin\beta, \tag{8}$$
$$\cos(\alpha+\beta)+\cos(\alpha-\beta)=2\cos\alpha\cos\beta, \tag{9}$$
$$\cos(\alpha+\beta)-\cos(\alpha-\beta)=-2\sin\alpha\sin\beta. \tag{10}$$

If we divide equations (7) and (9) by 2, and (10) by -2, we have

$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)], \qquad \text{XVII}$$
$$\cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)], \qquad \text{XVIII}$$
$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]. \qquad \text{XIX}$$

Formula XVII expresses the product of a sine and a cosine; XVIII, the product of two cosines; and XIX, the product of two sines, as the sum or difference of two sines or two cosines.

130. The Sum or Difference of Sines or Cosines. It is sometimes necessary to write the sum or the difference of two sines or two cosines as a product. If we put

$$\alpha+\beta = X, \qquad \alpha-\beta = Y,$$

and solve for α and β, we get

$$\alpha = \frac{X+Y}{2}, \qquad \beta = \frac{X-Y}{2}.$$

Substitute these values of $\alpha+\beta$, $\alpha-\beta$, α, and β in equations (7) to (10). The resulting equations are

$$\sin X + \sin Y = 2\sin\frac{X+Y}{2}\cos\frac{X-Y}{2}, \qquad \text{XX}$$
$$\sin X - \sin Y = 2\cos\frac{X+Y}{2}\sin\frac{X-Y}{2}, \qquad \text{XXI}$$
$$\cos X + \cos Y = 2\cos\frac{X+Y}{2}\cos\frac{X-Y}{2}, \qquad \text{XXII}$$
$$\cos X - \cos Y = -2\sin\frac{X+Y}{2}\sin\frac{X-Y}{2}. \qquad \text{XXIII}$$

EXAMPLE 1. Express as a sum or difference: $2\sin 3x\cos 4x$.

Use formula XVII, with $\alpha = 3x$ and $\beta = 4x$. We have

$$2\sin 3x\cos 4x = \sin 7x + \sin(-x) = \sin 7x - \sin x.$$

EXAMPLE 2. Prove the identity: $\dfrac{\sin x - \sin y}{\cos x + \cos y} = \tan\dfrac{x-y}{2}$.

From formulas XXI and XXII, we have

$$\frac{\sin x - \sin y}{\cos x + \cos y} = \frac{2\cos\frac{x+y}{2}\sin\frac{x-y}{2}}{2\cos\frac{x+y}{2}\cos\frac{x-y}{2}} = \frac{\sin\frac{x-y}{2}}{\cos\frac{x-y}{2}} = \tan\frac{x-y}{2}.$$

Exercises

1. State formulas XVII to XIX in words.
2. State formulas XX to XXIII in words.

Express each product as a sum or difference.

3. $2 \sin 55° \cos 15°$.

4. $2 \sin 15° \cos 55°$.

5. $2 \sin 55° \sin 15°$.

6. $2 \cos 55° \cos 15°$.

7. $\sin 3x \cos 2x$.

8. $\cos 5x \cos 4x$.

9. $\sin 3x \sin 5x$.

10. $\cos 7x \sin 2x$.

Express each sum or difference as a product.

11. $\sin 50° + \sin 20°$.

12. $\cos 40° - \cos 30°$.

13. $\cos 25° - \cos 75°$.

14. $\sin 80° + \sin 50°$.

15. $\sin 4x - \sin 2x$.

16. $\cos 3x + \cos 5x$.

17. $\cos 10x - \cos 4x$.

18. $\sin 9x - \sin 7x$.

19. $\sin 68° - \cos 34°$.

20. $\cos 53° + \sin 21°$.

HINT. $\sin 68° = \cos 22°$.

Find the values of each of the following quantities.

21. $\dfrac{\sin 70° + \sin 50°}{\cos 70° + \cos 50°}$.

22. $\dfrac{\cos 65° - \cos 25°}{\sin 65° - \sin 25°}$.

Prove the identities.

23. $\dfrac{\cos 2\theta - \cos 2\phi}{\cos 2\theta + \cos 2\phi} = \dfrac{\tan (\phi - \theta)}{\cot (\phi + \theta)}$.

24. $\dfrac{\sin 2\theta + \sin 2\phi}{\sin 2\theta - \sin 2\phi} = \dfrac{\cot (\theta - \phi)}{\cot (\theta + \phi)}$.

25. $\dfrac{\cos 3x - \cos 5x}{\sin 3x + \sin 5x} = \tan x$.

26. $\dfrac{\cos 3x - \cos x}{\sin 3x - \sin x} = \dfrac{2 \tan x}{\tan^2 x - 1}$.

27. $\dfrac{\cos 7x - \cos 9x}{\sin 6x - \sin 4x} = \dfrac{\sin 8x}{\cos 5x}$.

28. $\dfrac{\sin 8x + \sin 4x}{\cos 5x + \cos x} = 2 \sin 3x$.

29. $\dfrac{\sin (2\alpha - x) + \sin x}{\cos (2\alpha - x) + \cos x} = \tan \alpha$.

30. $\dfrac{\cos x - \cos (x - 2\alpha)}{\sin x + \sin (x - 2\alpha)} = -\tan \alpha$.

31. $\cot x + \cot 2x = \dfrac{\sin 3x}{\sin x \sin 2x} = \dfrac{2 \sin 3x}{\cos x - \cos 3x}$.

32. $1 + \tan 3x \tan 2x = \dfrac{\cos x}{\cos 3x \cos 2x} = \dfrac{2 \cos x}{\cos 5x + \cos x}$.

33. $\dfrac{\sin 2x + \sin x}{1 + \cos 2x + \cos x} = \tan x$.

34. $\dfrac{1 - \cos x + \sin x}{1 + \cos x + \sin x} = \tan \dfrac{x}{2}$.

35. $\sin 2x + \sin 4x + \sin 6x = 4 \cos x \cos 2x \sin 3x$.

36. $\cos 2x + \cos 4x + \cos 6x = 4 \cos x \cos 2x \cos 3x - 1$.

37. $\dfrac{\sin x + \sin 2x + \sin 3x}{\cos x + \cos 2x + \cos 3x} = \tan 2x$.

131. Equations of Condition. The process of solving an equation of condition involving two or more angles follows the methods outlined in Art. 121 with the additional restriction that we try to write the equation either in a form involving only functions of one angle or else to write it as a product of factors such that each factor contains only functions of one angle.

EXAMPLE 1. Solve the equation: $\cos 2x = \sin x$.

Since, by formula X, $\cos 2x = 1 - 2\sin^2 x$, we can write this equation in the form

$$1 - 2\sin^2 x = \sin x, \quad \text{or} \quad 2\sin^2 x + \sin x - 1 = 0.$$

On solving this quadratic in $\sin x$, we find that either,

$$\sin x = -1, \text{ giving } x = 270°, \quad \text{or} \quad \sin x = \tfrac{1}{2}, \text{ giving } x = 30° \quad \text{or} \quad 150°.$$

The required solutions are 30°, 150°, and 270°.

EXAMPLE 2. Solve the equation: $\sec x \cos 3x + 1 = 0$.

Multiply by $\cos x$: $\cos 3x + \cos x = 0$.

By formula XXII, this becomes: $2\cos 2x \cos x = 0$.

If we put $\cos x = 0$, we get $x = 90°$ or $270°$ but these are extraneous since they do not satisfy the given equation. They were introduced when we multiplied by $\cos x$.

If we put $\cos 2x = 0$, we get

$$2x = 90°, 270°, 450°, \quad \text{or} \quad 630°.$$

Hence $$x = 45°, 135°, 225°, \quad \text{or} \quad 315°.$$

The required solutions are $x = 45°, 135°, 225°$, and $315°$.

Exercises

Find all the positive or zero angles less than 360° that satisfy the given equation.

1. $\sin 2x = \sin x$.

2. $\sin 2x = \cos x$.

3. $\cos 2x + \cos x = 0$.

4. $\cos 2x + \sin x = 0$.

5. $\tan 2x + \tan x = 0$.

6. $\tan x \tan 2x = 1$.

7. $\cos x/2 - 1 = \cos x$.

8. $\sin x/2 + \cos x = 1$.

9. $\cos 3x - \cos x = \sin 2x$.

10. $\sin 3x + \sin 2x + \sin x = 0$.

11. $\cos 2x - \sin 2x = 1$.

12. $(\cos^2 3x - 1)(2\sin 2x + 1) = 0$.

13. $\cos 5x + 2\sin 2x = \cos x$.

14. $\sin 2x + \cos 2x + 2\sin x = 1$.

Chapter 17

The Inverse Trigonometric Functions

132. Notation. We know that $\sin 30° = \frac{1}{2}$. If, therefore, someone asks us to name an angle whose sine is $\frac{1}{2}$, we can answer that $30°$ is such an angle.

We shall need a symbol to denote an angle whose sine is a given number x. Two such symbols are in common use, namely,

$$\sin^{-1} x, \quad \text{and} \quad \text{arc} \sin x.$$

These two symbols are equivalent and may be used interchangeably. Both symbols may be read as either: *the inverse sine of x*, or *the arc sine of x*, or *an angle whose sine is x*. Throughout this chapter, and until the student is thoroughly familiar with these symbols, the last form of statement is to be preferred.

Observe that, in the symbol $\sin^{-1} x$, the -1 is *not* an exponent. The entire symbol $\sin^{-1} x$ is merely a short way of writing the statement, "an angle whose sine is x." When we wish to use -1 as an exponent for $\sin x$, we write the required expression in the form $(\sin x)^{-1}$.

Thus, the statement, $45° = \sin^{-1} \frac{\sqrt{2}}{2}$, should be read, "$45°$ is an angle whose sine is $\frac{\sqrt{2}}{2}$." Similarly, $240° = \text{arc} \sin\left(-\frac{\sqrt{3}}{2}\right)$, is read, "$240°$ is an angle whose sine is $-\frac{\sqrt{3}}{2}$," and so on.

Because the statement

$$\sin y = x \quad \text{implies} \quad y = \text{arc} \sin x,$$

and conversely, we say that each of these statements is the **inverse** of the other. Further, because of this inverse relation, we shall say that the two functions, arc sin x and sin x, are **inverse functions.**

A similar notation holds for each of the other five trigonometric functions. We have, in fact, the following six pairs of symbols:

$$\begin{array}{llll} \sin^{-1} x, & \text{or} \quad \text{arc} \sin x; & \csc^{-1} x, & \text{or} \quad \text{arc} \csc x; \\ \cos^{-1} x, & \text{or} \quad \text{arc} \cos x; & \sec^{-1} x, & \text{or} \quad \text{arc} \sec x; \\ \tan^{-1} x, & \text{or} \quad \text{arc} \tan x; & \cot^{-1} x, & \text{or} \quad \text{arc} \cot x. \end{array}$$

These symbols may be read, respectively, as, "an angle whose sine is x," "an angle whose cosine is x," "an angle whose tangent is x," and so on.

As in the case of the sine and the inverse sine, the statements

$$\begin{aligned} \cos y = x \quad &\text{and} \quad y = \text{arc} \cos x, \\ \tan y = x \quad &\text{and} \quad y = \text{arc} \tan x, \end{aligned}$$

and so on, are equivalent and the pairs of functions arc cos x and cos x, arc tan x and tan x are **inverse functions.**

Thus, since $\cos 120° = -\frac{1}{2}$, $120° = \text{arc cos}\,(-\frac{1}{2})$;
since $\tan 225° = 1$, $225° = \text{arc tan}\,1$;

and so on.

133. Multiple Valuedness of the Inverse Functions. We saw that $30° = \text{arc sin}\,\frac{1}{2}$. But, equally, since $\sin 150° = \frac{1}{2}$, we have also $150° = \text{arc sin}\,\frac{1}{2}$. Further, since $\sin x$ is periodic with period 360° (Art. 115), if we add to (or subtract from) either of these angles an integral multiple of 360°, we obtain an angle whose sine is $\frac{1}{2}$; that is, any one of the angles

$$-330°,\ -210°,\ 30°,\ 150°,\ 390°,\ 510°,$$

and so on in either direction, is an angle whose sine is $\frac{1}{2}$.

More generally, if y is an angle such that

$$\sin y = x, \quad \text{then, also,} \quad \sin\,(180° - y) = x,$$

and, further, if n is zero or any integer,

$$\sin\,(y \pm n360°) = x, \quad \text{and} \quad \sin\,(180° - y \pm n360°) = x.$$

It follows that *if y is one angle such that $y = \text{arc sin}\,x$, then any one of the angles*

$$y \pm n360°, \quad \text{and} \quad 180° - y \pm n360°, \tag{1}$$

where n is zero or any integer, is also an angle whose sine is x.

In a similar way, we find that, if y is any one angle such that

$$y = \text{arc csc}\,x,$$

then any one of the angles expressed by equations (1) is also an angle whose cosecant is x.

Again, if y is any angle such that $\cos y = x$, then, also,

$$\cos\,(\pm y \pm n360°) = x.$$

Hence, if y is one such angle, then *any one of the angles*

$$\pm y \pm n360°, \tag{2}$$

where n is zero or an integer, is an angle whose cosine is x.

Further, if y is any angle such that

$$y = \text{arc sec}\,x,$$

then any angle defined by the expression (2) is an angle whose secant is x.

Finally, in a similar way, we find that, if $\tan y = x$, then *any one of the angles*

$$y \pm n180°, \tag{3}$$

is an angle whose tangent is x and, if y is chosen so that $\cot y = x$, then (3) gives the angles whose cotangents are equal to x.

EXAMPLE 1. Find two positive and two negative values of arc sin $\left(-\frac{\sqrt{3}}{2}\right)$.

Since $\sin 240° = -\frac{\sqrt{3}}{2}$, $y = 240°$ is one such angle. Putting $y = 240°$ in (1), we find that $-120°$, $-60°$, $240°$, and $300°$ are four such angles.

EXAMPLE 2. Find two positive and two negative values of arc cot 1.

Since $\cot 45° = 1$, $y = 45°$ is one such angle. From (3), we now find that $-315°$, $-135°$, $45°$, and $225°$ are four such angles.

134. Principal Values. Among all the values an inverse trigonometric function may have for a given value of x, it has been found desirable to pick out one single value to represent the entire set. This one value, which has been chosen for its simplicity and usefulness, is called the **principal value** of the inverse function and is defined as follows:

The principal value of each of the functions $\sin^{-1} x$, $\csc^{-1} x$, $\tan^{-1} x$, *and* $\cot^{-1} x$ *is the numerically smallest (positive or negative) value of the function for the given value of* x.

The principal value of each of the functions $\cos^{-1} x$ *and* $\sec^{-1} x$ *is the smallest positive (or zero) value of the function for the given value of* x.

It follows from these definitions that, if the number x is positive, the principal values of all of the functions lie in the interval $0°$ to $90°$ but, if x is negative, the principal values of $\cos^{-1} x$ and $\sec^{-1} x$ lie in the interval $90°$ to $180°$ and those of all the others lie in the interval $0°$ to $-90°$.

When we wish to indicate that we are dealing with the principal value, only, of an inverse function, we shall begin the symbol with a capital letter.

Thus, $$\text{Arc sin } \sqrt{2}/2 = \text{Sin}^{-1} \sqrt{2}/2 = 45°.$$

These expressions (beginning with capital letters) must be carefully distinguished from arc sin $\sqrt{2}/2$ and $\sin^{-1} \sqrt{2}/2$ which may take any one of the values $45° \pm n360°$ or $135° \pm n360°$.

The student should verify, further, that

$$\text{Arc tan } (-\sqrt{3}) = \text{Tan}^{-1} (-\sqrt{3}) = -60°$$

and

$$\text{Arc sec } (-2) = \text{Sec}^{-1} (-2) = 120°.$$

EXAMPLE 1. Find the value of sin (Arc tan $\sqrt{3}/3$).

$$\sin (\text{Arc tan } \sqrt{3}/3) = \sin 30° = 1/2.$$

EXAMPLE 2. Find Arc sin (cos 120°).

$$\text{Arc sin } (\cos 120°) = \text{Arc sin } (-1/2) = -30°.$$

EXAMPLE 3. Express in terms of u and v: $\tan(\cos^{-1} u + \cos^{-1} v)$.

Let $\alpha = \cos^{-1} u$ and $\beta = \cos^{-1} v$. Then $\tan\alpha = \dfrac{\pm\sqrt{1-u^2}}{u}$ and $\tan\beta = \dfrac{\pm\sqrt{1-v^2}}{v}$

$$\tan(\cos^{-1} u + \cos^{-1} v) = \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$= \frac{\pm\dfrac{\sqrt{1-u^2}}{u} \pm \dfrac{\sqrt{1-v^2}}{v}}{1 \pm \dfrac{\sqrt{1-u^2}}{u}\dfrac{\sqrt{1-v^2}}{v}} = \frac{\pm v\sqrt{1-u^2} \pm u\sqrt{1-v^2}}{uv \pm \sqrt{1-u^2}\sqrt{1-v^2}}.$$

Exercises

Write two positive and two negative values of each of the following expressions.

1. arc sin $(-\sqrt{2}/2)$.
2. arc tan $\sqrt{3}$.
3. arc cos $(-1/2)$.
4. arc csc $2\sqrt{3}/3$.
5. arc cot (-1).
6. arc sec (-1).
7. $\sin^{-1} 0.9261$.
8. $\cot^{-1} 1.5013$.
9. $\tan^{-1} 0.1128$.
10. $\cos^{-1}(-0.5695)$.

Write each of the following principal angles in degrees and, for Exs. 11 to 16, write the angle also in radians.

11. Arc sec 2.
12. Arc cot $(-\sqrt{3})$.
13. Arc sin $(-\sqrt{3}/2)$.
14. Arc cos $(-\sqrt{2}/2)$.
15. $\text{Tan}^{-1} 0$.
16. $\text{Csc}^{-1}(-2)$.
17. $\text{Cot}^{-1} 1.3270$.
18. $\text{Sin}^{-1}(-0.9063)$.
19. $\text{Cos}^{-1}(-0.6602)$.
20. $\text{Tan}^{-1}(-1.1054)$.

Find the value of each of the following expressions.

21. sin (arc sin $\sqrt{3}/2$).
22. tan (arc cot 4).
23. tan (Arc cos $\sqrt{2}/2$).
24. cot (Arc sin 1).
25. sec (Arc sin $-1/2$).
26. sin (Arc sec -2).
27. $\text{Tan}^{-1}(\tan 225°)$.
28. Arc sin (tan 45°).
29. Arc cos (sin 315°).
30. Arc cot (sin 270°).
31. Arc sin (sin 240°).
32. Arc tan (tan 117°).

Express each of the following quantities in terms of u and v.

33. cos (2 arc sin u).
34. sin ($\frac{1}{2}$ arc cos u).
35. tan (2 arc tan u).
36. sin (arc tan u).
37. sin (2 arc sec u).
38. tan (arc sin u).
39. $\sin(\sin^{-1} u + \sin^{-1} v)$.
40. $\cos(\cos^{-1} u + \cos^{-1} v)$.
41. $\tan(\tan^{-1} u - \tan^{-1} v)$.
42. $\cot(\sin^{-1} u + \sin^{-1} v)$.

135. Graphs of the Inverse Trigonometric Functions. Since the equations,

$$y = \text{arc sin } x \quad \text{and} \quad x = \sin y,$$

are equivalent, the graph may be drawn from either form of the equation. But the graph of $x = \sin y$ differs from that of $y = \sin x$, which was given in Figure 46, only in that its position with respect to the x- and y-axes has been interchanged; that is, this curve (Fig. 52*a*) is a sine curve running along the y-axis. Similarly, the graph of

$$y = \text{arc cos } x \quad \text{or} \quad x = \cos y$$

is a cosine curve running along the y-axis (Fig. 52*b*).

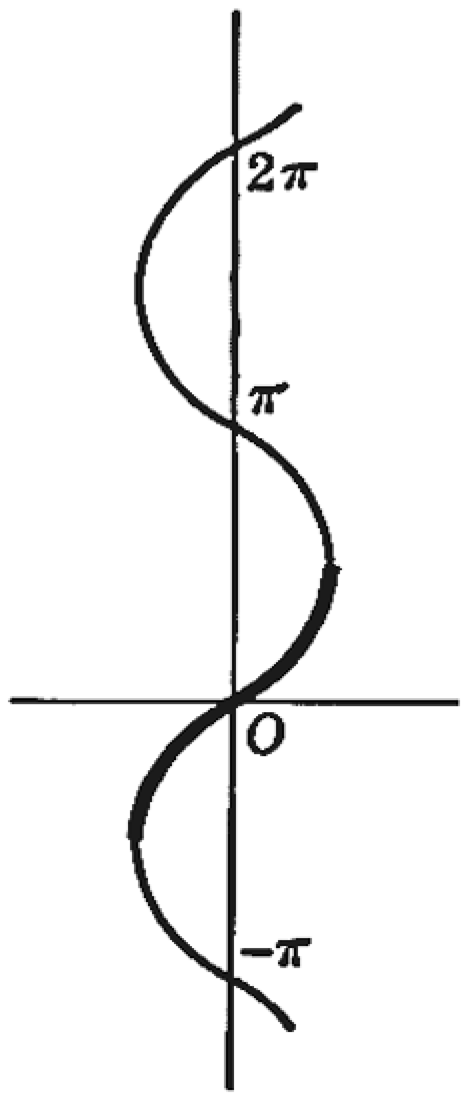


FIG. 52*a*

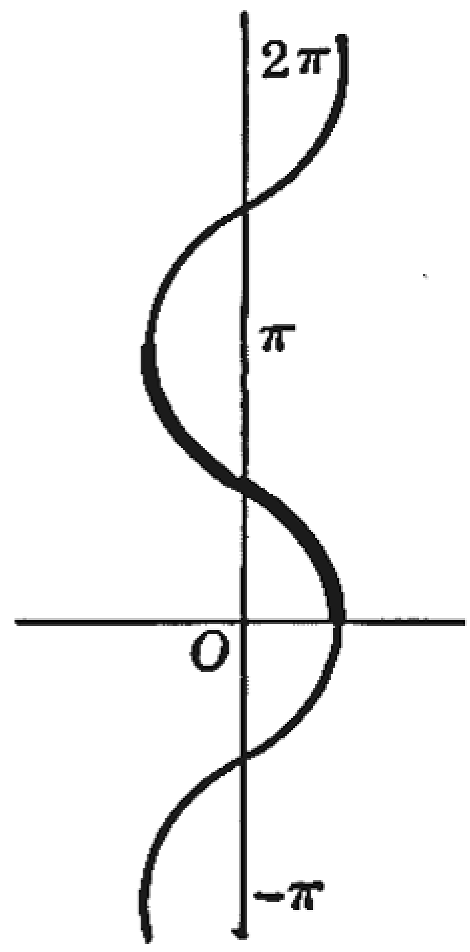


FIG. 52*b*

In these figures, we have indicated by a heavier line the portion of the curve corresponding to the principal values of the function; that is, to the portion corresponding to $y = \text{Arc sin } x$ in Figure 52*a* and to $y = \text{Arc cos } x$ in Figure 52*b*.

Exercises

Draw the graph of each of the following equations and indicate on it the part corresponding to the principal values of the function.

1. $y = \text{arc tan } x$. **2.** $y = \text{arc cot } x$.

3. $y = \text{arc sec } x$. **4.** $y = \text{arc csc } x$.

5. Find an interval on the x-axis for which arc sec x and arc csc x do not exist.

6. Is there any interval on the x-axis for which arc tan x and arc cot x do not exist?

Chapter 18

Solution of Oblique Triangles

136. Introduction. We shall denote the angles of the triangle by α, β, and γ, and the sides opposite these angles by a, b, and c, respectively. The six quantities α, β, γ, a, b, and c are the **parts** of the triangle. The problem of solving the triangle consists in finding three of these parts when the other three (of which one, at least, must be a side) are given. There are four cases which are numbered, according to the three parts that are given, as follows:

I. One side and two angles.
II. Two sides and the angle opposite one of them.
III. Two sides and their included angle.
IV. The three sides.

In this chapter, we shall set up suitable formulas and show how to carry through the solution of each of these four cases.

137. The Law of Sines. *In any triangle, the sides are proportional to the sines of the opposite angles;* that is,

$$\frac{a}{\sin\,\alpha} = \frac{b}{\sin\,\beta} = \frac{c}{\sin\,\gamma}. \tag{1}$$

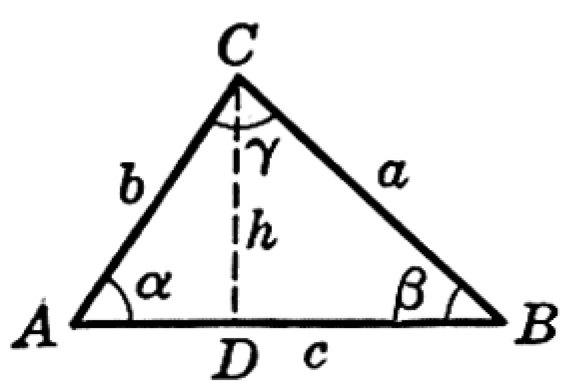


FIG. 53*a*

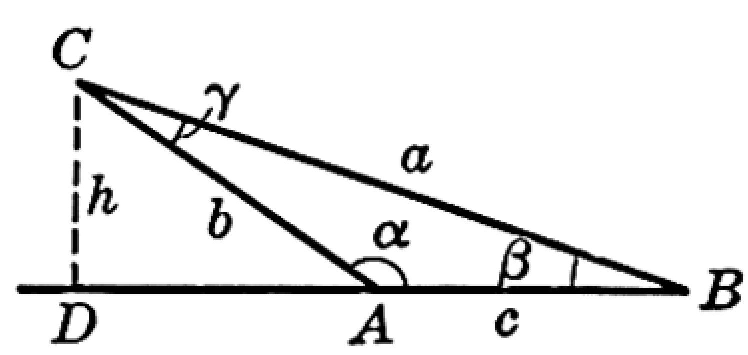


FIG. 53*b*

To prove the first of equations (1), drop a perpendicular CD from C to the opposite side (produced, if necessary) and denote this altitude by h.

From the right triangle BDC, we have $\sin\,\beta = h/a$, or $h = a\,\sin\,\beta$. From the right triangle ADC, we have $\sin\,\alpha = h/b$, or $h = b\,\sin\,\alpha$. Hence,

$$a\,\sin\,\beta = b\,\sin\,\alpha,$$

since each of these expressions is equal to h.

If we divide this equation by $\sin\,\alpha\,\sin\,\beta$, we have

$$\frac{a}{\sin\,\alpha} = \frac{b}{\sin\,\beta}.$$

In a similar way, if we drop a perpendicular from A to the side BC and equate the two values for this altitude, we find that

$$\frac{b}{\sin \beta} = \frac{c}{\sin \gamma}.$$

From these two equations, we have equations (1).

138. The Law of Tangents. Denote the common value of the three fractions in the law of sines by D.* We have

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} = D.$$

Solve for a and b:

$$a = D \sin \alpha, \quad b = D \sin \beta.$$

Subtract the second equation from the first; then add it to the first.

$$a - b = D(\sin \alpha - \sin \beta), \quad a + b = D(\sin \alpha + \sin \beta).$$

Divide these equations, member by member, then apply formulas XXI and XX of Art. 130.

$$\frac{a-b}{a+b} = \frac{\sin \alpha - \sin \beta}{\sin \alpha + \sin \beta} = \frac{2 \sin \frac{1}{2}(\alpha - \beta) \cos \frac{1}{2}(\alpha + \beta)}{2 \cos \frac{1}{2}(\alpha - \beta) \sin \frac{1}{2}(\alpha + \beta)},$$

or,

$$\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(\alpha - \beta)}{\tan \frac{1}{2}(\alpha + \beta)}.$$

Similarly,

$$\frac{b-c}{b+c} = \frac{\tan \frac{1}{2}(\beta - \gamma)}{\tan \frac{1}{2}(\beta + \gamma)}, \tag{2}$$

and

$$\frac{c-a}{c+a} = \frac{\tan \frac{1}{2}(\gamma - \alpha)}{\tan \frac{1}{2}(\gamma + \alpha)}.$$

These three formulas constitute the Law of Tangents. If b is greater than a, in the first formula, we can interchange a and b provided that we also interchange α and β. Similar statements hold for the other two formulas.

139. Case I. **Given One Side and Two Angles.** The third angle is found from the equation $\alpha + \beta + \gamma = 180°$. The two required sides are then found from the law of sines. As a check, we may use that formula of the law of tangents that involves the two computed sides. A figure, drawn to scale, will not only frequently be found helpful in indicating the method of solution but will also reveal any gross errors in the values of the computed parts.

The work may be arranged as shown on the following example. The entire form for the solution should be written out before any logarithms are looked for. If one logarithm, or the logarithms of two functions of

* It can be shown (Art. 146, Ex. 16) that D is the diameter of the circle circumscribed about the triangle ABC.

one angle, appears in two places in the computation, write it in both places at one opening of the table. Computations not involving logarithms may be made on scratch paper. Fill in the required parts at the top of the form as they are computed. If cologarithms are to be used, the form should be modified to correspond.

EXAMPLE. Solve the triangle: $a = 874.52$, $\alpha = 66° 18.4'$, $\beta = 42° 27.8'$.

Given:	Find:
$a = 874.52$,	$\gamma = 71° 13.8'$,
$\alpha = 66° 18.4'$,	$b = 644.74$,
$\beta = 42° 27.8'$.	$c = 904.24$.

$$\gamma = 180° - (\alpha + \beta),\ b = \frac{a \sin \beta}{\sin \alpha},\ c = \frac{a \sin \gamma}{\sin \alpha},\ \frac{c - b}{c + b} = \frac{\tan \frac{1}{2}(\gamma - \beta)}{\tan \frac{1}{2}(\gamma + \beta)}.$$

$$\begin{aligned} \log a &= 2.94177 \\ \log \sin \beta &= \underline{9.82938 - 10} + \\ &\ 12.77115 - 10 \\ \log \sin \alpha &= \underline{9.96176 - 10} - \\ \log b &= 2.80939 \end{aligned} \qquad \begin{aligned} \log a &= 2.94177 \\ \log \sin \gamma &= \underline{9.97627 - 10} + \\ &\ 12.91804 - 10 \\ \log \sin \alpha &= \underline{9.96176 - 10} - \\ \log c &= 2.95628 \end{aligned}$$

$$\begin{aligned} \log c - b &= 12.41414 - 10 \\ \log c + b &= \underline{3.19005} \quad - \\ &\ 9.22409 - 10 \end{aligned} \qquad \begin{aligned} \log \tan \tfrac{1}{2}(\gamma - \beta) &= 9.40900 - 10 \\ \log \tan \tfrac{1}{2}(\gamma + \beta) &= \underline{0.18494} \quad - \\ &\ 9.22406 - 10 \end{aligned}$$

Exercises

Solve the following triangles.

1. $c = 3.124$, $\alpha = 77° 15'$, $\gamma = 41° 24'$.
2. $b = 92.34$, $\alpha = 75° 24'$, $\gamma = 43° 58'$.
3. $a = 2.7368$, $\alpha = 34° 36.2'$, $\gamma = 67° 13.5'$.
4. $b = 84.291$, $\beta = 57° 15.2'$, $\gamma = 78° 18.3'$.
5. $c = 5716.3$, $\alpha = 26° 19.7'$, $\gamma = 41° 52.6'$.
6. $a = 463.71$, $\beta = 102° 34.1'$, $\gamma = 21° 32.8'$.
7. $b = 3.1847$, $\alpha = 76° 51.4'$, $\beta = 43° 27.4'$.
8. $c = 0.51386$, $\alpha = 26° 43.9'$, $\beta = 57° 50.3'$.

9. In a parallelogram $ABCD$, the angle at A is $32° 14'$. The length of the diagonal BD is 3.473 feet, and the angle ABD is $56° 41'$. Find, to four significant figures, the lengths of the sides.

10. Two shore batteries at A and B, 834.2 yards apart, are firing at a target at C. The angle ABC is $73° 21'$ and the angle BAC is $68° 52'$. Find the distances AC and BC to one decimal place.

11. A town B is 14.63 miles due north of A. The road from A to B runs N $27° 45'$ E to C, then N $34° 30'$ W to B. Find the distance by road from A to B.

140. Case II. **The Ambiguous Case. Given Two Sides and the Angle Opposite One of Them.** In this case, there may be two solutions, one solution, or no solution, as may be seen from the following considerations.

Let the given parts be α, a, and b.

First, let α be acute.

Construct the angle α at A, lay off $AC = b$, and draw CD perpendicular to the other side of the angle α. Then $DC = b \sin \alpha$. With C as center and radius a, draw a circle.

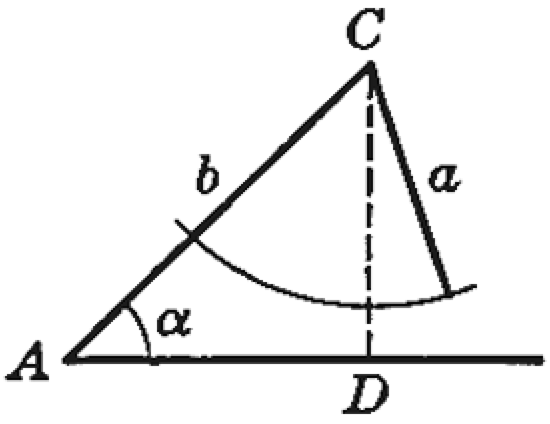


Fig. 54a

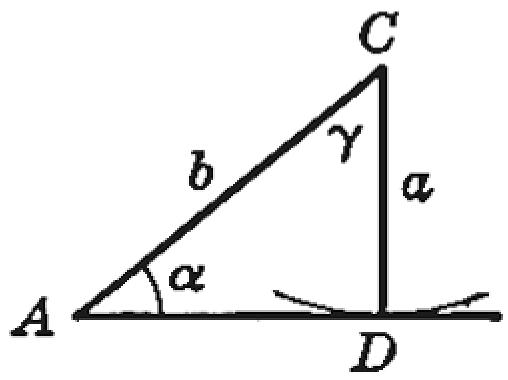


Fig. 54b

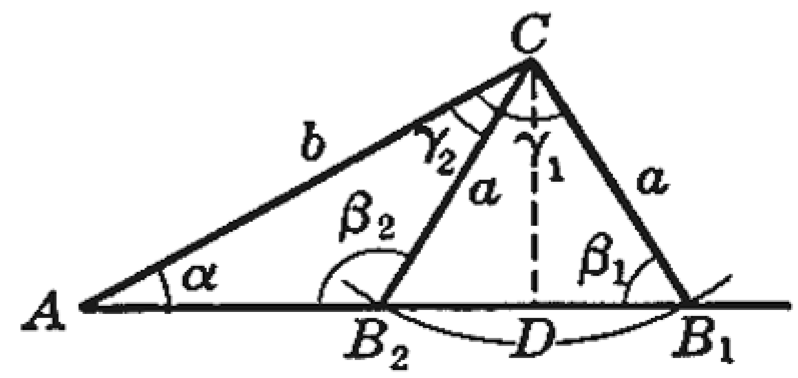


Fig. 54c

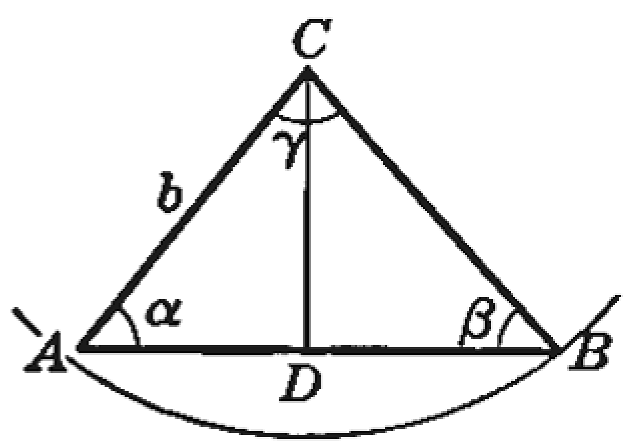


Fig. 54d

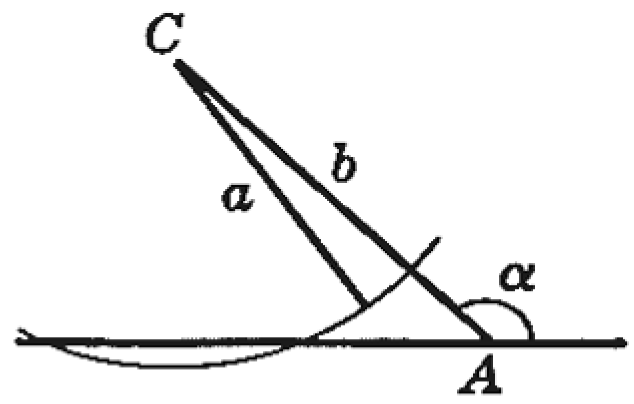


Fig. 54e

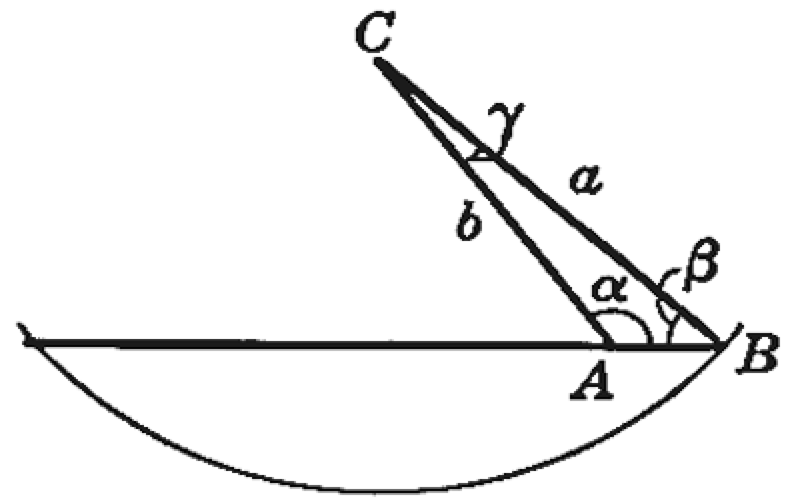


Fig 54f

1. If $a < b \sin \alpha$, the circle does not intersect the line AD and there is *no solution* (Fig. 54*a*).

2. If $a = b \sin \alpha$, the circle touches AD at D and there is *one solution*, the right triangle ADC (Fig. 54*b*).

3. If $b \sin \alpha < a < b$, the circle intersects the line AD in two points to the right of A. There are *two solutions*, AB_1C and AB_2C (Fig. 54*c*).

4. If $b < a$, the circle intersects the line AD in one point to the right and one to the left of A. There is *one solution* since the triangle with its vertex to the left of A does not have α as one of its angles (Fig. 54*d*).

Next, let α be obtuse.

5. If $a < b$, there is *no solution* (Fig. 54*e*).

6. If $b < a$, there is *one solution* (Fig. 54*f*).

In any given exercise, a figure drawn carefully to scale will usually show the number of solutions. If there still is doubt, make the computation for finding the logarithm of the sine of the first required angle, as in Example 2. If this logarithm is positive, there is no solution; if it is exactly zero; there is one solution; if it is negative (9 + mantissa − 10), there are two solutions.

A suitable form for the solution is shown in the following example.

Observe that, in finding a_1 and a_2, we first find log $b/\sin\beta$ then add each of the numbers log sin α_1 and log sin α_2 to this number.

EXAMPLE 1. Solve the triangle: $b = 5.3846$, $c = 7.3545$, $\beta = 37° 42.5'$.

Given:	Find:	
$b = 5.3846$,	$\gamma_1 = 56° 39.4'$,	$\gamma_2 = 123° 20.6'$,
$c = 7.3545$,	$\alpha_1 = 85° 38.1'$,	$\alpha_2 = 18° 56.9'$,
$\beta = 37° 42.5'$.	$a_1 = 8.7780$;	$a_2 = 2.8587$.

$$\sin\gamma_1 = \frac{c\sin\beta}{b},\quad \alpha_1 = 180° - (\beta+\gamma_1),\quad a_1 = \frac{b}{\sin\beta}\sin\alpha_1,$$

$$\gamma_2 = 180° - \gamma_1,\quad \alpha_2 = 180° - (\beta+\gamma_2),\quad a_2 = \frac{b}{\sin\beta}\sin\alpha_2.$$

$$\frac{a_1-c}{a_1+c} = \frac{\tan\frac{1}{2}(\alpha_1-\gamma_1)}{\tan\frac{1}{2}(\alpha_1+\gamma_1)},\qquad \frac{c-a_2}{c+a_2} = \frac{\tan\frac{1}{2}(\gamma_2-\alpha_2)}{\tan\frac{1}{2}(\gamma_2+\alpha_2)}.$$

$$\begin{aligned}
\log c &= 0.86655\\
\log\sin\beta &= \underline{9.78650 - 10}\,+\\
&\ 10.65305 - 10\\
\log b &= \underline{0.73116}\qquad -\\
\log\sin\gamma &= 9.92189 - 10
\end{aligned}
\qquad
\begin{aligned}
\log b &= 10.73116 - 10\\
\log\sin\beta &= \underline{9.78650 - 10}\,-\\
&\ 0.94466\\
\log\sin\alpha_1 &= 9.99874 - 10\,+\\
\log\sin\alpha_2 &= \underline{9.51150 - 10}\,+\\
\log a_1 &= 0.94340\\
\log a_2 &= 0.45616
\end{aligned}$$

$$\begin{aligned}
\log a_1 - c &= 10.15336 - 10\\
\log a_1 + c &= \underline{1.20768}\qquad -\\
&\ 8.94568 - 10
\end{aligned}
\qquad
\begin{aligned}
\log\tan\tfrac{1}{2}(\alpha_1-\gamma_1) &= 9.41235 - 10\\
\log\tan\tfrac{1}{2}(\alpha_1+\gamma_1) &= \underline{0.46665}\qquad -\\
&\ 8.94570 - 10
\end{aligned}$$

$$\begin{aligned}
\log c - a_2 &= 10.65281 - 10\\
\log c + a_2 &= \underline{1.00916}\qquad -\\
&\ 9.64365 - 10
\end{aligned}
\qquad
\begin{aligned}
\log\tan\tfrac{1}{2}(\gamma_2-\alpha_2) &= 10.11027 - 10\\
\log\tan\tfrac{1}{2}(\gamma_2+\alpha_2) &= \underline{0.46665}\qquad -\\
&\ 9.64362 - 10
\end{aligned}$$

EXAMPLE 2. Solve the triangle: $a = 4.1872$, $c = 3.7214$, $\gamma = 63° 17.4'$.

Since Figure 55 fails to show definitely the number of solutions, we shall carry through the computation of sin α to determine the number of solutions.

Given:

$a = 4.1872$,
$c = 3.7214$,
$\gamma = 63° 17.4'$.

We have $\sin\alpha = a\sin\gamma/c$.

$$\begin{aligned}
\log a &= 0.62192\\
\log\sin\gamma &= \underline{9.95099 - 10}\,+\\
&\ 0.57291\\
\log c &= \underline{0.57071}\qquad -\\
\log\sin\alpha &= 0.00220 = \log 1.0051.
\end{aligned}$$

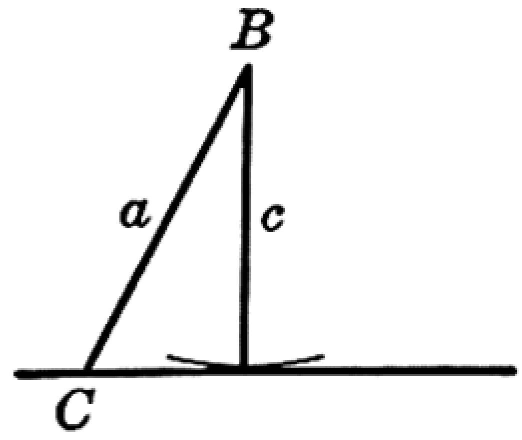


FIG. 55

It follows that $\sin\alpha = 1.0051$ which is impossible since the sine of an angle cannot exceed unity. There is no solution.

Exercises

Solve the following triangles.

1. $a = 5.6385$, $c = 9.2143$, $\alpha = 23° 41.2'$.
2. $a = 35.721$, $c = 43.285$, $\alpha = 41° 16.2'$.
3. $b = 2965.1$, $c = 3796.9$, $\beta = 43° 11.9'$.
4. $b = 24.588$, $c = 31.491$, $\gamma = 39° 17.4'$.
5. $a = 135.41$, $b = 148.52$, $\beta = 71° 29.6'$.
6. $a = 861.47$, $b = 579.28$, $\alpha = 117° 53.5'$.
7. $b = 43.692$, $c = 23.659$, $\gamma = 34° 28.6'$.
8. $a = 1.4437$, $c = 1.0342$, $\gamma = 147° 19.6'$.
9. $a = 0.26532$, $b = 0.38416$, $\alpha = 26° 49.2'$.
10. $b = 7953.8$, $c = 8147.6$, $\beta = 71° 18.7'$.

11. The angle at A of a parallelogram $ABCD$ is $42° 21.6'$. The side AB is 63.42 inches and the diagonal BD is 52.76 inches. Find the side AD.

12. Two houses, A and B, are 2736 feet apart. From a third house C, 1576 feet from A, they subtend an angle of $21° 16'$. Find the distance CB.

141. Case III. Given Two Sides and the Included Angle. If a, b, and γ are given, we first find $\frac{1}{2}(\alpha + \beta)$ from the relation

$$\tfrac{1}{2}(\alpha + \beta) = 90° - \tfrac{1}{2}\gamma,$$

which follows from the relation $\alpha + \beta + \gamma = 180°$. We can now compute $(\alpha - \beta)/2$ from the law of tangents. We next find

$$\alpha = \tfrac{1}{2}(\alpha + \beta) + \tfrac{1}{2}(\alpha - \beta), \quad \text{and} \quad \beta = \tfrac{1}{2}(\alpha + \beta) - \tfrac{1}{2}(\alpha - \beta).$$

Finally, we compute c twice, using the law of sines,

$$c = \frac{a \sin \gamma}{\sin \alpha}, \quad \text{and} \quad c = \frac{b \sin \gamma}{\sin \beta},$$

the second computation serving as a check on the accuracy of our results.

Example. Solve the triangle: $a = 87.326$, $b = 49.243$, $\gamma = 81° 52.4'$.

Given:	Find:
$a = 87.326$,	$\alpha = 66° 53.2'$,
$b = 49.243$,	$\beta = 31° 14.4'$,
$\gamma = 81° 52.4'$.	$c = 93.994$.

$$\frac{1}{2}(\alpha + \beta) = 90° - \frac{1}{2}\gamma, \quad \tan \frac{1}{2}(\alpha - \beta) = \frac{a - b}{a + b} \tan \frac{1}{2}(\alpha + \beta),$$

$$c = \frac{a \sin \gamma}{\sin \alpha} = \frac{b \sin \gamma}{\sin \beta}.$$

$$\begin{array}{rll} \log (a-b) = & 1.58073 & \\ \log \tan \frac{1}{2}(\alpha+\beta) = & \underline{0.06181} & + \\ & 11.64254-10 & \\ \log (a+b) = & \underline{2.13535} & - \\ \log \tan \frac{1}{2}(\alpha-\beta) = & 9.50719-10 & \\ \frac{1}{2}(\alpha+\beta) = & 49^\circ \ \ 3.8' & \\ \frac{1}{2}(\alpha-\beta) = & \underline{17^\circ\ 49.4'} & \\ \alpha = & 66^\circ\ 53.2' & \\ \beta = & 31^\circ\ 14.4' & \end{array}$$

$$\begin{array}{rll} \log a = & 1.94114 & \\ \log \sin \gamma = & \underline{9.99562-10} & + \\ & 11.93676-10 & \\ \log \sin \alpha = & \underline{9.96366-10} & - \\ \log c = & 1.97310 & \\ \log b = & 1.69235 & \\ \log \sin \gamma = & \underline{9.99562-10} & + \\ & 11.68797-10 & \\ \log \sin \beta = & \underline{9.71485-10} & - \\ \log c = & 1.97312 & \end{array}$$

Exercises

Solve the following triangles.

1. $a = 48.736$, $b = 31.437$, $\gamma = 56^\circ\ 26.6'$.
2. $b = 3.7508$, $c = 4.9156$, $\alpha = 123^\circ\ 38.3'$.
3. $a = 167.81$, $c = 318.58$, $\beta = 113^\circ\ 21.8'$.
4. $a = 95747$, $b = 69473$, $\gamma = 71^\circ\ 55.2'$.
5. $b = 12.471$, $c = 23.417$, $\alpha = 38^\circ\ 19.7'$.
6. $a = 0.36437$, $c = 0.21959$, $\beta = 99^\circ\ 41.5'$.
7. $b = 34.916$, $c = 37.254$, $\alpha = 16^\circ\ 51.3'$.
8. $a = 0.041752$, $b = 0.052761$, $\gamma = 28^\circ\ 46.2'$.

9. Towns B and C are, respectively, $11^\circ\ 53'$ and $75^\circ\ 32'$ east of north of A. The distance from A to B is 41.92 miles and from A to C is 17.63 miles. Find the distance from B to C.

10. Two forces, of magnitudes 217.6 and 358.3, make an angle of $57^\circ\ 41'$ with each other. Find the magnitude of the resultant force and the angle it makes with each given force.

142. The Law of Cosines. Place the triangle ABC so that the angle α is in standard position (Figs. 56). Let D be the foot of the perpendicular from C to the x-axis.

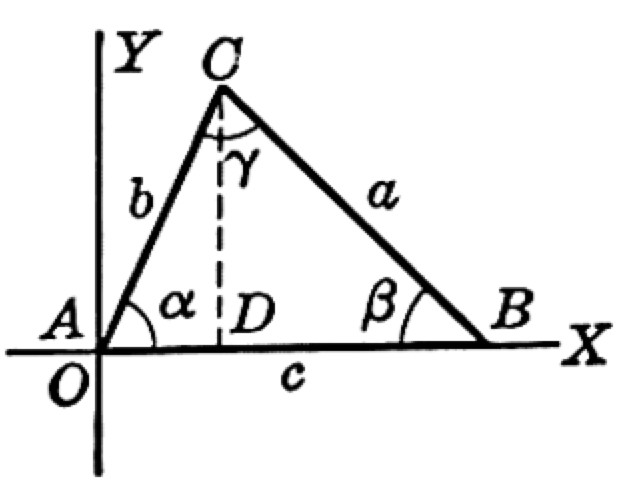


FIG. 56a

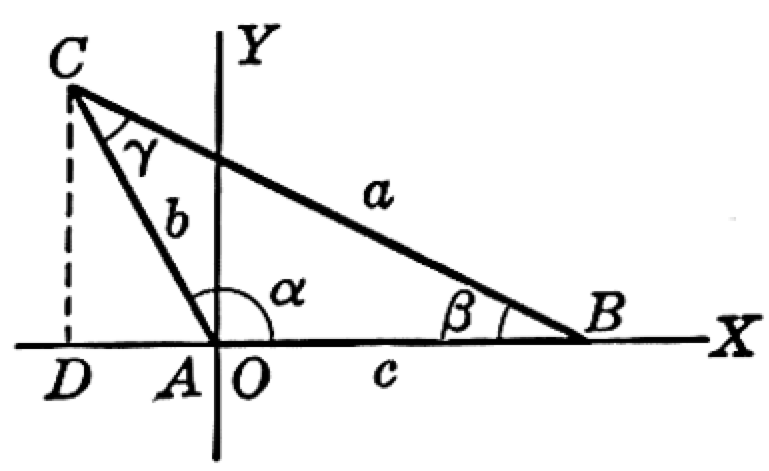


FIG. 56b

In either figure, using directed distances,

$$DC = b \sin \alpha, \quad AD = b \cos \alpha, \quad \text{and} \quad DB = AB - AD = (c - b \cos \alpha).$$

In the right triangle BDC,

$$\begin{aligned} a^2 = DC^2 + DB^2 &= b^2 \sin^2 \alpha + (c - b \cos \alpha)^2 \\ &= b^2 \sin^2 \alpha + c^2 - 2bc \cos \alpha + b^2 \cos^2 \alpha, \end{aligned}$$

or, since

$$\sin^2 \alpha + \cos^2 \alpha = 1,$$

$$\mathbf{a^2 = b^2 + c^2 - 2bc \cos \alpha,}$$

Similarly,

$$\mathbf{b^2 = c^2 + a^2 - 2ca \cos \beta,} \tag{3}$$

and

$$\mathbf{c^2 = a^2 + b^2 - 2ab \cos \gamma.}$$

These three formulas constitute the **law of cosines.**

These formulas are not adapted to logarithmic computation. They are, however, as we shall see in the next article, of importance in deriving other useful formulas. They may also be used, when the numbers involved are not too large, to solve the triangle using the natural functions. They are especially useful in this way if only one side, or one angle, is required.

Exercises

In the following exercises, use the natural functions and the law of cosines. Find the required sides to four significant figures and the required angles to the nearest minute.

1. Given $b = 5$, $c = 8$, $\alpha = 45°$, find a.
2. Given $a = 17$, $b = 12$, $\gamma = 37° \ 10'$, find c.
3. Given $a = 21.3$, $c = 34.2$, $\beta = 73° \ 50'$, find b.
4. Given $a = 9$, $b = 11$, $c = 14$, find α.
5. Given $a = 13$, $b = 10$, $c = 17$, find all the angles.
6. Given $a = 2.43$, $b = 3.15$, $c = 2.84$, find all the angles.

7. Two airplanes start from the same station at the same time in directions making angles of 41° 30′ with each other. At the end of an hour, one has gone 130 miles and the other 150 miles. How far are they apart?

8. A body is acted on by two forces, of 35 and 40 pounds, respectively, making an angle of 27° 40′ with each other. Find the magnitude of the resultant force.

143. The Half-angle Formulas. Denote half the sum of the sides of the triangle by s; that is,

$$s = \tfrac{1}{2}(a + b + c). \tag{4}$$

Then,

$$\begin{aligned} b + c - a &= a + b + c - 2a = 2(s - a), \\ c + a - b &= a + b + c - 2b = 2(s - b), \end{aligned}$$

and

$$a + b - c = a + b + c - 2c = 2(s - c).$$

From formula 13, Art. 128, and the law of cosines, we have

$$\sin^2\frac{\alpha}{2} = \frac{1}{2}(1-\cos\alpha) = \frac{1}{2}\left(1 - \frac{b^2+c^2-a^2}{2bc}\right)$$
$$= \frac{a^2-b^2+2bc-c^2}{4bc} = \frac{a^2-(b-c)^2}{4bc}$$
$$= \frac{(a-b+c)(a+b-c)}{4bc} = \frac{2(s-b)2(s-c)}{4bc}.$$

Hence,
$$\sin\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}. \tag{5}$$

In a similar way, we can derive the formulas

$$\sin\frac{\beta}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}, \qquad \sin\frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}. \tag{5'}$$

We have also,

$$\cos^2\frac{\alpha}{2} = \frac{1}{2}(1+\cos\alpha) = \frac{1}{2}\left(1 + \frac{b^2+c^2-a^2}{2bc}\right)$$
$$= \frac{b^2+2bc+c^2-a^2}{4bc} = \frac{(b+c)^2-a^2}{4bc}$$
$$= \frac{(b+c+a)(b+c-a)}{4bc} = \frac{2s\;2(s-a)}{4bc},$$

giving
$$\cos\frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{bc}}. \tag{6}$$

In a similar way, we derive

$$\cos\frac{\beta}{2} = \sqrt{\frac{s(s-b)}{ca}}, \quad \text{and} \quad \cos\frac{\gamma}{2} = \sqrt{\frac{s(s-c)}{ab}}. \tag{6'}$$

Since $\alpha/2$, $\beta/2$, and $\gamma/2$ are all acute angles, all the radicals in these formulas are to be taken as positive.

If we divide the value of $\sin\alpha/2$ from (5) by the value of $\cos\alpha/2$ from (6), we have

$$\tan\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}\,\frac{bc}{s(s-a)}} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$
$$= \sqrt{\frac{(s-a)(s-b)(s-c)}{s(s-a)^2}} = \frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

To simplify this expression, we put *

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}. \tag{7}$$

* It can be shown (Art. 146, Ex. 15), that the number r defined by equation (7) is the radius of the circle inscribed in the triangle ABC.

On substituting this value of r in the preceding equation, and writing the corresponding equations for $\tan \beta/2$ and $\tan \gamma/2$, we have

$$\tan\frac{\alpha}{2} = \frac{r}{s-a}, \qquad \tan\frac{\beta}{2} = \frac{r}{s-b}, \qquad \tan\frac{\gamma}{2} = \frac{r}{s-c}. \tag{8}$$

144. Case IV. **Given the Three Sides.** The solution by logarithms of the triangle when the three sides are given may be obtained by using formulas (4), (7), and (8) of Art 143. As a check, we observe that the sum of the half-angles is 90°.

Example. Solve the triangle: $a = 73.576$, $b = 51.835$, $c = 46.821$.

Given:	Find:
$a = 73.576,$	$\alpha = 96°\ 19.2',$
$b = 51.835,$	$\beta = 44°\ 26.8',$
$c = 46.821.$	$\gamma = 39°\ 14.0'.$
$2\,)\,172.232$	
$s = 86.116$	

$$s = \tfrac{1}{2}(a+b+c), \quad r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$$\tan\frac{\alpha}{2} = \frac{r}{s-a}, \quad \tan\frac{\beta}{2} = \frac{r}{s-b}, \quad \tan\frac{\gamma}{2} = \frac{r}{s-c}.$$

$$\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} = 90°.$$

$s - a = 12.540$	$\log(s-a) = 1.09830$
$s - b = 34.281$	$\log(s-b) = 1.53505$
$s - c = 39.295$	$\log(s-c) = 1.59434 +$
	4.22769
$s = 86.116$	$\log s = 1.93508 -$
	$2\,)\,2.29261$
	$\log r = 1.14630$
$\alpha/2 = 48°\ 9.6'$	$\log\tan\alpha/2 = 0.04800$
$\beta/2 = 22°\ 13.4'$	$\log\tan\beta/2 = 9.61125 - 10$
$\gamma/2 = 19°\ 37.0'$	$\log\tan\gamma/2 = 9.55196 - 10$

Exercises

Solve the following triangles.

1. $a = 7.4382,\quad b = 9.3745,\quad c = 6.8397.$
2. $a = 1.5637,\quad b = 2.0182,\quad c = 1.6921.$
3. $a = 2.2874,\quad b = 1.5138,\quad c = 3.2514.$
4. $a = 0.95863,\quad b = 1.6524,\quad c = 1.3749.$
5. $a = 48.632,\quad b = 34.723,\quad c = 45.218.$
6. $a = 0.061829,\quad b = 0.053925,\quad c = 0.039827.$
7. $a = 59.724,\quad b = 32.461,\quad c = 71.249.$
8. $a = 742850,\quad b = 943280,\quad c = 613590.$

9. A city block is bounded by three streets. If the length of the sides of the block are 271.6, 325.8, and 385.4 feet, find to the nearest minute the angles the streets make with each other.

10. Three circles, of radii 7.241, 4.837, and 6.495 inches, are tangent to each other externally. Find to the nearest minute the angles of the triangle formed by the lines joining their centers.

145. Area of a Triangle. Let S be the area of the triangle ABC and let h be the altitude from C on AB.

By elementary geometry,

$$S = \tfrac{1}{2}ch.$$

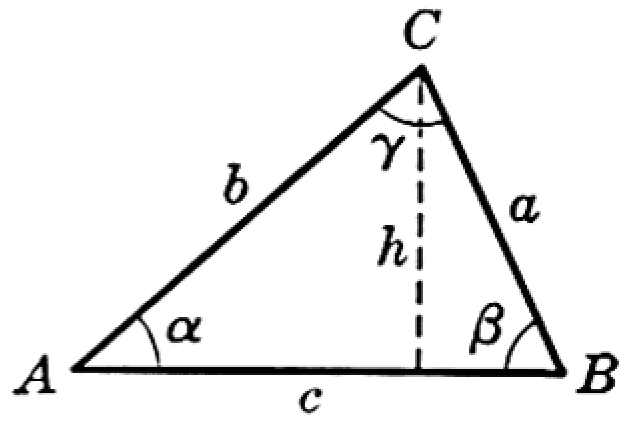


FIG. 57

From the definition of $\sin \alpha$,

$$h = b \sin \alpha.$$

It follows that

$$\boldsymbol{S = \tfrac{1}{2}bc \sin \alpha = \tfrac{1}{2}ca \sin \beta = \tfrac{1}{2}ab \sin \gamma.} \qquad (9)$$

In the second, third, and fourth members of equation (9), substitute the values of the second letter in terms of the first from the law of sines. We have

$$\boldsymbol{S = \frac{1}{2}b^2\frac{\sin \alpha \sin \gamma}{\sin \beta} = \frac{1}{2}c^2\frac{\sin \alpha \sin \beta}{\sin \gamma} = \frac{1}{2}a^2\frac{\sin \beta \sin \gamma}{\sin \alpha}.} \qquad (10)$$

From formula IX of Art. 127,

$$\sin \alpha = 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}.$$

Hence, from equation (9),

$$S = bc \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}.$$

If, in this equation, we substitute the values of $\sin \frac{\alpha}{2}$ and $\cos \frac{\alpha}{2}$ from equations (5) and (6) of Art. 143, we obtain

$$\boldsymbol{S = \sqrt{s(s-a)(s-b)(s-c)}.} \qquad (11)$$

Exercises

Find the area of the triangle.

1. $a = 925.36$, $c = 432.85$, $\beta = 42° 17.3'$.
2. $b = 1.3526$, $c = 2.4193$, $\alpha = 55° 49.7'$.
3. $a = 21.763$, $\beta = 31° 14.8'$, $\gamma = 67° 29.5'$.
4. $b = 4.7931$, $\alpha = 83° 42.7'$, $\gamma = 37° 21.3'$.
5. $a = 51.342$, $b = 29.571$, $c = 42.863$.
6. $a = 9347.6$, $b = 7134.2$, $c = 8563.2$.

146. Problems. In the following problems, find the required numbers to four significant figures and angles to the nearest minute.

1. Two roads leading to a village branch off from a main road at places 3764 feet apart. They meet at the village at an angle of 63° 47′ and the longer makes an angle of 29° 38′ with the main road. Find the lengths of the side roads.

2. From a point on a mountain directly above a tunnel, the angles of depression of the ends of the tunnel are 27° 43′ and 32° 29′. The distances from the point to the ends of the tunnel are 5943 and 4976 feet. Find the length of the tunnel.

3. To find the distance between two points A and B, separated by an obstruction, a point C was selected. The distances $AC = 3259$ feet and $CB = 2143$ feet, and the angle $ACB = 102° \ 17'$, were measured. Find AB.

4. Two landing places, A and B, on one side of a lake, are 2197 feet apart. C is a landing place on the opposite side of the lake such that the angle $BAC = 65° \ 48'$ and $ABC = 48° \ 31'$. Find AC and BC.

5. A balloon is directly above a straight road. Its angles of elevation from two consecutive mile posts on opposite sides of the balloon are 59° 37′ and 35° 52′. Find the height, in feet, of the balloon above the road.

6. A tower stands at the end of a road inclined 14° 39′ upwards from the horizontal. At a point on the road 153.5 feet from the foot of the tower, the angle of elevation of the top of the tower is 42° 17′. Find the height of the tower.

7. Two buildings of equal height are 100 feet apart. From a point on the ground between them, the angles of elevation of their tops are 63° 54′ and 41° 17′. Find the heights of the buildings and the distance of the point of observation from the nearest building.

8. The lengths of the diagonals of a parallelogram are 4372 and 3562 feet. They meet at an angle of 76° 52′. Find the lengths of the sides.

9. Two forces are of magnitudes 314.7 and 563.2 pounds. Their resultant is of magnitude 635.3 pounds. Find the angle each given force makes with the resultant.

10. Two forces, of magnitudes 253.7 and 382.7 pounds, make an angle of 58° 14′ with each other. Find the magnitude of their resultant and the angle it makes with each given force.

11. A flyer wishes to go to a place 247.5 miles northeast of his present position. If his plane travels 135 miles in still air and there is a wind of 27 miles an hour blowing from 10° west of north, in what direction should he point his plane and in how many minutes will he reach his destination?

12. In a quadrilateral $ABCD$, $AB = 643.7$ feet, $AC = 926.3$ feet, angle $BAC = 68° \ 14'$, $BAD = 104° \ 21'$, $ABD = 57° \ 53'$. Find AD and DC.

13. The walk leading to the bottom of a monument rises one foot vertically for every 12 feet measured horizontally. When the angle of elevation of the sun is 21° 43′, the length of the shadow of the monument, measured along the walk, is 117 feet. Find the height of the monument.

14. Prove the projection formulas:

$$a = b \cos \gamma + c \cos \beta, \quad b = c \cos \alpha + a \cos \gamma, \quad c = a \cos \beta + b \cos \alpha.$$

15. Show that the quantity r, defined in equation (7), Art. 143, is the radius of the circle inscribed in the triangle.

HINT. If O is the center and r' the radius of the inscribed circle, show that:

$$\text{area } ABC = \text{area } AOB + \text{area } BOC + \text{area } COA = r's.$$

Insert the value of area ABC from equation (11), Art. 145.

16. In equations (1) (the law of sines), show that the value of each of the equal fractions is $2R$, where R is the radius of the circle circumscribed around the triangle ABC.

HINT. Let the perpendicular to BC at C meet the circumcircle again at D. Show that $BD = 2R$ and that angle $BDC = \alpha$.

17. Show that R, the radius of the circumcircle, is given in terms of the sides by the formula

$$R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}.$$

HINT. Use the results of Ex. 16 and the values of $\sin \alpha/2$ and $\cos \alpha/2$ from equations (5) and (6), Art. 143.

18. Using the results of Ex. 15 and 17, find r and R, given $a = 71.34$, $b = 63.47$, and $c = 84.54$.

19. It was required to measure the distance from A to an inaccessible point B when no instruments for measuring angles were available. An accessible point C was chosen and the lines from B to A and to C were extended to D and E, respectively. The following distances were measured: $AC = 91$, $AD = 118$, $DC = 131$, $CE = 60$, $AE = 134$. Find AB.

Chapter 19

Definitions and Formulas

147. Directed Line Segments. When we defined the coördinates of a point in Art. 38, we brought in the concept of measuring line segments, on or parallel to the axes, *from* one point *to* another and of considering these distances as positive or negative according to the direction of measurement. From now on, we shall use this concept in dealing with segments that may, or may not, be parallel to either axis.

A **directed line** is one on which it has been agreed that distances measured in one direction are positive and those measured in the opposite direction are negative. The positive direction is sometimes indicated by an arrow, as in Figure 58. A **directed line segment** is one for which the direction of measurement has been selected. If the direction of measurement is from A to B, we read it, "the segment AB"; if it is measured in the opposite direction, we read it, "the segment BA." It follows from this definition that, for directed segments,

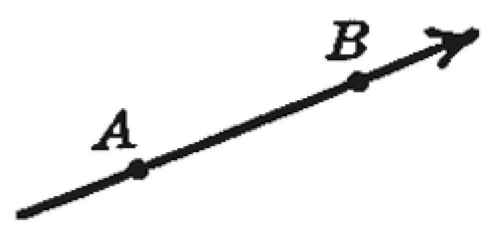


FIG. 58

$$BA = -AB, \quad \text{or} \quad AB + BA = 0.$$

In what follows, the segments considered will usually be directed segments. Care must be taken to read the segments correctly. The directed segment AB is not the same as the directed segment BA.

FIG. 59*a*

FIG. 59*b*

FIG. 59*c*

The use of directed segments frequently enables us to combine into a single formula results which, if undirected segments were used, would have to be treated as separate cases. For example, let A, B, and C be any three points, in any order, on a directed line. For all positions of these points relative to each other, the following relation holds for the directed segments connecting the points:

$$AB = AC + CB. \tag{1}$$

For, if C lies between A and B, then AB, AC, and CB all have the same signs and AB equals the sum of the other two. If C lies outside the segment AB, then AC and CB have opposite signs but their algebraic sum equals AB in magnitude and sign.

148. Segments on the Coördinate Axes. Let $L_1(x_1, 0)$ and $L_2(x_2, 0)$ be any two points on the x-axis (Fig. 60). Then

$$OL_1 = x_1, \quad \text{and} \quad OL_2 = x_2.$$

From equation (1), we have

$$L_1L_2 = L_1O + OL_2 = -OL_1 + OL_2 = -x_1 + x_2,$$

or

$$L_1L_2 = x_2 - x_1; \tag{2}$$

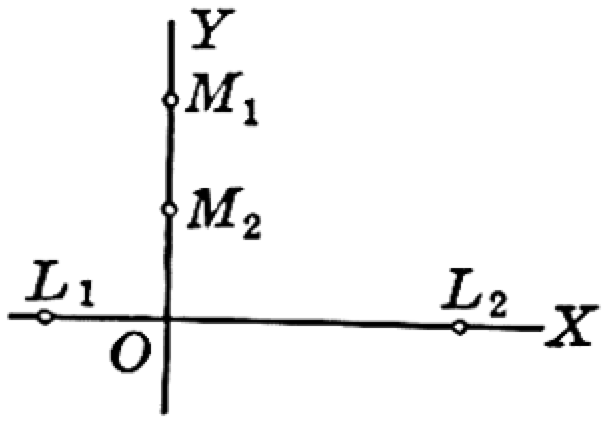


FIG. 60

that is, *the length of any directed segment L_1L_2 on the x-axis equals the abscissa of L_2 minus the abscissa of L_1.*

By similar reasoning, we find that, if $M_1(0, y_1)$ and $M_2(0, y_2)$ are any two points on the y-axis, then

$$M_1M_2 = y_2 - y_1; \tag{3}$$

that is, *the length of any directed segment M_1M_2 on the y-axis equals the ordinate of M_2 minus the ordinate of M_1.*

We shall have frequent occasion to use formulas (2) and (3) in the following articles.

Exercises

Find the length of the directed segment, by measurement, from a figure. Check your result by using equation (2).

1. $L_1(4, 0)$, $L_2(7, 0)$. **2.** $L_1(10, 0)$, $L_2(6, 0)$.
3. $L_1(3, 0)$, $L_2(-5, 0)$. **4.** $L_1(-4, 0)$, $L_2(2, 0)$.
5. $L_1(-2, 0)$, $L_2(-8, 0)$. **6.** $L_1(-9, 0)$, $L_2(-1, 0)$.

Find M_1M_2 by measurement from a figure and check by using equation (3).

7. $M_1(0, 3)$, $M_2(0, 8)$. **8.** $M_1(0, -9)$, $M_2(0, -5)$.
9. $M_1(0, -6)$, $M_2(0, 4)$. **10.** $M_1(0, 6)$, $M_2(0, 1)$.
11. $M_1(0, 5)$, $M_2(0, -2)$. **12.** $M_1(0, -2)$, $M_2(0, -6)$.

Let the feet of the perpendiculars from P_1 and P_2 on the x-axis be L_1 and L_2, and on the y-axis be M_1 and M_2, respectively. Find the lengths of the directed segments L_1L_2 and M_1M_2, given:

13. $P_1(3, 2)$, $P_2(7, 5)$. **14.** $P_1(-4, -1)$, $P_2(8, 4)$.
15. $P_1(-1, 5)$, $P_2(6, 3)$. **16.** $P_1(6, -1)$, $P_2(-2, 4)$.
17. $P_1(2, 8)$, $P_2(-3, -1)$. **18.** $P_1(7, -2)$, $P_2(-3, 5)$.

149. Distance between Two Points. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two given points (Fig. 61). Draw the undirected segment P_1P_2. It is required to find the length of this segment in terms of the coördinates of P_1 and P_2.

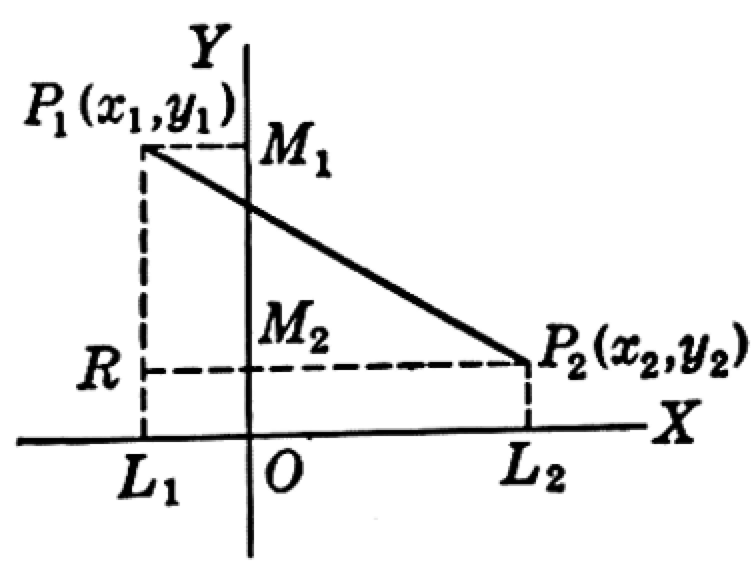


FIG. 61

Let the feet of the perpendiculars from P_1 on the x- and y-axes be L_1 and M_1, respectively, and from P_2 be L_2 and M_2, respectively. Let P_1L_1 and P_2M_2 (produced if necessary) intersect at R.

Since the angle at R is a right angle, we have,

$$(P_1P_2)^2 = (RP_2)^2 + (RP_1)^2$$
$$= (L_1L_2)^2 + (M_1M_2)^2. \qquad \text{(Why?)}$$

But

$$(L_1L_2)^2 = (x_2 - x_1)^2$$

and

$$(M_1M_2)^2 = (y_2 - y_1)^2.$$

Hence

$$(P_1P_2)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

or

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \qquad (4)$$

This formula gives the length of the undirected segment P_1P_2. If the line through P_1 and P_2 is directed (that is, if a positive direction has been chosen on it) and if the directed length P_1P_2 is desired, the proper sign must be determined by noticing whether the direction from P_1 to P_2 is positive or negative.

Exercises

Derive the distance formula, taking the figure so that:

1. P_1 is in the third quadrant and P_2 in the first.
2. P_1 is in the fourth quadrant and P_2 in the second.

Find the distance between the following pairs of points.

3. (3, 1), (7, 4). **4.** (− 7, − 3), (8, 5). **5.** (− 2, 5), (3, − 7).
6. (5, 11), (− 2, − 13). **7.** (2, − 5), (− 1, 7). **8.** (1, 8), (− 7, − 3).

Find the lengths of the sides of the triangle whose vertices are:

9. (3, 1), (7, 3), (5, 9). **10.** (− 1, − 2), (3, 5), (− 4, 7).
11. (3, − 5), (1, 7), (− 5, 2). **12.** (− 5, 0), (1, 3), (0, 8).

Show that the following triangles are isosceles.

13. (3, 5), (1, 2), (− 2, 4). **14.** (6, 8), (1, 3), (7, 1).
15. (2, 5), (8, 3), (3, − 2). **16.** (2, − 2), (− 3, − 1), (1, 6).

Show that the following triangles are right triangles.

17. (− 1, 3), (2, 5), (6, − 1). **18.** (5, − 1), (1, − 3), (2, 5).
19. (3, − 1), (− 1, 2), (2, 6). **20.** (5, − 4), (− 8, − 3), (4, − 7).

21. Find y, given that (5, y) is equidistant from (4, 3) and (1, − 2).

22. Find the coördinates of a point which is equidistant from (− 1, − 3) and (5, 3) and also from (2, 4) and (8, 0).

23. Find two points whose ordinates are 5 that lie at a distance 13 from the point (2, − 7).

24. State by an equation that the distance of the point (x, y) from the origin equals 5. Simplify the equation and draw its graph.

25. State by an equation that the point (x, y) is equidistant from $(4, 1)$ and $(2, -3)$. Simplify the equation and draw its graph.

150. Point Dividing a Given Segment in a Given Ratio. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be the end points of the given segment and let $P(x, y)$ be the point such that

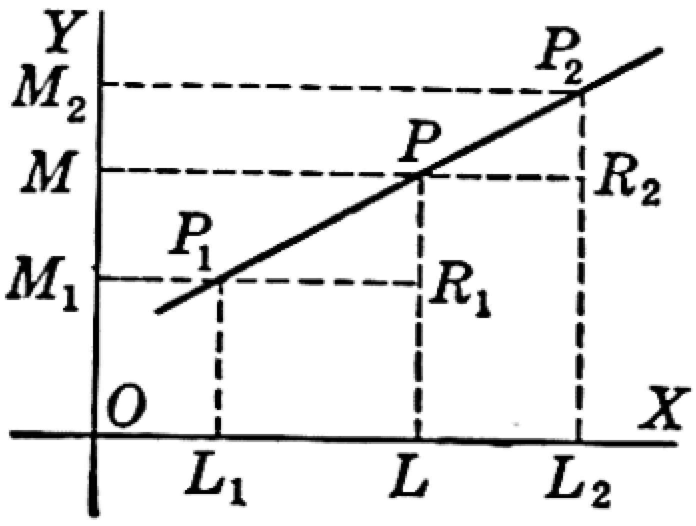


FIG. 62

$$\frac{P_1P}{PP_2} = \frac{n_1}{n_2},$$

where n_1/n_2 is the given ratio. Let L_1, M_1; L, M; and L_2, M_2 be the feet of the perpendiculars on the x- and y-axes from P_1, P, and P_2, respectively. Let R_1 be the intersection of the lines P_1M_1 and PL and R_2 the intersection of PM and P_2L_2. We have

$$P_1R_1 = L_1L = x - x_1 \quad \text{and} \quad PR_2 = LL_2 = x_2 - x.$$

The triangles P_1R_1P and PR_2P_2 are similar since their sides are respectively parallel. It follows that (both in magnitude and sign)

$$\frac{n_1}{n_2} = \frac{P_1P}{PP_2} = \frac{P_1R_1}{PR_2} = \frac{L_1L}{LL_2} = \frac{x - x_1}{x_2 - x}. \tag{5}$$

Using the same similar triangles, we find in the same way, that

$$\frac{n_1}{n_2} = \frac{P_1P}{PP_2} = \frac{R_1P}{R_2P_2} = \frac{M_1M}{MM_2} = \frac{y - y_1}{y_2 - y}. \tag{6}$$

If we equate the first and last members of equations (5) and (6) and solve for x and y, we get

$$x = \frac{n_2x_1 + n_1x_2}{n_1 + n_2}, \qquad y = \frac{n_2y_1 + n_1y_2}{n_1 + n_2}; \tag{7}$$

as the coördinates of the point $P(x, y)$ that divides the segment P_1P_2 in the ratio n_1/n_2.

In particular, if P is the midpoint of the segment P_1P_2, then $n_1 = n_2$ and equations (7) become

$$x = \frac{x_1 + x_2}{2}, \qquad y = \frac{y_1 + y_2}{2}; \tag{8}$$

that is, the coördinates of the midpoint of the segment are the half-sums of the coördinates of the end points.

Exercises

1. Find the coördinates of the midpoint of the segment from (*a*) $(4, -3)$ to $(-10, 7)$; (*b*) $(-5, -2)$ to $(9, -6)$.

2. Find the coördinates of the two points of trisection of the segment from $(-5, 3)$ to $(7, -6)$.

3. Find the three points of quadrisection of the segment from $(-1, 6)$ to $(11, 22)$.

4. Find the point which divides in the ratio $3:7$ the segment from $(6, 8)$ to $(16, -12)$.

5. In what ratio does the point $(4, -2)$ divide the segment from $(-1, 8)$ to $(13, -20)$?

6. The midpoint of a segment is $(1, 6)$ and one end point is $(9, 2)$. Find the other end point.

7. The vertices of a triangle are $(2, 5)$, $(10, 1)$, and $(12, 9)$. Find on each median the point twice as far from the vertex as from the midpoint of the opposite side. State the geometric theorem which shows that these points coincide.

8. Solve Ex. 7 for the triangle whose vertices are (x_1, y_1), (x_2, y_2), and (x_3, y_3).

151. The Inclination and Slope of a Line. The **inclination** of a line l (not parallel to the x-axis) is defined as *the smallest positive angle whose initial side extends in the positive direction along the x-axis and whose terminal side extends along l.* If l is parallel to the x-axis, its inclination is defined to be zero.

We shall usually deal, not with the inclination of the line, but with its **slope** which is defined as *the tangent of its inclination.* We shall customarily denote the inclination of a line by α and its slope by m, so that

$$m = \tan \alpha. \qquad (9)$$

If the inclination, α, is an acute angle, then $\tan \alpha$, or m, is *positive* and the line extends *upward to the right;* if α is obtuse, m is *negative* and the line extends *upward to the left* (Fig. 63). Finally, if $\alpha = 90°$, the line is perpendicular to the x-axis and $\tan \alpha$ does not exist; that is, *lines perpendicular to the x-axis have no slope.* When we speak of the slope of a line we shall suppose, accordingly, that the line is not perpendicular to the x-axis.

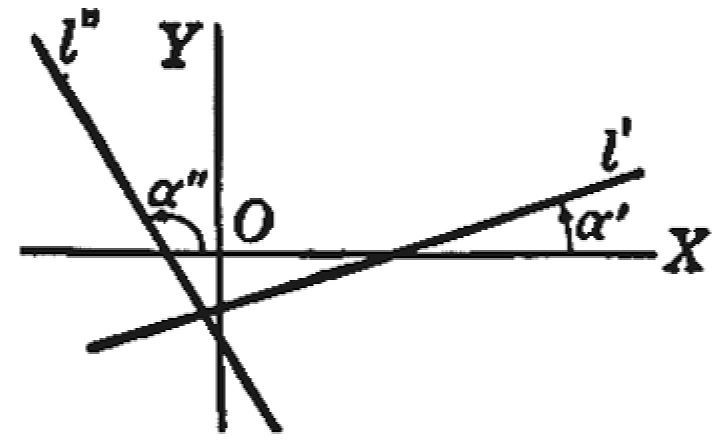


FIG. 63

152. Slope of a Line Through Two Given Points. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ $(x_2 \neq x_1)$ be the two given points and let l be the line passing through them. Through P_1 draw a line parallel to the x-axis and choose any point K on this line to the right of P_1. Denote by ϕ the smallest positive (or zero) angle having P_1K as its initial side and P_1P_2 as its terminal side.

If α is the inclination of l, we now have, either

$$\phi = \alpha, \quad \text{or} \quad \phi = 180° + \alpha,$$

according as $\phi < 180°$ (Fig. 64*a*) or $\phi \geq 180°$ (Fig. 64*b*).

In either case,

$$\tan \phi = \tan \alpha = m. \tag{10}$$

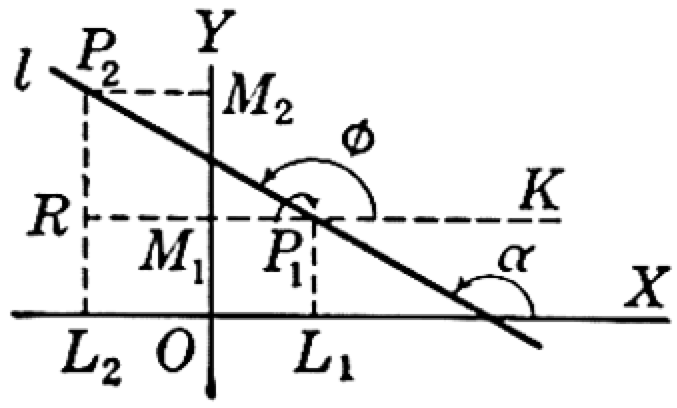


FIG. 64*a*

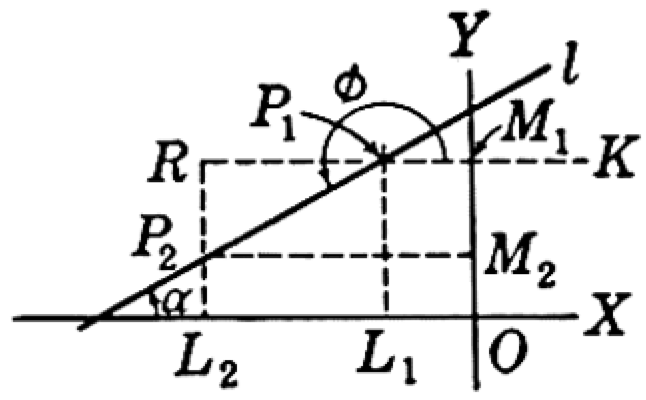


FIG. 64*b*

From Figs. 64*a* and 64*b* and the definition of the tangent of an angle

$$\tan \phi = \frac{RP_2}{P_1R} = \frac{M_1M_2}{L_1L_2} = \frac{y_2 - y_1}{x_2 - x_1};$$

or, since $\tan \phi = m$, by (10), the slope of the line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1}; \tag{11}$$

that is, *the slope of the line through two given points equals the ordinate of the second minus the ordinate of the first divided by the abscissa of the second minus the abscissa of the first.*

We have supposed throughout this article that $x_1 \neq x_2$. If $x_1 = x_2$, the line is parallel to the y-axis and has no slope (Art. 151).

EXAMPLE. Find the slope and the inclination of the line through (2, 1) and (− 2, 6).

From (11), we find, as the required slope,

$$m = \frac{6 - 1}{-2 - 2} = -\frac{5}{4} = -1.25.$$

To determine the inclination α, we substitute this value of m in (10). We obtain

$$\tan \alpha = -1.25, \quad \alpha = 128° 40'.$$

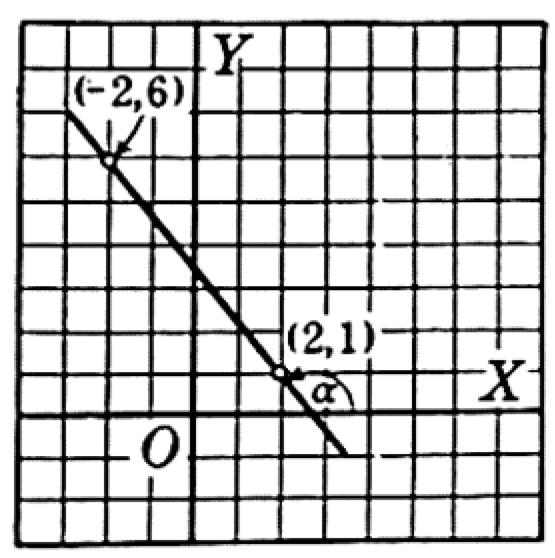


FIG. 65

Exercises

Find the slope of a line whose inclination is:

1. 30°. **2.** 135°. **3.** 26°. **4.** 147° 32′.

5. $\frac{\pi}{4}$. **6.** $\frac{5\pi}{6}$. **7.** $\frac{\pi}{3}$. **8.** $\frac{2\pi}{3}$.

Find the inclination of a line whose slope is:

9. $\sqrt{3}/3$. **10.** 1. **11.** $-\sqrt{3}$. **12.** 0.

13. 0.5095. **14.** − 1.4826. **15.** 0.5418. **16.** − 2.3825.

Find the slope and the inclination of the line through the given points.

17. (2, 1), (6, 4). **18.** (3, − 2), (− 1, 8).
19. (− 1, − 5), (3, 2). **20.** (1.4835, 2.9531), (3.5862, 6.1384).
21. (− 2.8573, 3.1762), (1.3857, − 2.9423).

Draw through the given point a line having the slope indicated.

22. (0, 0), $m = 1$. **23.** (1, − 5), $m = -1$. **24.** (− 4, − 2), $m = -2$.
25. (− 2, 5), $m = 3$. **26.** (1, − 7), $m = \frac{5}{2}$. **27.** (5, 1), $m = -\frac{4}{3}$.

28. An equilateral triangle has one vertex at the origin, another on the y-axis, and the third in the first quadrant. Find the slopes of two of its sides and show that the third side has no slope.

29. Find the slopes of the bisectors of the angles of the triangle in Ex. 28.

30. Three vertices of a square are (a, a), $(-a, a)$, and $(-a, -a)$. Find the fourth vertex and the slopes of the diagonals.

31. Express by an equation the condition that the slope of the line through (− 2, 1) and (x, y) equals 2. What is the graph of this equation?

153. Parallel and Perpendicular Lines. Let l_1 and l_2 be two lines, neither of which is parallel to the y-axis.

If the lines l_1 and l_2 are parallel to each other, their inclinations, and hence their slopes, are equal. (Why?) Conversely, if

$$m_1 = m_2, \quad \text{then} \quad \alpha_1 = \alpha_2,$$

and the lines are parallel.

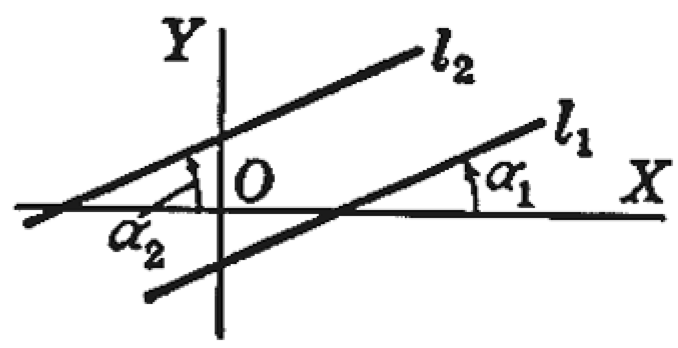


FIG. 66

Hence, *the condition that l_1 and l_2 are parallel is that*

$$m_1 = m_2. \tag{12}$$

If the lines l_1 and l_2 are perpendicular, we have either

$$\alpha_2 = \alpha_1 + 90°, \qquad \text{(Fig. 67}a\text{)}$$

or

$$\alpha_1 = \alpha_2 + 90°. \qquad \text{(Fig. 67}b\text{)}$$

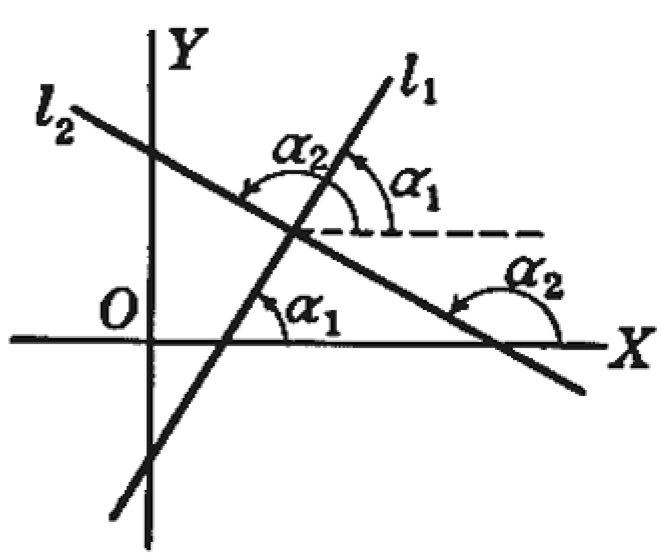


FIG. 67a

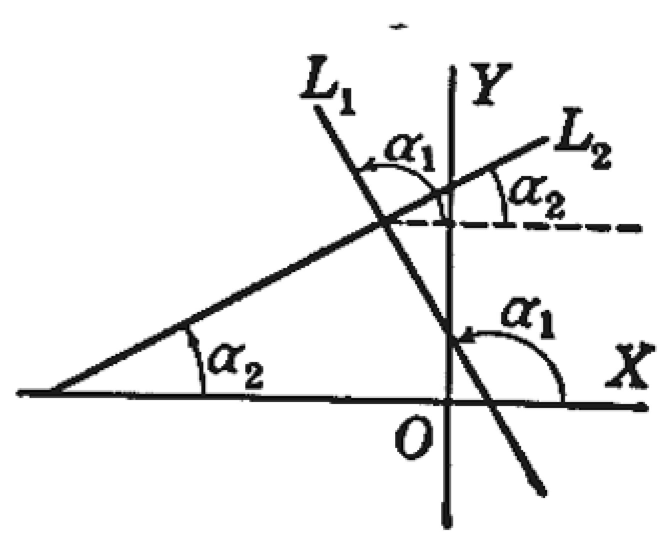


FIG. 67b

In either case

$$\tan \alpha_1 = -\cot \alpha_2 = -\frac{1}{\tan \alpha_2};$$

or, since $\tan \alpha_1 = m_1$ and $\tan \alpha_2 = m_2$,

$$m_1 = -\frac{1}{m_2} \quad \text{or} \quad m_1 m_2 = -1. \tag{13}$$

Conversely, if

$$m_1 = -\frac{1}{m_2}, \quad \text{then} \quad \tan \alpha_1 = -\frac{1}{\tan \alpha_2} = -\cot \alpha_2,$$

from which it follows that $\alpha_1 = \alpha_2 \pm 90°$ so that the given lines are perpendicular. Hence, *the condition that l_1 and l_2 are perpendicular is that the product of their slopes equals minus one.*

154. Angle from One Line to Another. In order to choose a definite one among all the angles formed by two given intersecting lines l_1 and l_2, we make the following definition: *the angle from the line l_1 to the line l_2 is the smallest positive angle through which l_1 must be rotated in order to coincide with l_2.* This angle is also frequently spoken of as *the angle l_2 makes with l_1.*

If ϕ is this angle and if m_1 and m_2 are the slopes of l_1 and l_2, respectively, we shall show that

$$\tan \phi = \frac{m_2 - m_1}{1 + m_1 m_2}. \tag{14}$$

We have, in fact,

Case I. If $\alpha_2 > \alpha_1$ (Fig. 68*a*)

$$\alpha_2 = \alpha_1 + \phi.$$

So that

$$\phi = \alpha_2 - \alpha_1.$$

Hence,

$$\tan \phi = \tan (\alpha_2 - \alpha_1) = \frac{\tan \alpha_2 - \tan \alpha_1}{1 + \tan \alpha_1 \tan \alpha_2}.$$

Case II. If $\alpha_1 > \alpha_2$ (Fig. 68*b*)

$$\alpha_1 = \alpha_2 + (180° - \phi).$$

So that

$$\phi = 180° + \alpha_2 - \alpha_1.$$

Hence,

$$\tan \phi = \tan (180° + \alpha_2 - \alpha_1) = \tan (\alpha_2 - \alpha_1) = \frac{\tan \alpha_2 - \tan \alpha_1}{1 + \tan \alpha_1 \tan \alpha_2}.$$

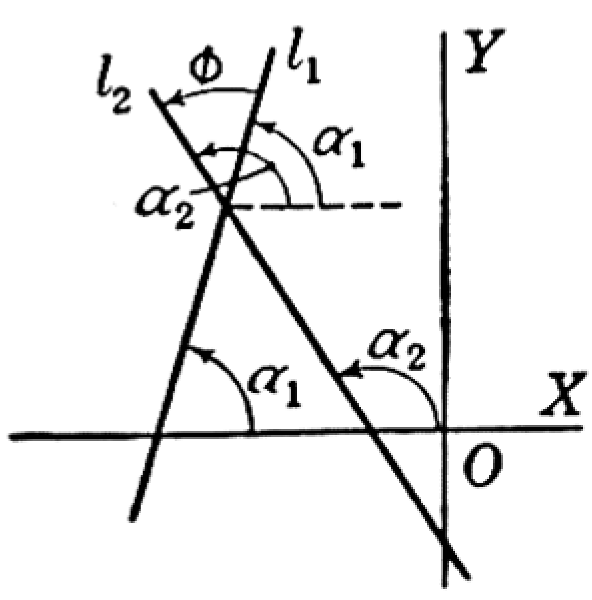


FIG. 68*a*

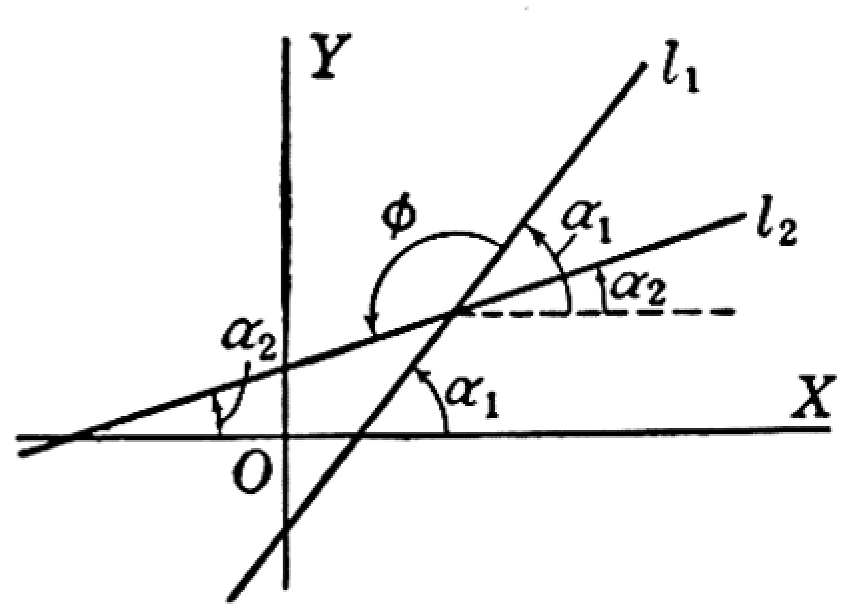


FIG. 68*b*

Since $\tan \alpha_1 = m_1$ and $\tan \alpha_2 = m_2$, we have, accordingly, in either case,

$$\tan \phi = \frac{m_2 - m_1}{1 + m_1 m_2},$$

which is the required formula (14).

EXAMPLE. Find, to the nearest minute, the angles of the triangle (Fig. 69) whose vertices are $A(-1, 2)$, $B(4, 1)$, and $C(7, 6)$.

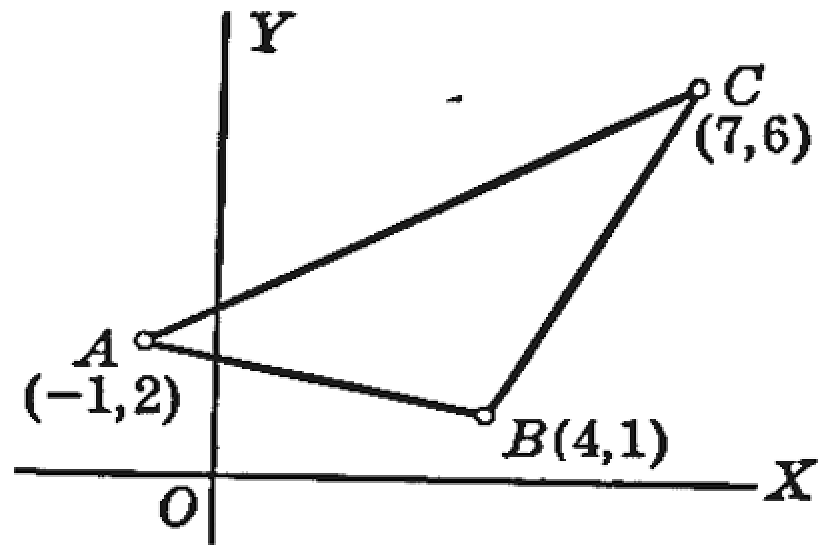


FIG. 69

The slopes m_1, m_2, and m_3, of BC, CA, and AB, respectively, are found by (11) to be

$$m_1 = \tfrac{5}{3}, \quad m_2 = \tfrac{1}{2}, \quad m_3 = -\tfrac{1}{5}.$$

To determine the interior angle of the triangle at A, for example, we notice that, if the line AB is turned around the vertex A through the required angle, it will coincide with AC. Hence

$$\tan A = \frac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{10}} = \tfrac{7}{9} = 0.7778. \qquad A = 37° \, 53'$$

$$\tan B = \frac{-\frac{1}{5}-\frac{5}{3}}{1-\frac{1}{3}} = \frac{-14}{5} = -2.8000. \qquad B = 109° \, 39'$$

$$\tan C = \frac{\frac{5}{3}-\frac{1}{2}}{1+\frac{5}{6}} = \tfrac{7}{11} = 0.6364. \qquad C = \underline{32° \, 28'}$$

$$180° \, 00'$$

Exercises

1. Show that $(-11, 12)$, $(6, -5)$, $(1, -8)$, and $(-6, 15)$ are the vertices of a rectangle and find its area.

2. Show that $(2, 5)$, $(7, 1)$, $(11, 6)$, and $(6, 10)$ are the vertices of a square and find the lengths of its diagonals.

3. Show that $(1, 5)$, $(5, -1)$ and $(9, 6)$ are the vertices of an isosceles triangle and find its equal angles.

4. Show by finding the slopes that the three points lie on a line.

(a) $(3, 8)$, $(5, 4)$, $(8, -2)$; (b) $(-3, -2)$, $(12, 7)$, $(2, 1)$.

5. Express by an equation the condition that the point (x, y) lies on the line through $(3, 4)$ and $(5, 7)$.

HINT. The slope of the line through (x, y) and $(3, 4)$ equals the slope of the line through $(5, 7)$ and $(3, 4)$.

6. Is the line through $(-2, 3)$ and $(4, 12)$ parallel to the line through $(2, -1)$ and $(6, 5)$?

Find the angle from l_1 to l_2, given:

7. $m_1 = \frac{5}{8}$, $m_2 = \frac{13}{3}$.
8. $m_1 = -\frac{5}{2}$, $m_2 = \frac{7}{3}$.
9. $m_1 = \frac{2}{5}$, $m_2 = -\frac{1}{3}$.
10. $m_1 = \frac{7}{2}$, $m_2 = \frac{2}{7}$.

Find the slope of l_2, given:

11. $m_1 = \frac{9}{5}$, $\phi = 45°$.
12. $m_1 = \frac{3}{8}$, $\phi = \tan^{-1}(-\frac{8}{3})$.

Find the angles of the triangle whose vertices are:

13. $(-2, 5)$, $(6, -1)$, $(4, 9)$. **14.** $(2, -3)$, $(5, -7)$, $(7, 6)$.
15. $(-3, 5)$, $(2, 1)$, $(5, 9)$. **16.** $(-2, -1)$, $(1, 6)$, $(5, 1)$.

17. The angle from the line through $(-4, 5)$ and $(3, y)$ to the line through $(-2, 4)$ and $(9, 1)$ is $135°$. Find y.

155. The Equation of a Locus. It was pointed out in Art. 40 that, if we have given an equation in x and y, then the locus (or graph) of this equation is the locus formed by the points whose coördinates satisfy this equation.

In what follows, we shall frequently meet the converse problem; that is, we shall be given a locus, defined, in the statement of the problem, as the locus of the points which satisfy a certain geometric condition. It will then be required to find the **equation of this locus**; that is, the equation which has this locus as its graph.

To find the equation of the locus, first take a point $P(x, y)$ on the locus. Next, state, by an equation in the coördinates of $P(x, y)$, the geometric condition that defines the locus. This equation, if stated so that it is satisfied by the coördinates of the points on the locus and no others, is the equation of the locus.

Frequently the equation, as obtained from the geometric definition, can be simplified. Care must be taken, in this process of simplification, that no solutions are lost and that no extraneous solutions are introduced.

EXAMPLE 1. Find the equation of the circle (Fig. 70) with center at $(2, -3)$ and radius 5.

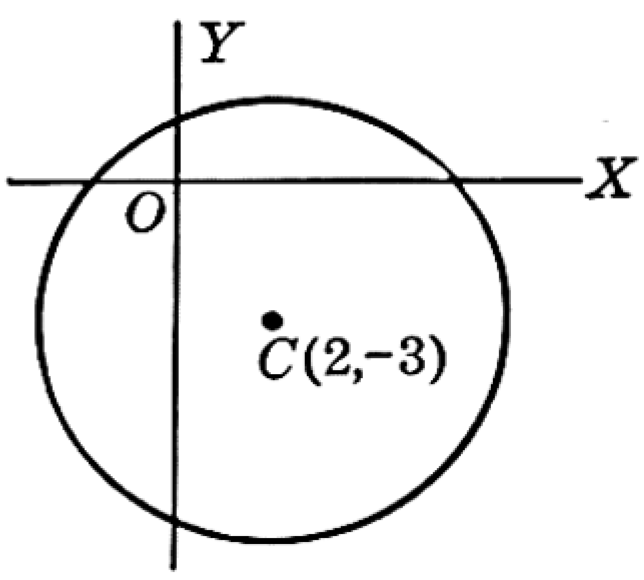


FIG. 70

By geometry, this circle is the locus of a point $P(x, y)$ whose distance from $(2, -3)$ is equal to 5. Hence, by the distance formula

$$\sqrt{(x-2)^2 + (y+3)^2} = 5,$$

or

$$(x-2)^2 + (y+3)^2 = 25.$$

This equation may be further simplified to the form

$$x^2 + y^2 - 4x + 6y - 12 = 0.$$

The coördinates of every point on the circle satisfy any one of these equations, and conversely, any point whose coördinates satisfy any one of the equations is at a distance 5 from $(2, -3)$ and lies on the circle. Hence, any one of these equations is an equation of the given circle.

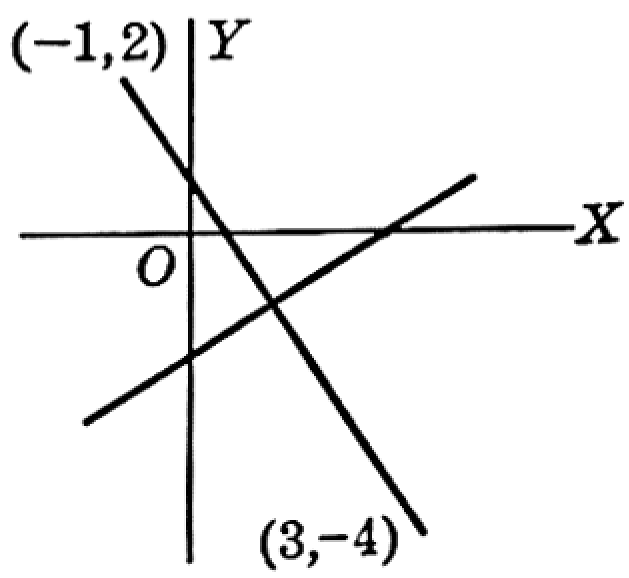


FIG. 71

EXAMPLE 2. Find the equation of the perpendicular bisector of the line segment (Fig. 71) joining the points $(-1, 2)$ and $(3, -4)$.

By geometry, this line is the locus of a point $P(x, y)$ whose distance from $(-1, 2)$ is equal to its distance from $(3, -4)$. Hence,

$$\sqrt{(x+1)^2 + (y-2)^2} = \sqrt{(x-3)^2 + (y+4)^2},$$

or $$x^2 + y^2 + 2x - 4y + 5 = x^2 + y^2 - 6x + 8y + 25.$$

This equation may be simplified to

$$2x - 3y - 5 = 0.$$

This equation is satisfied by the coördinates of every point on the bisector and not by the coördinates of any other point. Hence it is the required equation of the bisector.

Sometimes we shall be able to draw the graph of a given equation by recognizing the equation as the equation of a known locus.

Example 3. Find the locus of the equation $(x+2)^2 + (y-7)^2 = 16$.

If we write this equation in the form

$$\sqrt{(x+2)^2 + (y-7)^2} = 4,$$

we see that its graph is the locus of a point whose distance from $(-2, 7)$ is 4. Hence, the required graph is a circle with center at $(-2, 7)$ and radius 4.

Exercises

Find, and simplify whenever possible, the equation of each of the following loci.

1. The x-axis. **2.** The y-axis.

3. A line parallel to, and 3 units to the right of, the y-axis.

4. A line parallel to, and 5 units below, the x-axis.

5. A circle with center at the origin and radius 6.

6. A circle with center at $(-3, 5)$ and radius 4.

7. The perpendicular bisector of the line segment joining $(1, 3)$ and $(5, 1)$.

8. The perpendicular bisector of the line segment joining $(-2, -5)$ and $(3, 4)$.

9. The line through $(-1, 7)$ of slope -3.

Hint. Find the slope of the line joining (x, y) to $(-1, 7)$. Equate the result to -3.

10. The line through $(6, 5)$ of slope 2.

11. The line through $(4, 1)$ of inclination 45°.

12. The line through $(0, 6)$ of inclination 60°.

Identify the locus of each of the following equations.

13. $(x-1)^2 + (y-3)^2 = 49$. **14.** $(x+4)^2 + (y-2)^2 = 36$.

15. $\sqrt{(x-3)^2 + (y-6)^2} = \sqrt{(x+1)^2 + (y-2)^2}$.

16. $\sqrt{(x+2)^2 + y^2} = \sqrt{(x-4)^2 + (y-6)^2}$.

17. $\dfrac{y-1}{x-4} = 3$. **18.** $\dfrac{y+6}{x+3} = -2$. **19.** $y - 4 = 2(x+1)$.

Find the equation of the locus of a point that satisfies the following conditions. Identify each locus.

20. The sum of the squares of its distances from the axes is 36.

21. Its directed distances from the axes are equal.

22. Its directed distance from the x-axis equals -2 times its directed distance from the y-axis.

23. The square of its distance from the origin equals 6 times its directed distance from the y-axis.

Chapter 20

The Line

156. The Equation of a Line. In order that an equation be the equation of a line, it is necessary, first, that the coördinates of every point on the line shall satisfy the equation and, further, that every point whose coördinates satisfy the equation shall lie on the line.

In the applications, we must be able to write the equation of a line when enough geometric conditions are given to fix its position. This information may be given to us in any one of a number of ways; for example, we may be given its direction and the position of one point on it, or the positions of two of its points, and so forth. We shall, accordingly, begin this chapter by showing how the equation of a line may be found when its position has been fixed in various ways.

157. Lines Parallel to the Axes. If a line is parallel to the y-axis, it meets the x-axis in some point $(a, 0)$. It follows from the definition of the coördinates of a point (Art. 38) that the abscissa of any point on the line is $x = a$ and, further, that any point whose abscissa is $x = a$ lies on the line. Hence, the equation of any line parallel to the y-axis is

$$x = a.$$

In a similar way, we find that the equation of the line parallel to the x-axis that intersects the y-axis at $(0, b)$ is

$$y = b.$$

158. The Point-Slope Form. If the line l whose equation is required passes through a given point $P(x_1, y_1)$ (Fig. 72) and has for its slope a given number m, we shall show that its equation is

$$y - y_1 = m(x - x_1). \qquad (1)$$

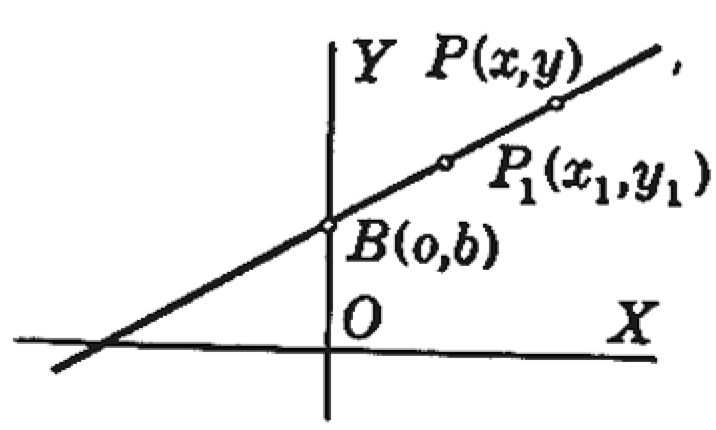


FIG. 72

This equation is called the **point-slope form** of the equation of the line.

To show that (1) is the equation of the line, we first observe that the point P_1 itself lies on l and that its coördinates satisfy the equation since, if we put $x = x_1$ and $y = y_1$, both members of (1) are zero.

Next, let $P(x, y)$ be any point on l other than P_1. Then the slope of the line through P_1 and P is m, that is

$$m = \frac{y - y_1}{x - x_1}, \qquad (2)$$

or

$$y - y_1 = m(x - x_1).$$

Hence, the coördinates of P satisfy (1).

Conversely, if the coördinates of any point P, other than P_1, satisfy (1), then they also satisfy (2). Hence the slope of the line P_1P is m and P lies on l.

159. The Slope-Intercept Form. If the given line intersects the y-axis at $B(0, b)$, the number b is called its **y-intercept.**

Since $B(0, b)$ is a fixed point on l, we may take its coördinates as (x_1, y_1) in (1). This gives

$$y - b = m(x - 0),$$

or

$$y = mx + b. \tag{3}$$

This is the **slope-intercept form** of the equation of the line.

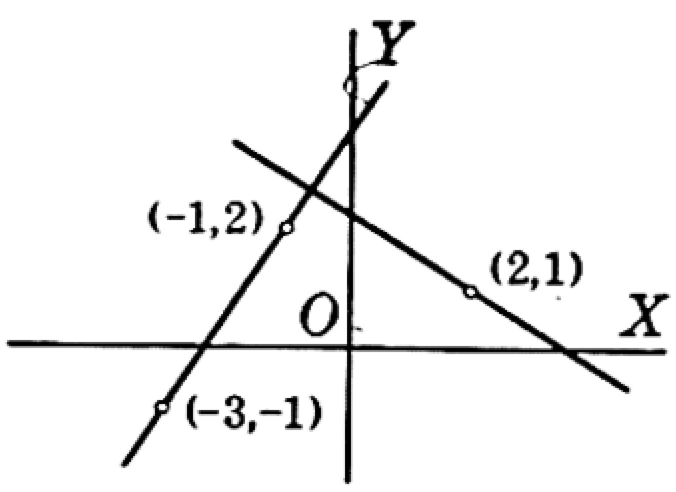


FIG. 73

EXAMPLE. Find the equation of the line through (2, 1) (Fig. 73) perpendicular to the line through (− 3, − 1) and (− 1, 2).

The slope of the line through (− 3, − 1) and (− 1, 2) is $\frac{3}{2}$. Hence (Art. 153) the slope of the required line is $-\frac{2}{3}$ and, since it passes through (2, 1), its equation is

$$y - 1 = -\tfrac{2}{3}(x - 2) \quad \text{or} \quad 2x + 3y - 7 = 0.$$

Exercises

Find the equation of the line through the given point having the given slope.

1. $(4, 1)$, $m = 3$. **2.** $(-2, 3)$, $m = -2$. **3.** $(-2, 0)$, $m = \frac{4}{3}$.
4. $(-3, -2)$, $m = -\frac{5}{3}$. **5.** $(3, 7)$, $m = 0$. **6.** $(0, -3)$, $m = \frac{2}{3}$.

Find the equation of the line through the given point having the given inclination.

7. $(1, -3)$, $\alpha = \pi/4$. **8.** $(-2, 5)$, $\alpha = \tan^{-1}(-\frac{4}{3})$.
9. $(1, 4)$, $\alpha = \tan^{-1}\frac{2}{3}$. **10.** $(7, -3)$, $\alpha = \pi/2$.

Find the equations of the following lines, given:

11. $m = \frac{3}{2}$, $b = 4$. **12.** $m = -3$, $b = \frac{2}{5}$.
13. $m = -\frac{4}{5}$, $b = -\frac{2}{3}$. **14.** $m = 0$, $b = -5$.

15. Find the equations of the lines through (− 2, 3) parallel to the lines in Ex. 11 to 14.

16. Find the equations of the lines through (4, 2) perpendicular to the lines in Ex. 11 to 14.

17. Write the equations of two lines parallel to the y-axis and at a distance from it numerically equal to 5.

18. Write the equations of the lines through (4, 2) parallel to the x-axis and to the y-axis.

Find the slope and the y-intercept of each of the following lines.

19. $y = 2x + 9$. **20.** $3y = 4x - 7$.
21. $2x + 5y = 15$. **22.** $3x + 4y + 6 = 0$.

23. Find the equation of the line through $(7, -3)$ parallel to the line through $(-1, 2)$ and $(5, 11)$.

24. Find the equation of the line through $(-6, 1)$ perpendicular to the line through $(4, 1)$ and $(-2, 5)$.

25. Find the equation of the line through $(3, -1)$ such that the angle from it to the line $y = 2x + 6$ is $45°$.

160. The Two-Point Form. If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are two given points on a line, and if $x_1 \neq x_2$, the slope of the line through these two points is, by Art. 152,

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

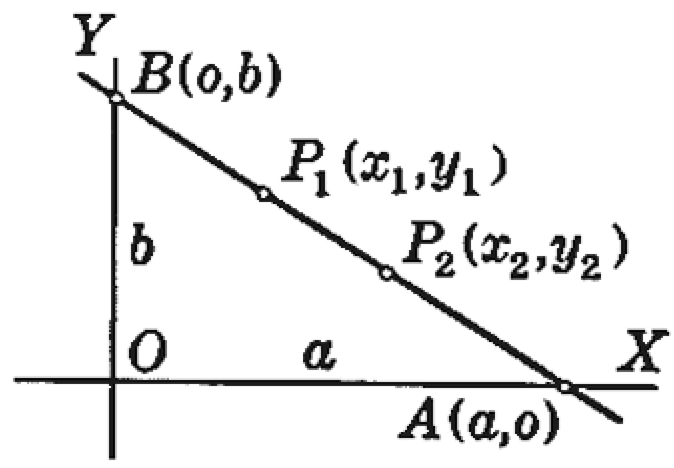


FIG. 74

If we substitute this value of m in equation (1), we have

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1). \qquad (4)$$

This is the **two-point form** of the equation of a line.

If $x_1 = x_2$, the line through P_1 and P_2 is parallel to the y-axis and its equation is, by Art. 157,

$$x = x_1.$$

161. The Intercept Form. The directed distances, OA and OB (Fig. 74) from the origin to the intersections of a line with the axes are called the x- and y-intercepts, respectively, of the line. We shall denote them by a and b.

Let a and b both be different from zero. Then $A(a, 0)$ and $B(0, b)$ are two fixed points on the line and we find, by taking the coördinates of these points as (x_1, y_1) and (x_2, y_2) in (4), that the equation of this line is

$$y - 0 = \frac{b - 0}{0 - a}(x - a),$$

or

$$-ay = bx - ab.$$

If we rearrange this equation, and divide by ab which, by hypothesis, is different from zero, we obtain

$$\frac{x}{a} + \frac{y}{b} = 1, \qquad (5)$$

which is the **intercept form** of the equation of the line.

EXAMPLE 1. Find the equation of the line through the intersection of $x + 2y - 4 = 0$, $x - 3y + 1 = 0$ and also through the midpoint of the segment joining $(2, 5)$ and $(4, 3)$.

The point of intersection of the given lines is found, by solving their equations as simultaneous, to be (2, 1) and the midpoint of the given segment, by Art. 150, is (3, 4). The equation of the line through these two points is

$$y - 1 = \frac{4 - 1}{3 - 2}(x - 2),$$

or

$$y = 3x - 5.$$

EXAMPLE 2. Find the intercepts of the line $3x - 2y - 12 = 0$ and write its equation in the intercept form.

The x-intercept is found, by putting $y = 0$ and solving for x, to be $a = 4$. Similarly, by putting $x = 0$, we find for the y-intercept $b = -6$. Hence the intercept form of the equation is $\frac{x}{4} + \frac{y}{-6} = 1$.

Exercises

Find the equation of the line through the two given points.

1. (3, 5), (7, − 1).
2. (− 2, − 3), (4, 7).
3. (− 1, 2), (5, − 4).
4. (2, 0), (7, 6).
5. (0, 3), (6, − 1).
6. (− 2, − 4), (3, 7).

Write each of the following equations in (*a*) the intercept form and (*b*) the slope-intercept form.

7. $2x + 7y = 14$.
8. $4x - 5y = 20$.
9. $3x - 4y + 15 = 0$.
10. $2x + 5y + 9 = 0$.

Write the equations of two lines through the given point, one parallel and the other perpendicular to the given line.

11. (8, 6), $2x + 3y - 7 = 0$.
12. (4, − 1), $3x - 5y + 8 = 0$.
13. (− 5, − 2), $4x - 5y - 10 = 0$.
14. (7, − 3), $2x + 3y - 9 = 0$.

15. Write the equation, in the intercept form, of the line through (3, − 2) parallel to the line through (5, 1) and (− 1, 10).

16. Find the point on the line $2x - 3y + 13 = 0$ that is equidistant from (3, − 4) and (5, 8).

17. Find the equations of the sides of the parallelogram whose vertices are (− 1, 3), (1, 8), (9, 7), (7, 2).

18. Three vertices of a rectangle are (2, 3), (1, 8), (− 4, 7). Find the equations of its sides and the coördinates of its fourth vertex.

19. Find the equations of the sides of the triangle whose vertices are (3, 4), (13, 8), (9, − 4).

20. Find the equations of the altitudes of the triangle in Ex. 19 and find the coördinates of their common point.

21. Find the equations of the perpendicular bisectors of the sides of the triangle in Ex. 19 and find the coördinates of their common point.

22. Find the equation of a line, given that its slope is m and its x-intercept is a.

162. The General Form. An equation of the form,

$$Ax + By + C = 0, \tag{6}$$

where A, B, and C are constants and A and B are not both zero, is *an equation of the first degree in x and y*. We shall show that:

Every equation of the first degree in x and y, with real coefficients, is the equation of a line.

There are two cases, according as $B \neq 0$ or $B = 0$.

If $B \neq 0$, we can solve the equation for y, giving

$$y = -\frac{A}{B}x - \frac{C}{B}. \tag{7}$$

By Art. 159, this is the equation of a line in the slope-intercept form

$$y = mx + b,$$

for which the slope, m, and the y-intercept, b, have the values

$$m = -\frac{A}{B}, \quad \text{and} \quad b = -\frac{C}{B}. \tag{8}$$

If $B = 0$, we can solve equation (6) for x, giving

$$x = -\frac{C}{A}.$$

By Art. 157, this is the equation of a line parallel to the y-axis.

Hence, in both cases, equation (6) is the equation of a line. It is called the **general form** of the equation of a line.

Because of the theorem of this article, equation (6) is usually spoken of as a **linear equation** in x and y.

From equations (7) and (8), we have the following useful result: *If the equation of a line is solved for y, the coefficient of x is the slope, and the constant term is the y-intercept, of the line.*

The student should show further, by putting $y = 0$ and solving for x, that the x-intercept of the line (6) is

$$a = -\frac{C}{A}. \tag{9}$$

EXAMPLE. Given the line $3x + 4y - 24 = 0$. Find its slope and its intercepts and reduce its equation to the slope intercept and to the intercept form.

We first solve the equation for y, giving

$$y = -\tfrac{3}{4}x + 6.$$

This is the slope-intercept form. From it we obtain at once $m = -\frac{3}{4}$ and $b = 6$. To find a, we put $y = 0$ and solve for x. We obtain $x = 8 = a$. Hence

$$\frac{x}{8} + \frac{y}{6} = 1$$

is the intercept form of the equation of the given line.

163. The Linear Function. An expression of the form

$$mx + b, \qquad m \neq 0 \tag{10}$$

is called a **linear function** of x.

To find the graph of a linear function, we equate it to y and plot the graph of the resulting equation. We thus find that *the graph of a linear equation is a line.*

In the applications of mathematics, two variable quantities are often related in such a way that a change of a given amount in one produces a proportional change in the other. Under such circumstances, the second is a linear function of the first.

EXAMPLE. An automobile, traveling at a constant rate of 40 miles per hour, has gone 100 miles by noon. Express the distance traveled as a function of the time after noon and draw the graph.

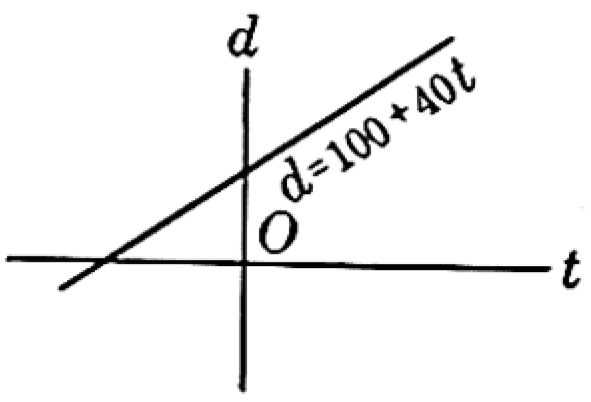


FIG. 75

We have

$$d = 100 + 40t$$

where d is the distance traveled in miles and t is the time in hours.

The graph of d, as a function of t, is given in Figure 75. In drawing this graph, we have used a unit on the t-axis 60 times as large as on the d-axis.

Exercises

Find the slopes and intercepts of the following lines.

1. $4x - y = 8$. **2.** $5x - 3y = 30$.

3. $3x + 2y = 12$. **4.** $4x + 3y = 7$.

5. $4x + 7y + 10 = 0$. **6.** $2x + 5y = 0$.

7. Find the equation of a line, given that its x-intercept is -4 and its inclination is 135°.

8. Find the angle from the line $x + 3y = 7$ to the line $2x - 5y = 12$.

9. The sides of a parallelogram are parallel to $2y = 3x - 1$ and $2x + 5y = 9$. Two vertices are (5, 8) and (3, 1). Find the equations of the sides of the parallelogram.

10. Two sides of a rectangle are parallel to $3x + 5y = 7$ and have x-intercepts 2 and -5, respectively. The other two sides have y-intercepts 3 and 7, respectively. Find the equations of the sides of the rectangle.

11. The equations of two sides of a parallelogram are $3x - 2y + 5 = 0$ and $3x + 5y - 9 = 0$. If $(4, -2)$ is a vertex, find the equations of the other two sides.

12. The equations of two sides of a rectangle are $x - 2y = 4$ and $x - 2y + 11 = 0$. The equation of one diagonal is $8x - y = 17$. Find the equations of the other two sides.

13. Find the slope of a line in terms of its intercepts a and b.

14. Find the x-intercept of a line in terms of its slope m and its y-intercept b.

15. Show that the condition that the lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ are parallel is $A_1B_2 - A_2B_1 = 0$.

16. Show that the condition that the lines in Ex. 15 are perpendicular is $A_1A_2 + B_1B_2 = 0$.

17. Show that, if $C_1 \neq C_2$, the lines $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$ are parallel.

18. Show that the lines $Ax + By + C_1 = 0$ and $Bx - Ay + C_2 = 0$ are perpendicular.

164. The Normal Form. Let l (Fig. 76) be the given line. Draw through O a line ON perpendicular to l and let ω be the *inclination of this perpendicular*. From Art. 151, we have, $0° \leqq \omega < 180°$. We shall consider ON as a directed line, its positive direction being that of the terminal half-line of ω.

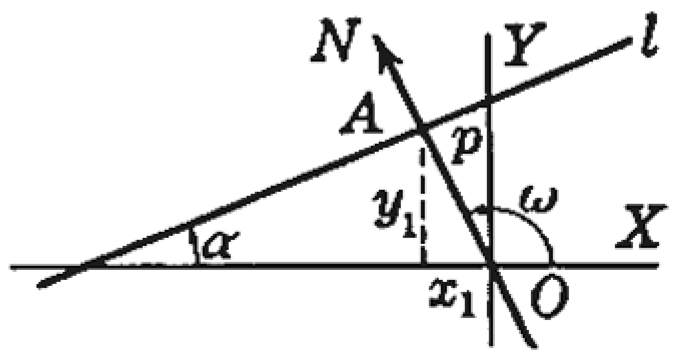


FIG. 76

Let $A(x_1, y_1)$ be the intersection of ON with l and denote the length of the directed segment OA by p. Then p is positive if A lies above O and negative if it lies below.

From the definitions of sin ω and cos ω, we find that

$$x_1 = p \cos \omega, \qquad y_1 = p \sin \omega.$$

Let m be the slope of l. Since ON is perpendicular to l, we have

$$m = -\cot \omega.$$

If we substitute these values of x_1, y_1, and m in the point slope equation (1) of a line, we have

$$y - p \sin \omega = -\cot \omega (x - p \cos \omega).$$

In this equation, we replace cot ω by its value $\dfrac{\cos \omega}{\sin \omega}$ and multiply through by sin ω. We thus obtain

$$y \sin \omega - p \sin^2 \omega = -x \cos \omega + p \cos^2 \omega,$$

or
$$x \cos \omega + y \sin \omega - p(\sin^2 \omega + \cos^2 \omega) = 0,$$

that is
$$\mathbf{x \cos \omega + y \sin \omega - p = 0.} \tag{11}$$

This is the **normal form** of the equation of a line. Its importance arises chiefly from the fact (which we shall prove in Art. 166) that, whenever we are required to determine the distance from a line to a point, we shall need the equation of the line in the normal form.

EXAMPLE 1. Find the normal form of the equation of the line through $A(-3, -4)$ perpendicular to the line through A and the origin.

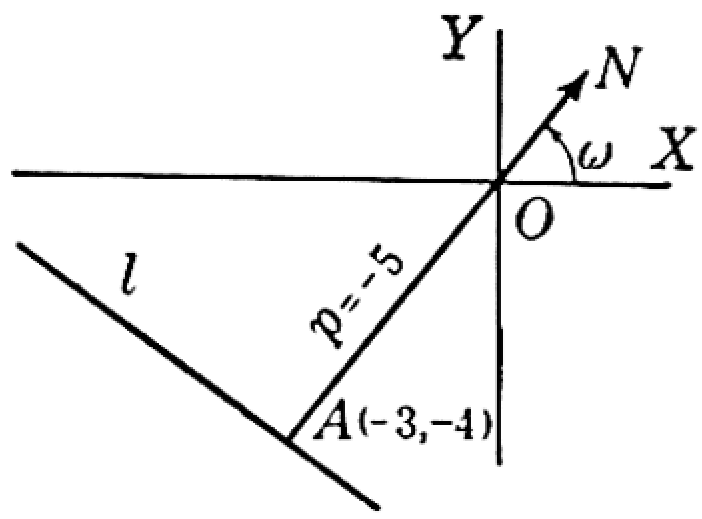


FIG. 77

Draw the line ON through O and A. From the figure, ω is an acute angle and OA, or p, is negative. Hence we have

$$p = -\sqrt{(-3)^2 + (-4)^2} = -5, \quad \sin\omega = \frac{-4}{-5} = \frac{4}{5},$$

$$\cos\omega = \frac{-3}{-5} = \frac{3}{5},$$

and the required equation is

$$\frac{3x}{5} + \frac{4y}{5} + 5 = 0.$$

EXAMPLE 2. Two lines of inclination 60° lie at a distance from the origin numerically equal to 10. Find their equations in the normal form.

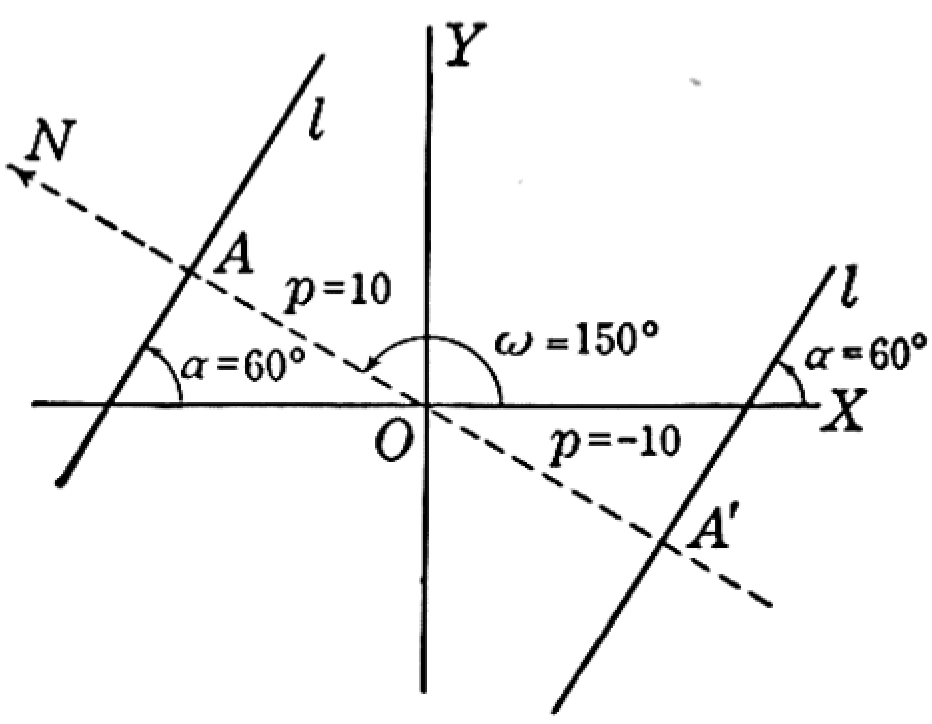


FIG. 78

From Figure 78 we find $\omega = 60° + 90° = 150°$, so that $\sin\omega = \sin 150° = \frac{1}{2}$ and $\cos\omega = \cos 150° = -\frac{\sqrt{3}}{2}$. For one of the required lines, $p = 10$ and, for the other, $p = -10$. Hence the required equations are

$$-\frac{\sqrt{3}}{2}x + \frac{1}{2}y - 10 = 0 \text{ and } -\frac{\sqrt{3}}{2}x + \frac{1}{2}y + 10 = 0.$$

165. Reduction of the Equation of a Line to the Normal Form. Let

$$Ax + By + C = 0 \tag{12}$$

be the equation of a line and let

$$x\cos\omega + y\sin\omega - p = 0 \tag{13}$$

be the normal form of the equation of the same line. To reduce the first equation to the second, we multiply it by a constant k

$$kAx + kBy + kC = 0, \tag{14}$$

and determine k so that the coefficients in this equation and in (13) are equal, that is,

$$kA = \cos\omega, \qquad kB = \sin\omega, \qquad kC = -p. \tag{15}$$

If we square the members of the first two of equations (15) and add, we obtain

$$k^2A^2 + k^2B^2 = \cos^2\omega + \sin^2\omega = 1.$$

If we solve this equation for k, we find that

$$k = \frac{1}{\pm\sqrt{A^2+B^2}}. \tag{16}$$

To determine the sign of the radical, we note that, since $0° \leqq \omega < 180°$, $\sin\omega$ is always positive or zero. Hence, by the second equation of (15), if $B \neq 0$, k and B have the same signs. If, however, $B = 0$, then $\sin\omega = 0$, so that $\omega = 0$, $\cos\omega = 1$ and k agrees in sign with A.

If we substitute the value of k from (16) in (14), we obtain, as the normal form of the equation of the line defined by (12),

$$\frac{A}{\pm\sqrt{A^2+B^2}}x + \frac{B}{\pm\sqrt{A^2+B^2}}y + \frac{C}{\pm\sqrt{A^2+B^2}} = 0, \tag{17}$$

where the signs before the radicals agree with that of B if $B \neq 0$, and with that of A if $B = 0$.

Hence, *to reduce the equation of a line to the normal form, divide each term by* $\pm\sqrt{A^2+B^2}$, *choosing the sign before the radical so as to make the coefficient of* y *positive if* $B \neq 0$, *and the coefficient of* x *positive if* $B = 0$.

Use the opposite sign of constant

EXAMPLE. Reduce the equation $x - 3y - 7 = 0$ (Fig. 79) to the normal form and find the values of ω and p.

Since $B = -3 < 0$, divide each term by $-\sqrt{1^2+(-3)^2} = -\sqrt{10}$. The required equation is thus found to be

$$\frac{-x}{\sqrt{10}} + \frac{3y}{\sqrt{10}} + \frac{7}{\sqrt{10}} = 0.$$

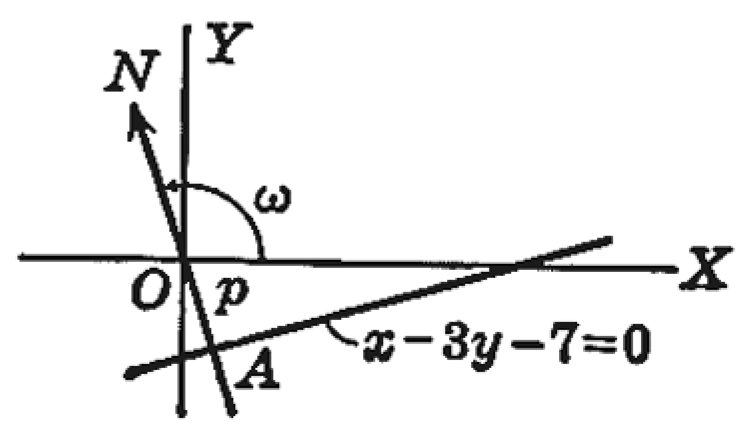


FIG. 79

Since this equation is in the normal form, we have, by comparing it with equation (11),

$$\cos\omega = -\frac{1}{\sqrt{10}}, \qquad \sin\omega = \frac{3}{\sqrt{10}}, \qquad \text{and} \qquad p = -\frac{7}{\sqrt{10}}.$$

From the tables, we find $\omega = 108° 26'$.

Exercises

Write the equations of the following lines in the normal form.

1. $\omega = 45°$, $p = 5$.
2. $\omega = 150°$, $p = 6$.
3. $\omega = 60°$, $p = -4$.
4. $\omega = \pi/6$, $p = -2$.
5. $\omega = 2\pi/3$, $p = 7$.
6. $\omega = 3\pi/4$, $p = -9$.

Write the equations of the following lines in the normal form and find the values of ω and p.

7. $5x + 5\sqrt{3}y - 6 = 0$.
8. $x + y + 11 = 0$.
9. $x - y = 0$.
10. $\sqrt{3}x - y + 8 = 0$.
11. $3x - 4y - 15 = 0$.
12. $8x + 15y = 0$.

13. Find the equation of the line through $(3, -7)$ for which (*a*) $\omega = 135°$, (*b*) $\omega = \pi/6$.

14. Find the equation of the line through $(-12, 5)$ which is perpendicular to the line joining this point to the origin.

15. Find the equation of a line whose y-intercept is 8 and for which $\omega = 150°$.

16. Find the equations of two lines of inclination 30° each of which is tangent to the circle of radius 3 with its center at the origin.

HINT. If a line is tangent to a circle, it is perpendicular to the radius at its point of tangency and its distance from the center is equal to the radius.

17. Solve Ex. 16 when the inclination is 135° and the radius is 7.

18. Find the equations of two lines whose x-intercepts are 13 and which are tangent to the circle of radius 5 with its center at the origin.

19. Solve Ex. 18 when the y-intercept is 5 and the radius is 4.

166. The Distance from a Line to a Point. Let $P_1(x_1, y_1)$ be the given point and let the normal form of the equation of the given line be

$$x \cos \omega + y \sin \omega - p = 0, \tag{18}$$

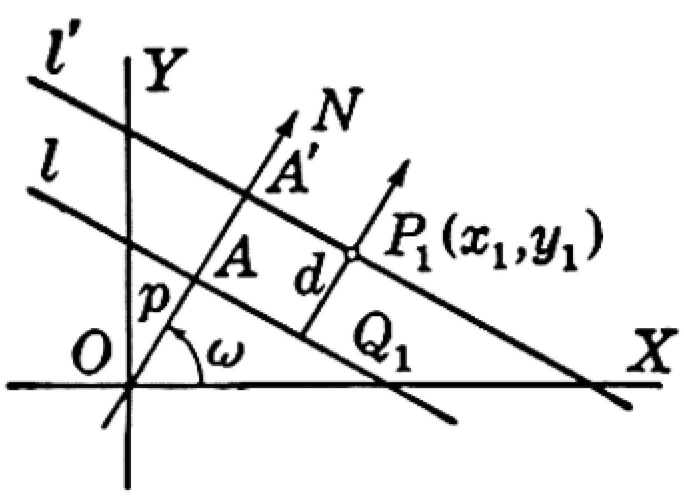


FIG. 80

in which p is the directed distance OA (Fig. 80). If l' is the line through P_1 parallel to l, we may write its equation in the normal form

$$x \cos \omega + y \sin \omega - p' = 0,$$

where ω has the same value as in equation (18) (Why?) and p' is the directed distance OA'. Since P_1 lies on l', its coördinates satisfy this equation; that is,

$$x_1 \cos \omega + y_1 \sin \omega - p' = 0. \tag{19}$$

Let Q_1 be the foot of the perpendicular from P_1 on l and let us choose the positive direction on the line through Q_1 and P_1 to agree with that on ON. Then we have, for the required directed distance d,

$$d = Q_1P_1 = AA' = OA' - OA = p' - p.$$

If we substitute for p' in this equation its value from (19), we have

$$\mathbf{d = x_1 \cos \omega + y_1 \sin \omega - p;} \tag{20}$$

that is, *to find the distance from a line to a point, substitute the coördinates of the point in the first member of the normal form of the equation of the line. The resulting number is the required distance.*

The distance d, or Q_1P_1, determined by (20), is a directed distance. It is positive if Q_1P_1 agrees in direction with the positive direction on ON and it is negative in the contrary case.

If the equation of the line is given in the general form

$$Ax + By + C = 0$$

and we wish to find the distance of the point $P_1(x_1, y_1)$ from it, we first reduce the equation of the line to the normal form, as in equation (17), and then substitute the coördinates of P_1 in it. The result is

$$d = \frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}} \tag{21}$$

wherein the sign before the radical agrees with that of B if $B \neq 0$, and agrees with that of A if $B = 0$.

EXAMPLE. Find the distance from the line $2x - y - 4 = 0$ to each of the points $P_1(-1, -2)$, $P_2(5, 0)$, and $P_3(3, 5)$.

We first reduce the given equation to the normal form by dividing through by $-\sqrt{2^2 + (-1)^2} = -\sqrt{5}$. The result is

$$\frac{2x - y - 4}{-\sqrt{5}} = 0.$$

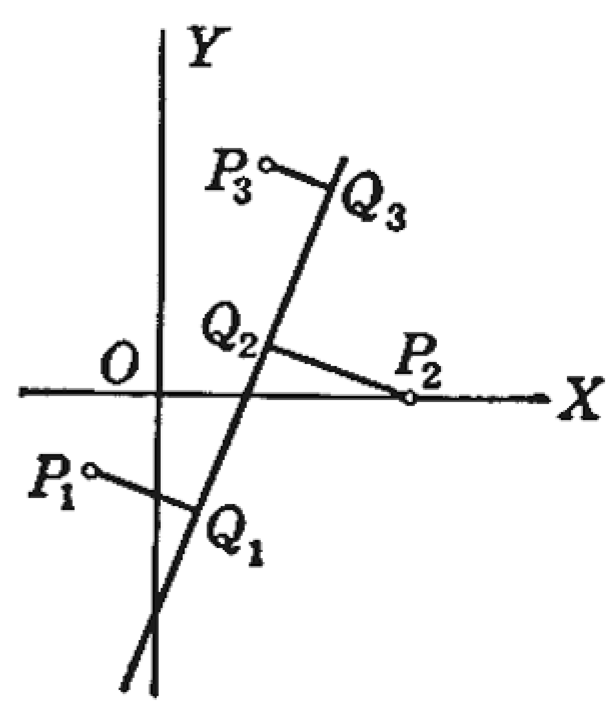


FIG. 81

By substituting the coördinates of the given points in the first member of this equation, we find as the required distances

$$d_1 = \frac{4}{\sqrt{5}}, \qquad d_2 = -\frac{6}{\sqrt{5}}, \qquad d_3 = \frac{3}{\sqrt{5}}.$$

Since d_1 and d_3 are positive and d_2 is negative, the points P_1 and P_3 lie *above* the line and P_2 lies *below* it (Fig. 81).

Exercises

Find the distance from the given line to the given point and state whether the point lies above the line or below it.

1. $15x + 8y - 14 = 0$, $(5, 3)$.
2. $3x - 4y + 8 = 0$, $(-5, 2)$.
3. $5x - 12y + 3 = 0$, $(7, 1)$.
4. $24x + 7y = 0$, $(3, -11)$.
5. $3x + 8y = 0$, $(-5, 4)$.
6. $2x - 9y - 14 = 0$, $(-4, -3)$.

7. Show that the points $(-2, -1)$ and $(11, 2)$ lie on opposite sides of the line $2x - 5y - 9 = 0$.

8. Show that the point $(5, -3)$ lies between the parallel lines $2x + 3y - 9 = 0$ and $6x + 9y + 13 = 0$.

Find the distance between the following pairs of parallel lines.

9. $3x + 4y - 9 = 0$, $3x + 4y + 6 = 0$.
10. $12x - 5y + 4 = 0$, $36x - 15y - 40 = 0$.
11. $2x + y - 7 = 0$, $4x + 2y + 11 = 0$.
12. $3x - 7y + 2 = 0$, $6x - 14y + 21 = 0$.

13. Write the equation of the locus of a point whose directed distance from $4x + 7y - 5 = 0$ is (*a*) 3, (*b*) -7.

14. Find two points on the line $x + 7y = 22$ that lie at a distance from $3x - 4y + 9 = 0$ numerically equal to 5.

15. Find two lines parallel to $4x + 3y + 5 = 0$ that lie at a distance from $(3, -7)$ numerically equal to 8.

16. Show that the equations of the bisectors of the pairs of vertical angles formed by the lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ are

$$\frac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2}} = \pm \frac{A_2x + B_2y + C_2}{\sqrt{A_2^2 + B_2^2}}.$$

HINT. One bisector is the locus of the points whose distances from the given lines are equal in magnitude and sign; the other is the locus of the points whose distances are equal in magnitude but opposite in sign.

17. Using the results of Ex. 16, find the equations of the bisectors of the angles formed by the lines $2x + 11y + 30 = 0$ and $x - 2y + 3 = 0$ and state which bisects the angle in which $(-2, 5)$ lies.

18. Find the equations of the bisectors of the interior angles of the triangle formed by the lines $7x + 6y - 11 = 0$, $9x - 2y + 7 = 0$, and $6x - 7y - 16 = 0$. Show that these lines meet in a point (the center of the inscribed circle) and find its coördinates.

167. The Area of a Triangle. Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, and $P_3(x_3, y_3)$ be the vertices of a triangle. If we consider the side P_2P_3 as the base, the length of the base is, by the distance formula,

$$b = \sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2}.$$

To find the altitude, we first find the equation of the line through P_2 and P_3. By equation (4), this is

$$(y_2 - y_3)x - (x_2 - x_3)y + x_2y_3 - y_2x_3 = 0.$$

The altitude, h, is the distance of P_1 from this line. Hence,

$$h = \frac{(y_2 - y_3)x_1 - (x_2 - x_3)y_1 + x_2y_3 - y_2x_3}{\pm\sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2}}.$$

For the area, we have $S = \frac{1}{2}bh$. Replace b and h by their values just given. We have

$$S = \pm\, \tfrac{1}{2}(x_1y_2 - x_1y_3 - x_2y_1 + x_3y_1 + x_2y_3 - x_3y_2).$$

This formula is more easily remembered if it is written as a determinant, as follows,

$$S = \pm \tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}. \tag{22}$$

The sign should be chosen so as to make S positive.

Exercises

Find the areas of the triangles whose vertices are:

1. $(2, -3)$, $(-9, 8)$, $(8, 4)$.
2. $(9, 5)$, $(1, 6)$, $(2, 1)$.
3. $(6, -1)$, $(-7, 9)$, $(9, 11)$.
4. $(3, 5)$, $(8, 1)$, $(9, 4)$.
5. $(5, 3)$, $(-1, 7)$, $(3, 1)$.
6. $(5, 2)$, $(1, 3)$, $(-6, 2)$.
7. $(-2, 3)$, $(4, -1)$, $(6, 2)$.
8. $(-1, 2)$, $(2, 3)$, $(6, -3)$.

Show by areas that the three points lie on a line. Check your results by finding the slopes.

9. $(1, 3)$, $(4, -2)$, $(-5, 13)$.
10. $(-1, -5)$, $(5, 4)$, $(9, 10)$.

168. Families of Lines. Parameters. If, in the equation

$$3x + 5y + k = 0, \tag{23}$$

we substitute for k any number we please, we determine a line. For example, if we put $k = 10$, we obtain

$$3x + 5y + 10 = 0$$

which is the equation of a line; and similarly for any other value we may assign to k (Fig. 82).

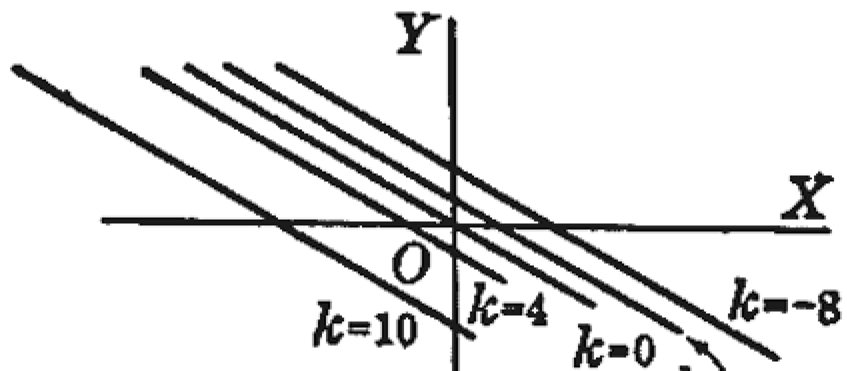


Fig. 82

All the lines that can be determined by substituting values for k in (23) are parallel since their slopes all equal $-3/5$. Moreover, by substituting a suitable value for k in (23), we can obtain the equation of any given line of slope $-3/5$. Equation (23) is consequently called the equation of the **family** of lines of slope $-3/5$ and k is the **parameter** of the family.

Similarly, the equation,

$$(3x - 2y + 4) + k(x - 5y + 2) = 0, \tag{24}$$

in which k is a parameter, defines a family of lines. All the lines of this family pass through the point of intersection of the lines

$$3x - 2y + 4 = 0, \quad \text{and} \quad x - 5y + 2 = 0. \tag{25}$$

For, the coördinates of the point of intersection of these two lines satisfy both of equations (25) and hence, when substituted in (24), reduce that equation to $0 + k0 = 0$, which is true for all values of k. Equation (24)

is called the equation of the family of lines through the point of intersection of the lines (25).

More generally, if the coefficients in the equation of a line contain a quantity k such that, by letting k run through all possible values, we obtain a whole system of lines satisfying some geometric condition, then we say that the given equation defines a **family** of lines and that k is the **parameter** of the family.

For our purposes, the importance of the consideration of families of lines lies in the fact that, if the required line is known to belong to a given family, we may first write the equation of this family and then find from the conditions of the problem the value of k that fixes the required line. The following examples will indicate the procedure.

EXAMPLE 1. Find the equations of the lines parallel to $8x + 15y - 10 = 0$ (Fig. 83) that lie at a distance from (2, 1) numerically equal to 2.

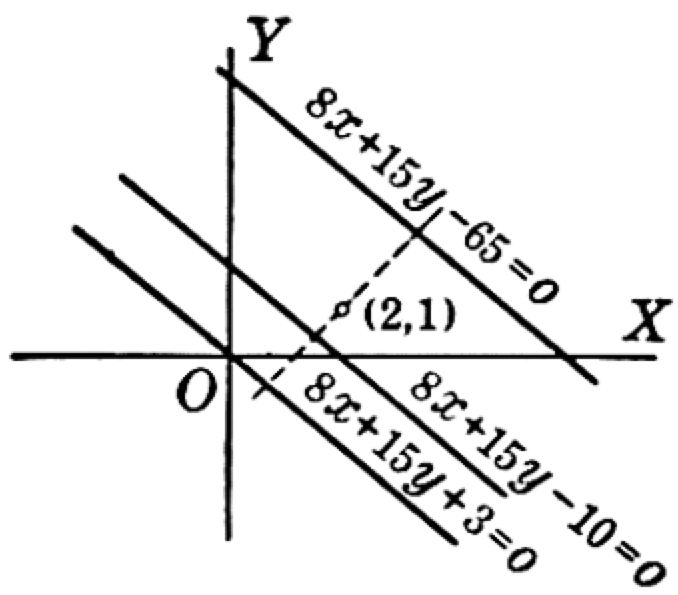


FIG. 83

The equation of the family of lines parallel to the given line is

$$8x + 15y + k = 0.$$

Since we have to do with the distance from this line to a point, we first reduce its equation to the normal form

$$\frac{8}{17}x + \frac{15}{17}y + \frac{k}{17} = 0.$$

The condition that the distance from this line to (2, 1) is equal to ± 2 is now found, by substituting the coördinates of (2, 1) in the first member of the above equation and equating the result to ± 2, to be $\dfrac{31 + k}{17} = \pm 2$. Hence $k = 3$ or $k = -65$ and the required lines are

$$8x + 15y + 3 = 0 \quad \text{and} \quad 8x + 15y - 65 = 0.$$

EXAMPLE 2. Find the equation of the line of inclination 135° through the intersection of $8x - 2y - 5 = 0$ and $5x + 10y + 7 = 0$.

The equation of the family of lines through the intersection of the given lines is

$$8x - 2y - 5 + k(5x + 10y + 7) = 0,$$

or

$$(8 + 5k)x + (-2 + 10k)y + (-5 + 7k) = 0.$$

Since the slope of the required line is $m = \tan 135° = -1$, we have, by equation (8), Art. 162,

$$-\frac{8 + 5k}{-2 + 10k} = -1.$$

Hence, $k = 2$. The line of the family for which $k = 2$ is

$$2x + 2y + 1 = 0.$$

This is the line required.

EXAMPLE 3. Find the equations of the lines through $(6, -1)$ for which the product of the intercepts equals 3.

Denote the intercepts by a and b. From the statement of the problem, $ab = 3$ or $b = 3/a$. By substituting this value of b in the intercept form, $\frac{x}{a} + \frac{y}{b} = 1$, of the equation of a line, we get

$$\frac{x}{a} + \frac{ay}{3} = 1, \quad \text{or} \quad 3x + a^2y = 3a.$$

This is the equation of the family of lines for which the product of the intercepts equals 3.

The condition that a line of this family passes through $(6, -1)$ is that the coördinates of this point satisfy the equation of the line. On putting $x = 6$, $y = -1$ in the preceding equation, we get

$$18 - a^2 = 3a.$$

Hence, $a = 3$ or $a = -6$. By substituting these values of a in the equation of the family, we find, as the equations of the required lines,

$$x + 3y - 3 = 0, \quad \text{and} \quad x + 12y + 6 = 0.$$

Exercises

Write the equations of the following families of lines. Assume, in each case, four values of the parameter and draw the corresponding lines.

1. Of slope -3.
2. Parallel to $2x - 5y + 1 = 0$.
3. Having $\omega = 3\pi/4$.
4. Perpendicular to $3x + 5y = 0$.
5. Through $(2, -9)$.
6. Having equal intercepts.
7. With y-intercepts 7.
8. With x-intercepts -7.
9. Through the intersection of $3x + 5y - 2 = 0$ and $2x - 7y + 8 = 0$.
10. Tangent to the circle with center at $(0, 0)$ and radius 3.

Draw four lines of each of the following families and state a geometric property common to all the lines of the family.

11. $\frac{x}{a} + \frac{y}{3} = 1$.
12. $y = mx + 9$.
13. $x = a$.
14. $x + y - 5 + k(x - 3y) = 0$.
15. $y + 3 = m(x - 2)$.
16. $x \cos \omega + y \sin \omega \pm 6 = 0$.

17. Find the line of the family $y = mx - 3$ that passes through the point (*a*) $(2, 5)$, (*b*) $(-4, 9)$.

18. Find the line through the intersection of $7x + 4y - 10 = 0$ and $9x + 7y - 5 = 0$ that is (*a*) parallel, and (*b*) perpendicular to $2x + 3y - 6 = 0$.

19. Find the line through $(2, 5)$ and the intersection of the lines $2x - 3y - 2 = 0$ and $2x + 5y - 3 = 0$.

20. Find two lines through $(10, -3)$ for which the x-intercept exceeds the y-intercept by 2.

21. Find the line through $(2, 5)$ for which the sum of the reciprocals of the intercepts is $\frac{1}{3}$.

Chapter 21

The Circle

169. The Standard Form of the Equation of a Circle. A circle is defined as the locus of a point whose distance from a fixed point, the center, is equal to a constant, the radius.

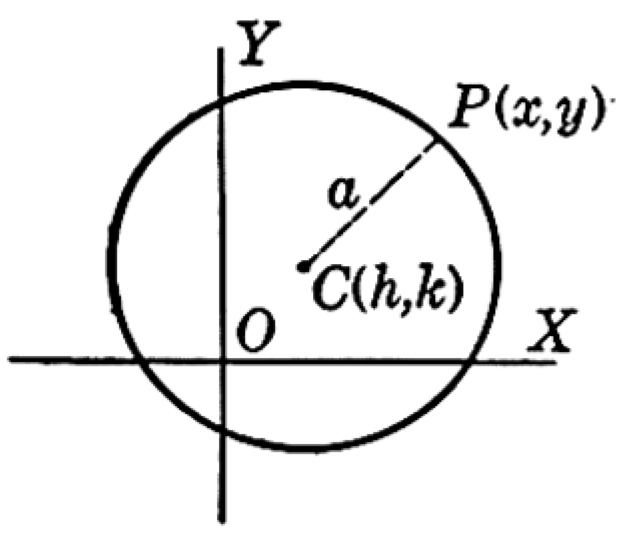


FIG. 84

To find the equation of the circle from its definition, let $C(h, k)$ be the center and let a be its radius. Let $P(x, y)$ be any point on the circle. The distance from P to the center C is equal to a; that is, by the distance formula,

$$\sqrt{(x-h)^2+(y-k)^2}=a. \tag{1}$$

or

$$(x-h)^2+(y-k)^2=a^2. \tag{2}$$

Conversely, if the coördinates of a point satisfy (2), they also satisfy (1). It follows that the distance from P to C is a and that P lies on the circle. Equation (2) is thus the equation of a circle with center at $C(h, k)$ and radius a.

In particular, if the center is at the origin, so that h and k are zero, the equation of the circle reduces to the simple form

$$x^2+y^2=a^2. \tag{3}$$

170. The General Form of the Equation of a Circle. If we expand equation (2), we get

$$x^2+y^2-2hx-2ky+h^2+k^2-a^2=0. \tag{4}$$

This equation is of the form

$$x^2+y^2+Dx+Ey+F=0, \tag{5}$$

so that every circle has an equation of the form of equation (5).

To find whether, conversely, every equation of the form of equation (5), with real coefficients, is the equation of a circle, we first write (5) in the form

$$(x^2+Dx\qquad)+(y^2+Ey\qquad)=-F$$

and complete the squares of the terms in the two parentheses by adding $D^2/4$ and $E^2/4$ to both sides of the equation. The resulting equation may be written in the form

$$\left(x+\frac{D}{2}\right)^2+\left(y+\frac{E}{2}\right)^2=\frac{D^2+E^2-4F}{4}. \tag{6}$$

The first member of equation (6) is the square of the distance of the point $P(x, y)$ on the locus from the fixed point $(-D/2, -E/2)$ and the second member is a constant. The appearance of the graph will depend on the value of this constant, in the following way:

(*a*) If $D^2 + E^2 - 4F > 0$, the second member of equation (6) is positive so that the locus of (6), and hence of (5), is a circle of

$$\text{Center}\left(-\frac{D}{2}, -\frac{E}{2}\right), \quad \text{and} \quad \text{radius } \tfrac{1}{2}\sqrt{D^2 + E^2 - 4F}. \tag{7}$$

(*b*) If $D^2 + E^2 - 4F = 0$, there is only one point on the locus, the center $(-D/2, -E/2)$. In this case, we say that equation (5) defines a **point circle** or **circle of zero radius.**

(*c*) If $D^2 + E^2 - 4F < 0$, there can be no points on the graph since neither term in the first member of equation (6) can be negative. In this case, we say that equation (5) defines an **imaginary circle,** with the center and radius defined by equation (7).

If the definition of a circle is extended as in (*b*) and (*c*), equation (5) defines a circle for all real values of D, E, and F. This equation is called the **general form** of the equation of a circle. It is often a more convenient form to work with than equation (2) is because the coefficients enter in it to the first power only.

EXAMPLE 1. Find the center, the radius, and the coördinates of the intersections with the axes, of the circle $x^2 + y^2 - 4x + 8y + 11 = 0$.

To complete the squares of the terms in x and of the terms in y, we first write the equation in the form

$$(x^2 - 4x \qquad) + (y^2 + 8y \qquad) = -11.$$

To complete the square in the first parentheses, we add $(-2)^2$, and, in the second, $(4)^2$. After making these additions to both sides, we may write the resulting equation in the form

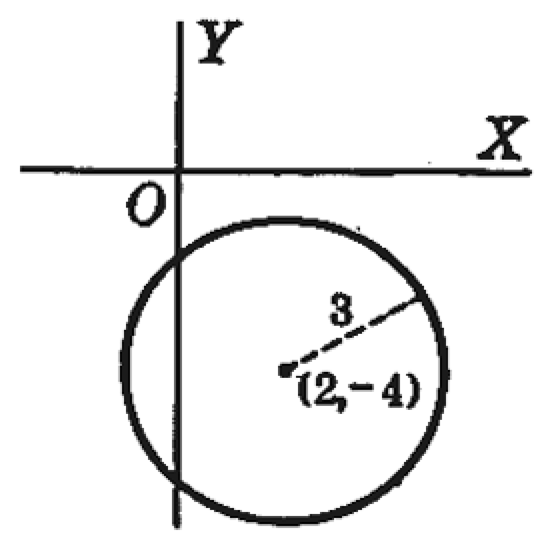


FIG. 85

$$(x - 2)^2 + (y + 4)^2 = 9.$$

The given equation thus defines a circle with center $(2, -4)$ and radius 3 (Fig. 85).

To find the intersections with the x-axis, put $y = 0$ and solve for x.

We have $$x^2 - 4x + 11 = 0.$$

Hence $$x = 2 \pm \sqrt{-7}.$$

Since these values of x are imaginary, the curve does not intersect the x-axis (Fig. 85).

By putting $x = 0$, we obtain   y intersection

$$y^2 + 8y + 11 = 0$$

so that $$y = -4 \pm \sqrt{5}.$$

Hence the intersections with the y-axis are $(0, -4 + \sqrt{5})$ and $(0, -4 - \sqrt{5})$.

EXAMPLE 2. Find the equation of the circle of radius 5 that lies in the first quadrant and is tangent to both axes.

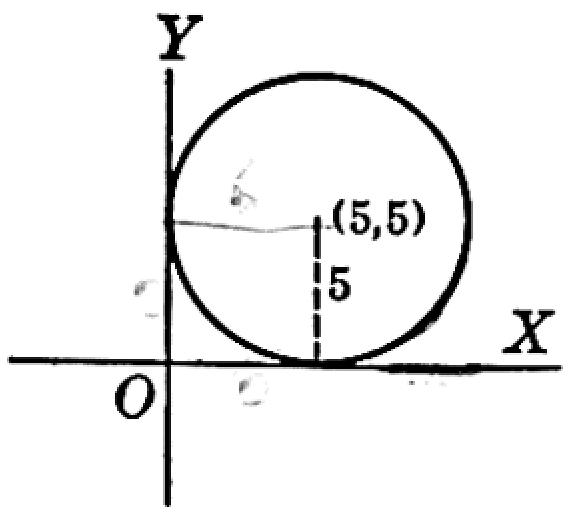


FIG. 86

Since the distance from the center of a circle to a tangent is numerically equal to the radius, the coördinates of the center of the required circle are (5, 5) (Fig. 86), and its equation is, from (2),

$$(x-5)^2+(y-5)^2=5^2$$

or

$$x^2+y^2-10x-10y+25=0.$$

To verify that the circle defined by this equation is tangent to the x-axis, we put $y=0$ in the final equation, giving

$$x^2-10x+25=0.$$

Since the roots of this equation are equal, the circle is tangent to the x-axis. The point of tangency is found, by solving the above equation, to be (5, 0). Similarly, by putting $x=0$, we find that it touches the y-axis at (0, 5).

Exercises

Write the equation of the circle having the given point as center and satisfying the given conditions.

1. (4, 7), radius 5.
2. (− 3, 2), radius 4.
3. (− 2, 8), radius 7.
4. (4, − 3), radius 3.
5. (6, − 6), radius 6.
6. (− 1, 2), radius $\frac{5}{2}$.
7. (12, − 5), through (0, 0).
8. (2, 7), through (6, 4).
9. (− 4, 3), tangent to $x=0$.
10. (5, 7), tangent to $y=0$.
11. (4, − 2), tangent to $4x-3y+8=0$.
12. (4, 7), tangent to $2x+3y-3=0$.

Find the center and radius of each of the following circles. State whether the circle is a real circle, a point circle, or an imaginary circle.

13. $x^2+y^2-10x+24y=0$.
14. $x^2+y^2+6x-14y-15=0$.
15. $x^2+y^2-8x+10y-17=0$.
16. $x^2+y^2+12x+8y+52=0$.
17. $x^2+y^2-4x-6y+22=0$.
18. $x^2+y^2+3x-7y+9=0$.
19. $5x^2+5y^2-4x+12y-37=0$.
20. $2x^2+2y^2-7x-5y-1=0$.

Write the equations of the following circles.

21. Having (2, − 7) and (8, 1) as ends of a diameter.
22. Passing through (2, − 5) and concentric with $x^2+y^2-6x+4y=0$.
23. Find the points of intersection of the line $x-7y+16=0$ with the circle $x^2+y^2-4x+2y-20=0$.
24. Find the points of intersection of the circles

$$x^2+y^2-8x+2y-17=0,\quad x^2+y^2-12x-2y+3=0.$$

Write the equations of the following families of circles.

25. With center at (0, 0). **26.** With center at $(-3, 7)$.

27. With center on the y-axis and passing through the origin.

28. With center on the line $y = x$ and passing through the origin.

171. Circles Determined by Three Conditions. Each of the forms of the equation of a circle,

$$(x - h)^2 + (y - k)^2 = a^2 \quad \text{and} \quad x^2 + y^2 + Dx + Ey + F = 0,$$

contains three constants. In order to find the equation of a required circle, we must, accordingly, be given enough information about the circle so that we can set up three equations from which to determine these constants. We can, for example, determine the constants if we know that the circle passes through three given points, or that it passes through two known points and that its center lies on a given line, and so on.

In any given problem, we must first decide which of the above two standard forms of the equation of the circle to use. If the coördinates of the center and the radius can be determined conveniently from the statement of the problem, it is usually best to find these numbers and substitute them in the first standard form. In most other cases, it is easier to use the general form of the equation of the circle, since the constants D, E, and F enter in this equation to the first power only.

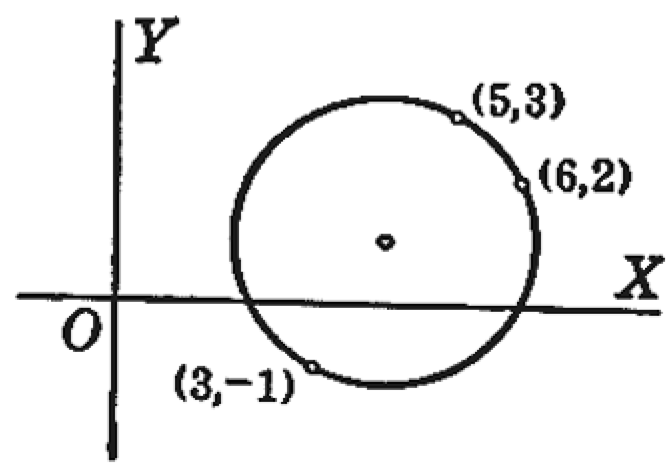


FIG. 87

EXAMPLE 1. Find the equation of the circle through the points (5, 3), (6, 2), and $(3, -1)$ (Fig. 87).

In problems of this type, we shall use the general form (5).

To find D, E, and F, we impose the condition that the coördinates of each of the given points satisfy the equation of the circle, that is,

$$\begin{aligned} 25 + 9 + 5D + 3E + F &= 0, \\ 36 + 4 + 6D + 2E + F &= 0, \\ 9 + 1 + 3D - E + F &= 0. \end{aligned}$$

On solving these equations for D, E, and F, we find that

$$D = -8, \quad E = -2, \quad F = 12.$$

When these values of D, E, and F are substituted in equation (5), we have, as the required equation of the circle through the given points,

$$x^2 + y^2 - 8x - 2y + 12 = 0.$$

EXAMPLE 2. Find the equations of the circles that pass through $(-1, 1)$, and (1, 3) and are tangent to the line $x + 3y = 0$ (Fig. 88).

In this case, we shall use equation (2).

Since the coördinates of each of the given points satisfy the equation of the circle, we have

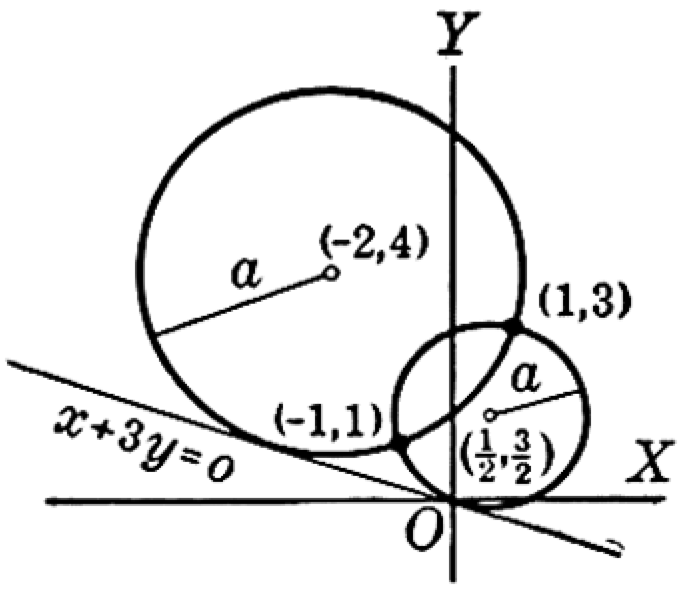


FIG. 88

$$(-1-h)^2+(1-k)^2=a^2$$

and

$$(1-h)^2+(3-k)^2=a^2.$$

Since the distance from the center to the given tangent line is numerically equal to the radius, we have also

$$\frac{h+3k}{\sqrt{10}}=\pm a$$

By subtracting the second equation from the first, and simplifying, we obtain

$$h+k-2=0. \tag{8}$$

If we substitute the value of a from the third equation in the first, and simplify, we have

$$9h^2-6hk+k^2+20h-20k+20=0. \tag{9}$$

On solving (8) and (9) as simultaneous and substituting the resulting solutions in the first of the above equations to find a, we obtain

$$h=-2,\ k=4,\ a=\sqrt{10}\ \text{ and }\ h=\frac{1}{2},\ k=\frac{3}{2},\ a=\frac{\sqrt{10}}{2}.$$

By substituting these sets of values of h, k, and a in (2), we obtain, as the equations of the required circles,

$$(x+2)^2+(y-4)^2=10 \quad\text{and}\quad (x-\tfrac{1}{2})^2+(y-\tfrac{3}{2})^2=\tfrac{5}{2}$$

or

$$x^2+y^2+4x-8y+10=0 \quad\text{and}\quad x^2+y^2-x-3y=0.$$

Exercises

Find the equation of the circle through the given points.

1. (0, 0), (6, 0), (0, − 8).

2. (0, 0), (4, 2), (1, 3).

3. (2, 3), (4, − 1), (5, 2).

4. (4, 5), (2, 9), (− 2, − 3).

5. (4, 3), (2, 7), (− 3, − 8).

6. (9, 4), (3, 2), (5, 6).

7. (4, 9), (6, 5), (2, − 3).

8. (3, 1), (− 2, 5), (1, 4).

Find the equations of the circles that satisfy the given conditions.

9. Circumscribed about the triangle defined by the lines $3x+2y=13$, $x+2y=3$, and $x+y=5$.

10. Circumscribed about the triangle defined by $x+y=1$, $x+3y+5=0$, and $x+2y+4=0$.

11. With center at the intersection of the lines $3x+5y=7$ and $x-2y+5=0$ and passing through $(2,-4)$.

12. Passing through $(6, 3)$ and $(-2, 7)$ and having its center on $3x-5y=16$.

13. Concentric with the circle $x^2 + y^2 + 10x - 2y = 0$ and tangent to the line $5x - 12y = 15$.

14. Tangent to $2x - 5y - 18 = 0$ at $(4, -2)$ and having its center on $4x - 7y + 13 = 0$.

15. Tangent to $3x + 2y - 18 = 0$ at $(2, 6)$, and passing through $(-3, 7)$.

16. Tangent to $3x - 4y = 13$ at $(7, 2)$; radius 10.

17. Passing through $(5, 7)$ and $(-2, 14)$; radius 13.

18. Tangent to the lines $x + y + 4 = 0$ and $7x - y + 4 = 0$ and having its center on $4x + 3y - 2 = 0$.

19. Passing through $(2, 3)$ and $(3, 6)$ and tangent to $2x + y = 2$.

20. Tangent to the circle $x^2 + y^2 = 25$; center at $(6, 8)$.

21. Inscribed in the triangle $x = 0$, $y = 0$, $3x + 4y = 24$.

22. Inscribed in the triangle $11x - 2y = 0$, $2x - 11y = 0$, $x + 2y = 24$.

172. The Family of Circles $S + kS' = 0$. Let

$$S = x^2 + y^2 + Dx + Ey + F = 0, \tag{10}$$

and

$$S' = x^2 + y^2 + D'x + E'y + F' = 0,$$

be the equations of two circles. Consider the family of curves

$$S + kS' = x^2 + y^2 + Dx + Ey + F + k(x^2 + y^2 + D'x + E'y + F') = 0. \tag{11}$$

For all values of k except $k = -1$, the curves of this family are circles, since, if $k \neq -1$, we may write the equation in the form

$$x^2 + y^2 + \frac{D + kD'}{1 + k}x + \frac{E + kE'}{1 + k}y + \frac{F + kF'}{1 + k} = 0 \tag{12}$$

which is the equation of a circle.

If the circles $S = 0$ and $S' = 0$ intersect (Fig. 89*a*), all the circles (12) pass through their points of intersection. For, let $P_1(x_1, y_1)$ be a

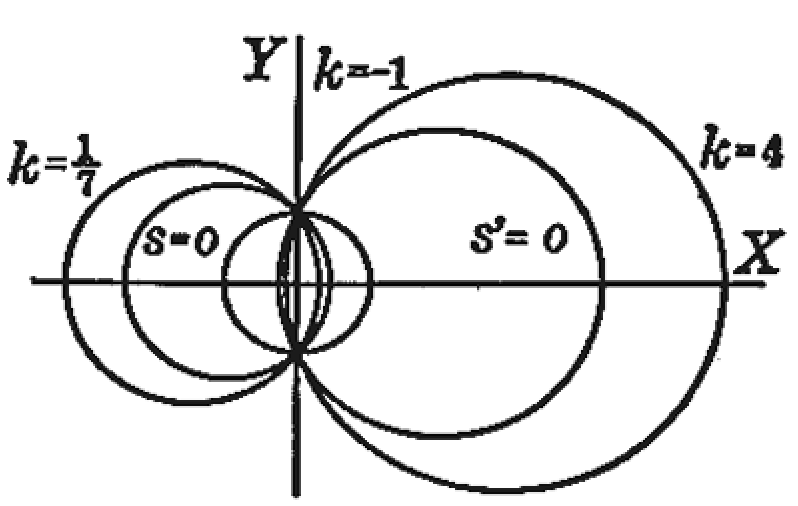


FIG. 89*a*

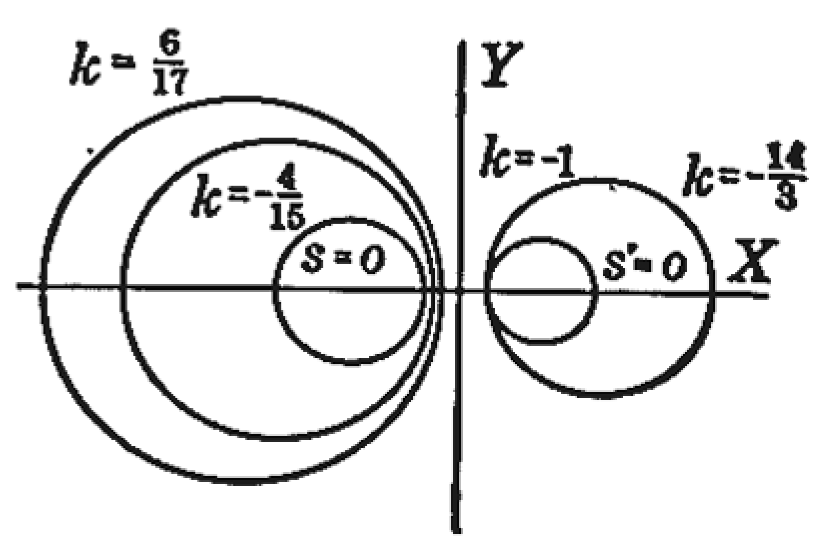


FIG. 89*b*

point the coördinates of which satisfy both equations (10). When we substitute the coördinates of P_1 in (11), the resulting equation reduces to $0 + k \cdot 0 = 0$, which is true for all values of k, so that P_1 lies on all of the curves (11).

If $S = 0$ and $S' = 0$ do not intersect (Fig. 89*b*), the family of circles (11) still exists but no two circles of the family intersect.

If the circles $S = 0$ and $S' = 0$ are not concentric, the curve of the family (11) defined by putting $k = -1$ is the line

$$(D - D')x + (E - E')y + F - F' = 0. \qquad (13)$$

This line is the **radical axis** of the family of circles (11).

If $S = 0$ and $S' = 0$ intersect in two points, the radical axis (13) passes through these two points and is thus the common chord of all the circles of the family (11).

Exercises

Draw four circles and the radical axis of the family $S + kS' = 0$, given

1. $S = x^2 + y^2 + 8y - 18 = 0, \quad S' = x^2 + y^2 - 16y + 6 = 0.$

2. $S = x^2 + y^2 - 20x = 0, \quad S' = x^2 + y^2 + 30x + 125 = 0.$

3. In Ex. 1, find the points of intersection of $S = 0$ and $S' = 0$. Show that every circle through these two points belongs to the family $S + kS' = 0$.

4. In Ex. 2, show that there are two point circles in the family and find their equations.

5. Given $S = x^2 + y^2 + 12x + 2y - 16 = 0$ and $S' = x^2 + y^2 + 3x - y - 7 = 0$, find the circle $S + kS' = 0$ that passes through $(-1, 4)$.

6. Find the circle of the family defined in Ex. 5 that has its center on the line $3x + 7y - 23 = 0$.

Write the equation $S + kS' = 0$ of the family, find the equation of the radical axis and the coördinates of the points (if there are any) common to all the curves of the family, given:

7. $S = x^2 + y^2 - 10x - 8y + 21 = 0, \quad S' = x^2 + y^2 + 2x - 12y + 17 = 0.$

8. $S = x^2 + y^2 - 6x - 4y + 5 = 0, \quad S' = x^2 + y^2 - 14x + 2y + 21 = 0.$

9. $S = x^2 + y^2 - 14x - 6y + 33 = 0, \quad S' = x^2 + y^2 + 4x + 2y - 11 = 0.$

10. If $S = 0$ and $S' = 0$ are not concentric, show that the centers of all the circles of the family $S + kS' = 0$ lie on the line through the centers of $S = 0$ and $S' = 0$. This line is the *line of centers* of the family.

11. Show that the line of centers (Ex. 10) is perpendicular to the radical axis.

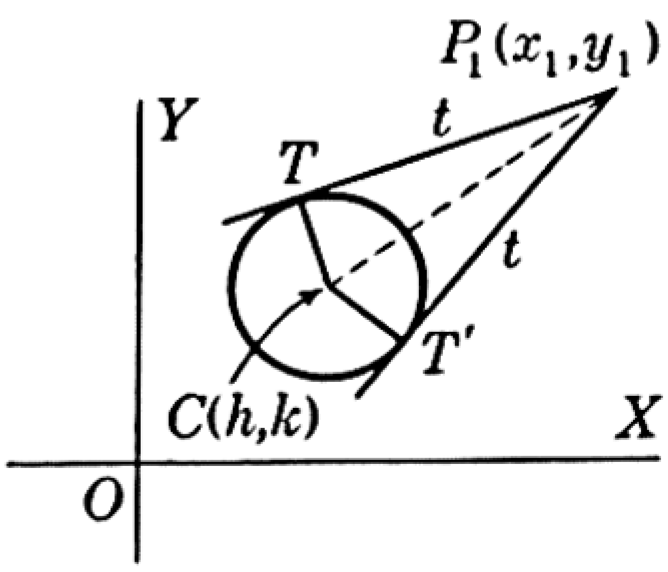


FIG. 90

12. Let $P_1(x_1, y_1)$ be a point external to the circle $(x - h)^2 + (y - k)^2 - a^2 = 0$. Let T be the point of tangency of either tangent from P_1 to this circle (Fig. 90). Show that $P_1T^2 = (x_1 - h)^2 + (y_1 - k)^2 - a^2$. The undirected length P_1T is called the *length of the tangent* from P_1 to the circle.

HINT. CTP_1 is a right triangle.

13. If the lengths of the tangents (Ex. 12) from P_1 to two circles $S = 0$ and $S' = 0$ are equal, show that P_1 lies on the radical axis of $S + kS' = 0$, and conversely.

173. Loci Problems. In some of the following problems, no coördinate axes are indicated in the statement of the problem. In such cases, the student should choose for himself a set of axes that will make the computations as simple as possible.

To find the equation of the required locus, follow the directions for finding the equation of a locus outlined in Art. 155.

EXAMPLE 1. Given two points A and B such that the length of the segment $AB = 2c$. Find the locus of a point such that the sum of the squares of its distances from A and B is equal to $4c^2$.

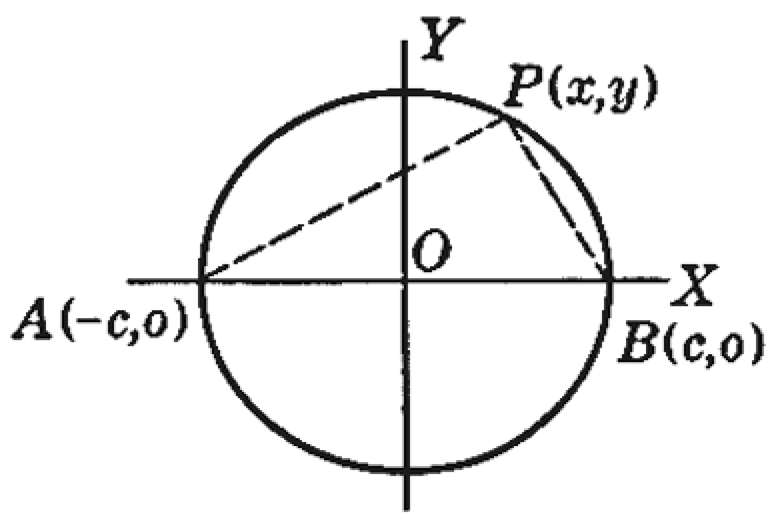


FIG. 91

We choose the line through A and B as x-axis and the line perpendicular to it through the midpoint of the segment AB as y-axis (Fig. 91). Then the coördinates of A are $(-c, 0)$ and of B $(c, 0)$.

Let $P(x, y)$ be a point anywhere on the locus. From the statement of the problem, we have

$$AP^2 + BP^2 = 4c^2$$

and, on replacing AP^2 and BP^2 by their values from the distance formula, we obtain, as the equation of the required locus,

$$(x + c)^2 + (y - 0)^2 + (x - c)^2 + (y - 0)^2 = 4c^2.$$

This equation may be simplified to

$$x^2 + y^2 = c^2.$$

The required locus is thus a circle on AB as a diameter.

If a problem imposes two geometric conditions on a point, we may find the equation of the locus of a point that satisfies each given condition separately. The points of intersection of these two loci will then be the points that satisfy both given conditions.

EXAMPLE 2. Given the base of a triangle and the lengths of the altitude and the median from the vertex to the base. Find the vertex of the triangle.

Take the midpoint of the base as origin and the line that contains the base as x-axis. Denote the length of the base, the altitude, and the median by $2c$, h, and m, respectively. Then the coördinates of the ends of the base are $A(-c, 0)$ and $B(c, 0)$ (Fig. 92).

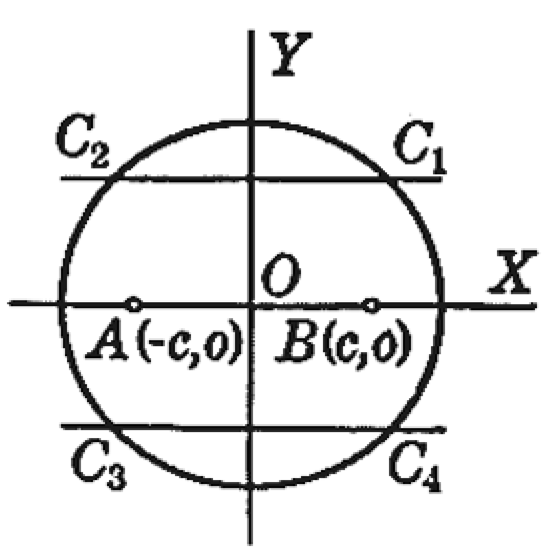


FIG. 92

Let $C(x, y)$ be the required vertex.

The statement of the problem imposes two conditions on C. We shall determine the locus of a point that satisfies each of these conditions separately. Then any point of intersection of the two loci so obtained may be taken as the required vertex.

Since the distance of C from the midpoint $(0, 0)$ of the base is equal to m, the length of the median, C lies on the circle

$$x^2 + y^2 = m^2. \tag{14}$$

Since the distance of C from the x-axis is *numerically* equal to h, the length of the altitude, C lies on one of the lines

$$y = h \quad \text{or} \quad y = -h.$$

The intersections of these lines with the circle (14), that is, the points

$$(\sqrt{m^2 - h^2}, h),\ (-\sqrt{m^2 - h^2}, h),\ (\sqrt{m^2 - h^2}, -h),\ (-\sqrt{m^2 - h^2}, -h)$$

are the required positions of the vertex C.

The four triangles having the segment AB as base, and any one of the above four points as vertex, are congruent.

If $m^2 - h^2 < 0$, the triangle cannot be constructed.

Problems

Find the equation of the locus of a point satisfying the given conditions.

1. Its distance from $(4, 6)$ equals three times its distance from $(1, 3)$.
2. The sum of the squares of its distances from $2x - y - 7 = 0$ and $x + 2y + 4 = 0$ equals 16.
3. The square of its distance from the origin equals 20 times its directed distance from $3x - 4y + 11 = 0$.
4. The sum of the squares of its distances from $x + y - 3 = 0$ and $x - y + 1 = 0$ equals 34 times its directed distance from $15x + 8y - 21 = 0$.
5. Find the locus of the midpoint of the segment joining the origin to any point on the circle $x^2 + y^2 - 10x - 16y + 80 = 0$.
6. Find the locus of the midpoint of a segment of length 20 having its end points on the coördinate axes.

Choose a suitable set of coördinate axes and find the equation of the locus of a point satisfying the given conditions.

7. The sum of the squares of its distances from two adjacent sides of a rectangle equals three times the sum of the squares of its distances from the other two sides.
8. Its distance from a fixed point A is k times its distance from a fixed point B.
9. The lines joining it to two fixed points intersect at a constant angle.
10. The sum of the squares of its distances from three given points equals a constant.
11. The feet of the perpendiculars from it to the sides of a given triangle lie on a line.

In the following problems, two vertices of a triangle are given. Find the third vertex as a point of intersection of two loci, given:

12. The base, the area, and the length of one side.

13. The base, a base angle, and the median to the base.

14. The base, the angle at the vertex, and the altitude.

15. The base, the angle at the vertex, and the length of the median to the base.

16. The base, a base angle, and the radius of the circumcircle.

Chapter 22

Polar Coördinates

174. Introduction. Instead of fixing the position of a point by its directed distances from two fixed lines, as in rectangular coördinates, it is sometimes preferable to locate it by its distance and direction from a fixed point. When its position is fixed in this way, the point is said to be located by means of **polar coördinates.**

In principle, the method of fixing the position of a point by its polar coördinates is not unfamiliar. We are accustomed, for example, to such statements as that Cleveland is about 300 miles northwest of Washington, or that Buffalo is about 400 miles west of Boston.

175. Polar Coördinates. Let O be a fixed point, the **origin,** or **pole,** and let OI be a fixed line through O, the **initial line,** or **polar axis.** The line through the pole perpendicular to the polar axis is the **90° axis.**

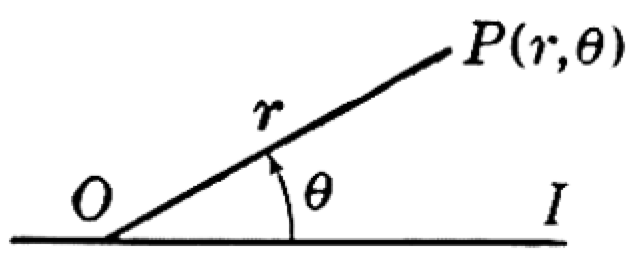


FIG. 93

Let P be any point in the plane (other than O) * and draw the line through O and P. The position of P is fixed if we know the length r of the segment OP and the angle θ that has the half-line OI for its initial side and the half-line OP for its terminal side. The quantities (r, θ) are called the **polar coördinates** of P; r is the **radius vector** and θ is the **vectorial angle.**

If the polar coördinates (r, θ) are given, the point P is definitely fixed; but, for a given point P, we can find as many pairs of polar coördinates as we please. For, if we add to θ, or subtract from it, any number of complete revolutions, we do not change the terminal side nor the position of P. Thus, (r, θ), $(r, \theta + 2\pi)$, $(r, \theta - 2\pi)$, etc., are all polar coördinates of the same point P.

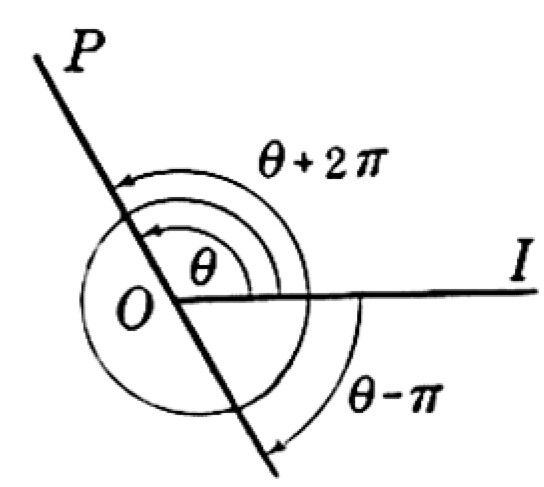


FIG. 94

Moreover, we can also fix the position of P by choosing, for the terminal side of the vectorial angle, the half-line extending from O in the opposite direction from P and considering the length of the radius vector as negative. Thus, $(-r, \theta - \pi)$, $(-r, \theta + \pi)$, $(-r, \theta + 3\pi)$, etc., are also polar coördinates of the point (r, θ).

For plotting points, or drawing graphs, in polar coördinates, it will be found that both speed and accuracy are improved by the use of **polar coördinate paper,** as in Figure 95.

* The polar coördinates of the origin O are defined by taking the radius vector, r, equal to zero and the vectorial angle, θ, to be of any magnitude we please.

EXAMPLE. Plot the points having the polar coördinates $(3, 0°)$, $(4, -240°)$, $(2, 180°)$, $\left(3, \frac{\pi}{4}\right)$, $\left(1, \frac{3\pi}{2}\right)$.

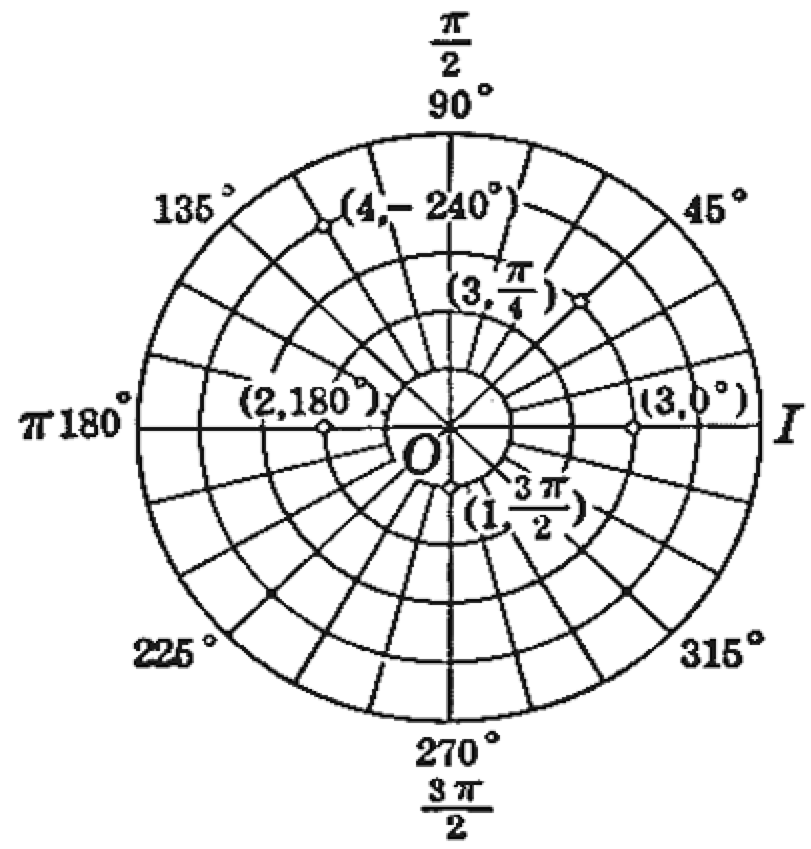


FIG. 95

From OI, as initial side, we measure off the given angle θ and, on its terminal side, lay off the given length of the radius vector. The resulting points are shown in Figure 95.

Exercises

Plot the points whose polar coördinates are given. Find, for each point, one other pair of polar coördinates for which r is positive and one for which r is negative.

1. $(9, 60°)$.	**2.** $(4, 135°)$.	**3.** $(8, -30°)$.
4. $(5, -225°)$.	**5.** $(3, 300°)$.	**6.** $(10, 270°)$.
7. $(8, 76°)$.	**8.** $(10, -73°)$.	**9.** $(12, 147°)$.
10. $(2, \pi/4)$.	**11.** $(6, 5\pi/6)$.	**12.** $(5, \pi/2)$.
13. $(7, -\pi/3)$.	**14.** $(7, 2)$.	**15.** $(4, 5)$.

16. Show that the distance between the points $P_1(r_1, \theta_1)$ and $P_2(r_2, \theta_2)$ is $d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_2 - \theta_1)}$.

HINT. Apply the law of cosines (Art. 142) to the triangle OP_1P_2.

17. Show that the area of the triangle whose vertices are the origin and the points $P_1(r_1, \theta_1)$ and $P_2(r_2, \theta_2)$ is $S = \frac{1}{2}r_1r_2 \sin(\theta_2 - \theta_1)$.

Using the results of Ex. 16 and 17, find the distance P_1P_2 and the area of the triangle OP_1P_2, given:

18. $P_1(5, 45°)$, $P_2(8, 90°)$.	**19.** $P_1(4, 60°)$, $P_2(10, 150°)$.
20. $P_1(12, 180°)$, $P_2(9, 30°)$.	**21.** $P_1(6, 37°)$, $P_2(3, 79°)$.

176. Relations between Polar and Rectangular Coördinates. If a system of polar coördinates is so related to a system of rectangular coordinates that they have the same origin and the directions OI on the initial line and OX on the x-axis coincide, as in Figure 96, then the relations between the polar coördinates (r, θ) and the rectangular coördinates (x, y) of a given point P may be found in the following way.

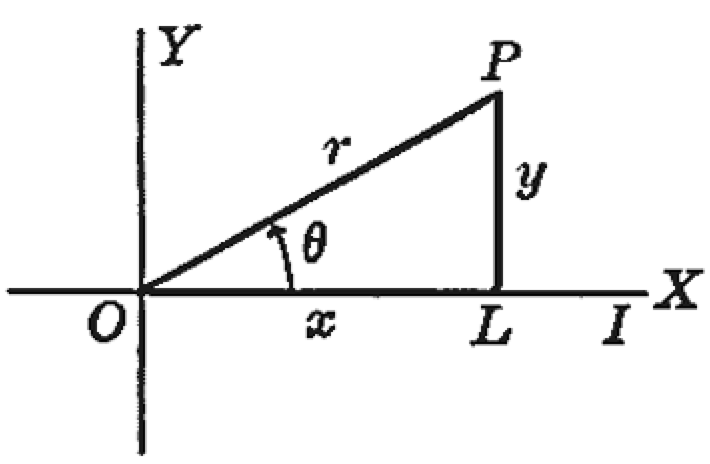


FIG. 96

From the definitions of $\sin\theta$ and $\cos\theta$, we have

$$\cos\theta = \frac{x}{r} \quad \text{and} \quad \sin\theta = \frac{y}{r},$$

so that the values of x and y in terms of r and θ are

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{1}$$

$r = \frac{x}{\sqrt{x^2+y^2}}$

To find r and θ in terms of x and y, we notice (Fig. 96) that $\tan\theta = \dfrac{y}{x}$ and that r is the hypotenuse of a right triangle whose legs are x and y. Hence

$$r = \sqrt{x^2 + y^2}; \quad \tan\theta = \frac{y}{x} \quad \text{or} \quad \theta = \tan^{-1}\frac{y}{x}. \tag{2}$$

When we determine θ from the last of equations (2), we must bear in mind that there are two angles, differing by 180°, for which $\tan\theta$ has the given value. Before we can determine θ from this equation we must accordingly first find out, by plotting the point on the figure, in what quadrant the given point lies.

EXAMPLE 1. Find the rectangular coördinates of a point given that its polar coördinates are (4, 60°).

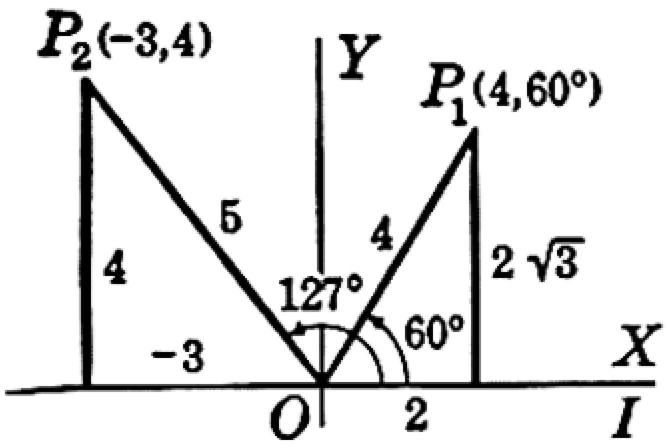


FIG. 97

Since $\cos 60° = \frac{1}{2}$ and $\sin 60° = \dfrac{\sqrt{3}}{2}$, we have, from equations (1),

$$x = 4 \cdot \tfrac{1}{2} = 2, \qquad y = 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}.$$

Hence, the required coördinates are $(2, 2\sqrt{3})$ (Fig. 97).

EXAMPLE 2. Find the polar coördinates for a point, given that its rectangular coördinates are $(-3, 4)$.

From equations (2), we have

$$r = \sqrt{(-3)^2 + 4^2} = 5, \qquad \tan\theta = -\tfrac{4}{3} = -1.3333.$$

Moreover, it is seen from Figure 97 that P_2 lies in the second quadrant. Hence we may take $\theta = 126°\ 52'$ and the required coördinates are (5, 126° 52′).

EXAMPLE 3. Transform the equation $3x - y - 5 = 0$ to polar coördinates.

By substituting for x and y their values in terms of r and θ from (1), we have

$$3r\cos\theta - r\sin\theta - 5 = 0 \quad \text{or} \quad r(3\cos\theta - \sin\theta) - 5 = 0.$$

EXAMPLE 4. Transform the polar equation $r = 2\cos\theta + 3\sin\theta$ to rectangular coördinates.

In this case, it will be convenient, first, to multiply the given equation by r, giving

$$r^2 = 2r\cos\theta + 3r\sin\theta.$$

From equations (1) and (2) we now obtain

$$x^2 + y^2 = 2x + 3y.$$

The student should show that the graph of this equation is a circle with center at (1, 3/2) (in rectangular coördinates) and radius $\sqrt{13}/2$.

Exercises

Find the rectangular coördinates of the following points.

1. $(6, 30°)$. 2. $(7, 90°)$. 3. $(8, -120°)$.
4. $\left(4, \frac{\pi}{4}\right)$. 5. $(3, -\pi)$. 6. $\left(-10, \frac{2\pi}{3}\right)$.

Find a pair of polar coördinates for the points whose rectangular coördinates are:

7. $(2, 2\sqrt{3})$. 8. $(3, -3)$. 9. $(0, 4)$.
10. $(-\sqrt{3}, 1)$. 11. $(-2, -2)$. 12. $(5, -3)$.

13. Show that $(4, 45°)$, $(4, 135°)$, $(4, 225°)$, and $(4, 315°)$ are the vertices of a square. Find the length of a side of the square.

14. The center of a regular hexagon is at the origin. One vertex is at $(5, 0°)$. Find the polar coördinates of the other vertices.

Write the following equations in polar coördinates and draw the graphs.

15. $y = -4$. 16. $x = 7$. 17. $2x - 3y = 6$.
18. $x^2 + y^2 = 81$. 19. $x^2 + y^2 = 8x$. 20. $x^2 + y^2 + 4y = 0$.

Write the following equations in rectangular coördinates and draw the graphs.

21. $r = 3$. 22. $\theta = 45°$. 23. $r \sin \theta = 2$.
24. $r = 2 \sin \theta$. 25. $r = 2 \sec \theta$. 26. $r^2 \sin 2\theta = 8$.

177. The Polar Equation of a Line. By substituting the values of x and y in terms of r and θ from equations (1) in the general form, $Ax + By + C = 0$ of the equation of a line, we obtain, as the **general polar form** of the equation of a line,

$$r(A \cos \theta + B \sin \theta) + C = 0. \tag{3}$$

If we make the same substitution in the normal form, $x \cos \omega + y \sin \omega - p = 0$, we have

$$r(\cos \theta \cos \omega + \sin \theta \sin \omega) - p = 0.$$

This simplifies to the **polar normal form**

$$r \cos(\theta - \omega) - p = 0. \tag{4}$$

The following special cases of equation (4) arise frequently. If the given line is perpendicular to the initial line, $\omega = 0$ and equation (4) reduces to:

$$r \cos \theta = p, \quad \text{or} \quad r = p \sec \theta. \tag{5}$$

If the given line is parallel to the initial line, $\omega = \pi/2$ and equation (4) reduces to:

$$r \sin \theta = p, \quad \text{or} \quad r = p \csc \theta. \tag{6}$$

Exercises

Find the rectangular equation and draw the lines.

1. $r(5 \cos \theta - 3 \sin \theta) = 9$. **2.** $r(4 \cos \theta + 9 \sin \theta) = 12$.
3. $r \sin \theta = 6$. **4.** $r \cos (\theta - \pi/6) = 4$.
5. $r + 4 \sec (\theta - \pi/3) = 0$. **6.** $4 \tan \theta + 3 = 0$.

Write the following equations in polar form.

7. $3x - 7y + 9 = 0$. **8.** $x + 5y - 10 = 0$.
9. $4x - 9y + 11 = 0$. **10.** $y - 2x = 0$.

Write the polar equation of a line:

11. Parallel to the initial line and passing through (*a*) $(2, -\pi/2)$, (*b*) $(6, \pi/6)$, and (*c*) $(4, \pi/4)$.

12. Perpendicular to the initial line and passing through (*a*) $(5, 0)$, (*b*) $(2, \pi)$, (*c*) $(7, 2\pi/3)$.

13. Passing through $(5, 5\pi/6)$ and perpendicular to the line joining the origin to this point.

14. Passing through $(4, -\pi/2)$ and making an angle $\pi/6$ with the initial line.

15. Passing through $(3, 0)$ and making an angle $3\pi/4$ with the initial line.

16. Passing through $(3, 0)$ and perpendicular to the line in Ex. 15.

17. Find polar coördinates of the point of intersection of the lines $r \cos \theta = 2$ and $r \sin \theta = 2\sqrt{3}$.

18. Find the slope of the line $r \cos (\theta - 2\pi/3) = 6$.

19. Find the length of the segment of the line $r \cos (\theta - 3\pi/4) = 6$ that is included between the polar axis and the 90° axis.

178. The Polar Equation of a Circle. Let $C(c, \gamma)$ be the center and a the radius of the circle and let $P(r, \theta)$ be any point on the circle. If we apply the law of cosines (Art. 142) to the triangle COP, we obtain

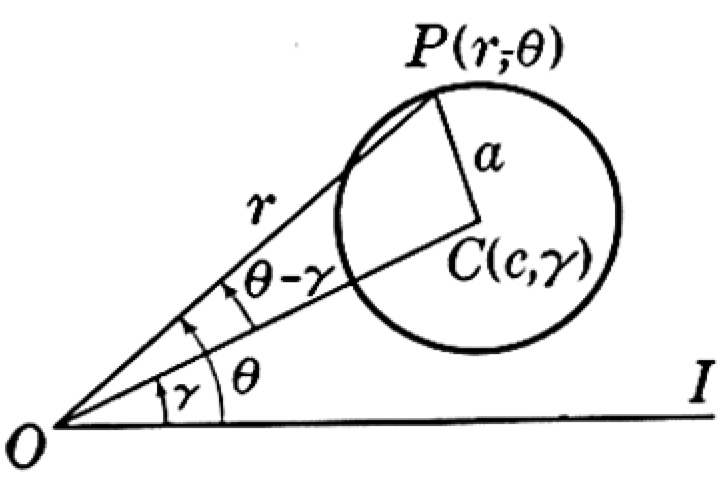


FIG. 98

$$r^2 - 2cr \cos (\theta - \gamma) + c^2 = a^2. \quad (7)$$

This is the polar equation of a circle with center at (c, γ) and radius a.

The following special cases are of importance.

(*a*) If the center lies on the polar axis and the circle passes through the origin, the coördinates of the center are $(a, 0)$ or (a, π) according as the center lies to the right or left of the origin. Equation (7) now reduces to

$$r = 2a \cos \theta, \quad \text{or} \quad r = -2a \cos \theta, \quad (8)$$

the first equation holding if C is to the right of the origin and the second if C is to the left.

(*b*) If the center lies on the 90° axis and the circle passes through the origin, the coördinates of the center are $(a, \pm \pi/2)$ and equation (7) reduces to

$$r = 2a \sin \theta, \quad \text{or} \quad r = -2a \sin \theta, \tag{9}$$

according as the center lies above or below the origin.

(*c*) If the center lies at the origin, $c = 0$ and the equation may be simplified to

$$r = a. \tag{10}$$

Exercises

Find the polar equation of the circle having its center at the given point and having the given radius.

1. $(0, 0)$, radius 7.
2. $(5, \pi/2)$, radius 5.
3. $(2, \pi)$, radius 2.
4. $(4, \pi/3)$, radius 3.
5. $(6, \pi/6)$, radius 4.
6. $(2, -4\pi/3)$, radius 5.
7. $(a, \pi/4)$, radius $2a$.
8. $(2a, 7\pi/6)$, radius a.

Find the center and radius of each of the following circles.

9. $r = 6 \cos \theta$.
10. $r = -10 \sin \theta$.
11. $r = 8 \cos (\theta - \pi/6)$.
12. $r = -4 \cos (\theta - \pi/4)$.
13. $r^2 - 6r \cos \theta = 7$.
14. $r^2 - 8r \sin \theta = 9$.
15. $r^2 - 4r \cos (\theta - \pi/4) = 12$.
16. $r^2 + 12r \cos (\theta - \pi/3) = 28$.

Write the equations of the following circles in rectangular coördinates.

17. $r = -4 \cos \theta$.
18. $r = 12 \sin \theta$.
19. $r = 14 \cos (\theta - \pi/3)$.
20. $r^2 - 4r \cos \theta = 5$.
21. $r^2 - 6r \cos (\theta - \pi/6) + 5 = 0$.
22. $r^2 - 10r \cos (\theta - \pi/4) = 11$.

Write the equations of the following circles in polar coördinates.

23. $x^2 + y^2 = 121$.
24. $x^2 + y^2 = 10y$.
25. $x^2 + y^2 + 6x = 0$.
26. $x^2 + y^2 - 6x - 6y = 0$.
27. $x^2 + y^2 + 10x - 10\sqrt{3}y - 44 = 0$.
28. $x^2 + y^2 - 9x - 9y + 27 = 0$.

Write each of the following equations in the form of equation (7) and state the coördinates of the center and the radius.

29. $r = 10 \cos \theta + 24 \sin \theta$.
30. $r = 30 \cos \theta - 16 \sin \theta$.

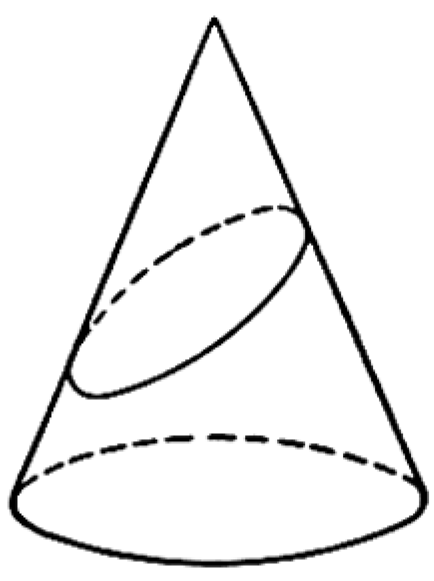
Find the points of intersection of the two loci.

31. $r = 12 \cos \theta, r \cos \theta = 3$.
32. $r = 4 \sin \theta, r \cos \theta = \sqrt{3}$.

Chapter 23

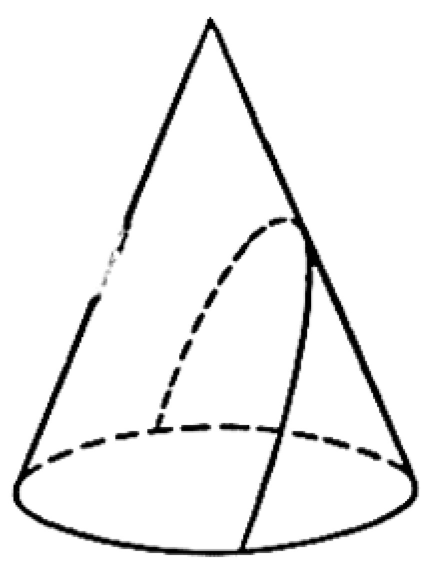
The Conic Sections

179. Plane Sections of a Right Circular Cone. The curve of section of a right circular cone by any plane is called a **conic section** or, simply, a **conic.** If the cutting plane does not pass through the vertex of the cone, the conic belongs to one of the following three types.



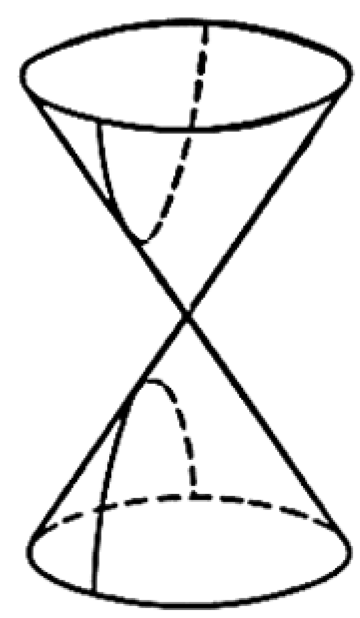
Ellipse

FIG. 99*a*



Parabola

FIG. 99*b*



Hyperbola

FIG. 99*c*

(*a*) If the cutting plane cuts entirely across one nappe of the cone (Fig. 99*a*), the conic is an **ellipse.**

(*b*) If the cutting plane is parallel to one rectilinear element of the cone (Fig. 99*b*), the conic is a **parabola.**

(*c*) If the plane cuts both nappes of the cone (Fig. 99*c*), the conic is **a hyperbola.**

These three types of curves were named, and discussed very thoroughly, by the Greek mathematicians who studied them by the aid of methods similar to those we now use in elementary geometry. In the course of these studies they found properties which will serve to define these conics as loci in their planes. It is these definitions, instead of the one just given, that we shall use to derive the equations of these curves.

Moreover, in the present chapter, we shall choose the position of the coördinate axes with respect to the curve very carefully so as to get the equation of the curve in as simple a form as possible. This simplest form is called the **standard form** of the equation of the conic. The study of the conics when their equations are not in the standard form will be taken up in Chapter 25.

The Parabola

180. Definitions. *A parabola is the locus of a point whose undirected distance from a fixed point is equal to its undirected distance from a fixed line.*

Let F (Fig. 100) be the fixed point and $D'D$ the fixed line. The locus of a point P such that

$$FP = RP,$$

where R is the foot of the perpendicular from P to $D'D$, is a parabola. By plotting a number of these points and drawing a smooth curve through them, we obtain a suitable figure to represent the curve.

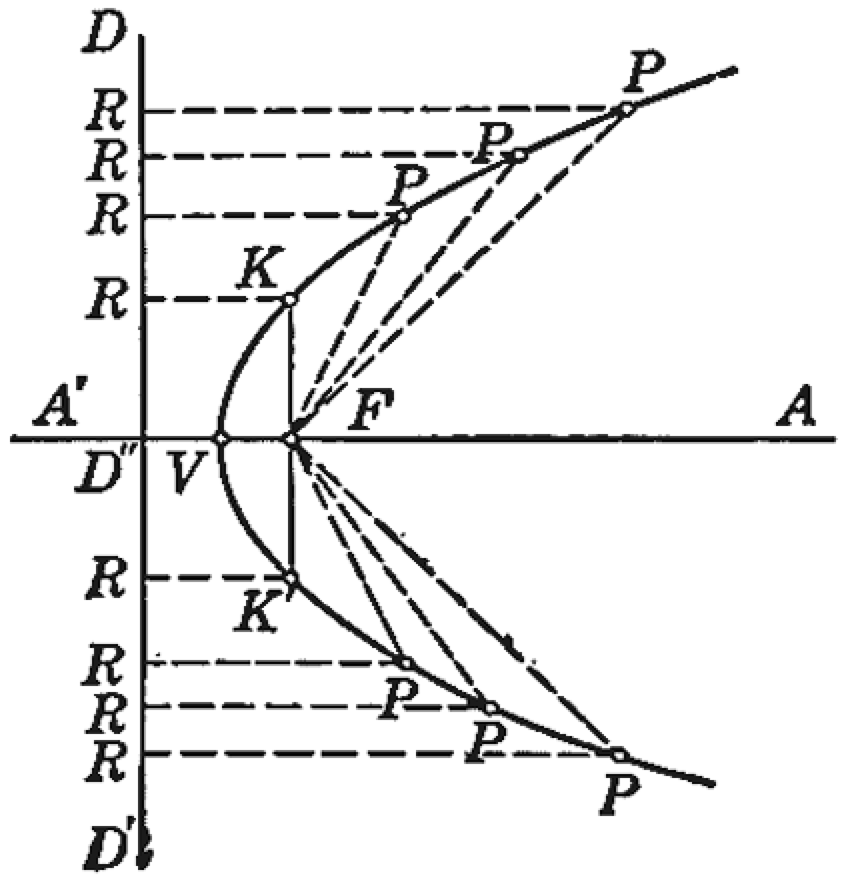


Fig. 100

The fixed point F is the **focus** and the fixed line $D'D$ is the **directrix** of the parabola. The line $A'A$, through the focus perpendicular to the directrix, is the **principal axis** of the curve. The point V in which the principal axis intersects the parabola is the **vertex.** The chord $K'K$, through the focus, parallel to the directrix, and terminated by the curve, is the **latus rectum.**

181. Standard Forms of the Equation of the Parabola. To derive a standard form of the equation of the parabola, we take the principal

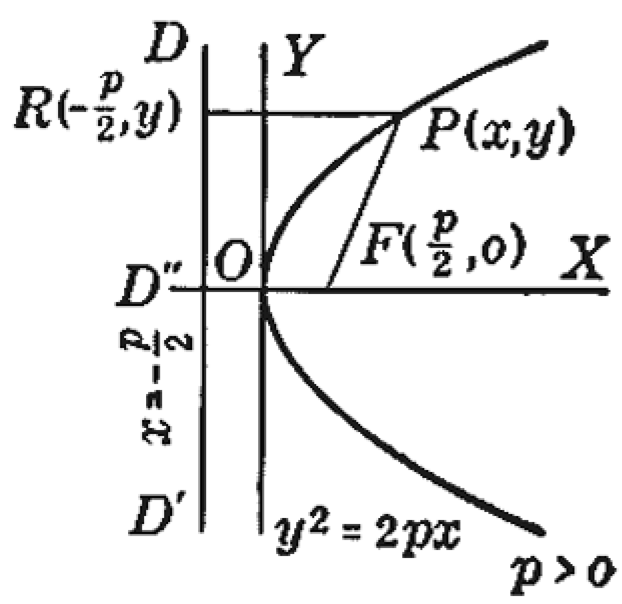


Fig. 101a

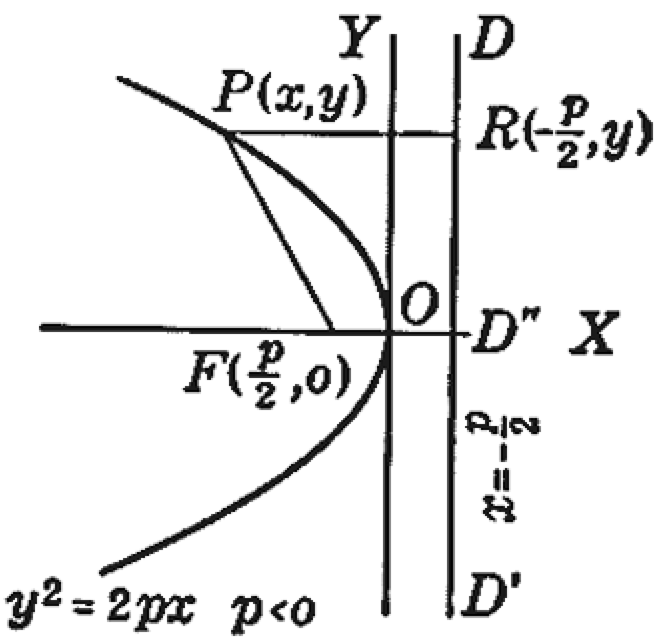


Fig. 101b

axis as the x-axis and the vertex as the origin (Figs. 101a and b). Let F be the focus and $D'D$ the directrix. Let D'' be the intersection of the principal axis with the directrix and let the length of the directed segment $D''F$ be p.

Since the vertex O lies on the curve, $D''O = OF = p/2$. Hence, the coördinates of F are $(p/2, 0)$, those of D'' are $(-p/2, 0)$, and the equation of the directrix is $x = -p/2$.

Let $P(x, y)$ be any point on the parabola. Draw FP and RP, where R is the foot of the perpendicular from P to the directrix. From the definition of a parabola,

$$FP = RP, \quad \text{or} \quad FP^2 = RP^2. \tag{1}$$

From the distance formula, we have

$$FP^2 = \left(x - \frac{p}{2}\right)^2 + (y - 0)^2.$$

Also,
$$RP^2 = \left(x + \frac{p}{2}\right)^2. \qquad \text{(Why?)}$$

By substituting these values of FP^2 and RP^2 in (1), we obtain

$$\left(x - \frac{p}{2}\right)^2 + y^2 = \left(x + \frac{p}{2}\right)^2. \tag{2}$$

On simplifying this equation, we have

$$\boldsymbol{y^2 = 2px.} \tag{3}$$

Conversely, if the coördinates of P satisfy (3), we find, by adding $\left(x - \frac{p}{2}\right)^2$ to both sides of the equation, that they satisfy equation (2). It follows that $FP^2 = RP^2$ or, since FP and RP, being undirected, are both positive, that $FP = RP$, so that P lies on the parabola.

Equation (3) is the equation of the parabola when the axis of the parabola is the x-axis and the vertex is the origin. If p is positive, the focus is to the right of the origin (Fig. 101a) and, if p is negative, the focus is to the left (Fig. 101b).

If we take the principal axis of the parabola as the y-axis, the vertex remaining at the origin, we find in a similar way that the equation of the parabola is

$$\boldsymbol{x^2 = 2py,} \tag{4}$$

where the focus lies above the origin if p is positive and below if p is negative.

Equations (3) and (4) are the **standard forms** (Art. 179) of the equation of the parabola.

182. Discussion of the Equation. If we solve equation (3) for y, we obtain

$$y = \pm \sqrt{2px}.$$

If p is positive, any negative value assigned to x would make y imaginary. Hence, there are no points on the curve to the left of the origin (Fig. 101a). Similarly, if p is negative, there are no points on the curve to the right of the origin (Fig. 101b).

If $x = 0$, then $y = 0$ so that the origin, which we have taken at the vertex, lies on the curve. To each value of x that agrees in sign with p, there correspond two values of y which are numerically equal but opposite in sign; that is, the curve is symmetric (Art. 41) to the x-axis. As x increases numerically, the corresponding values of y also increase numerically and the curve extends indefinitely far from both axes.

Since the latus rectum (Art. 180) is parallel to the directrix, its length is the sum of the numerical values of ordinates of its end points. To find these ordinates, put $x = p/2$ (Why?) in the equation of the curve. We then have

$$y^2 = p^2, \quad \text{or} \quad y = \pm p.$$

Hence, *the length of the latus rectum is the numerical value of $2p$.*

EXAMPLE 1. Locate the vertex, focus, axis, and directrix and find the length of the latus rectum of the parabola $y^2 = -6x$.

Since the equation is in the standard form (3), with $p = -3$, the coördinates of the vertex are (0, 0) and, of the focus, are $(-\frac{3}{2}, 0)$. The equation of the axis of the parabola is $y = 0$ and of its directrix is $x - \frac{3}{2} = 0$. The length of the latus rectum is 6 (Fig. 102).

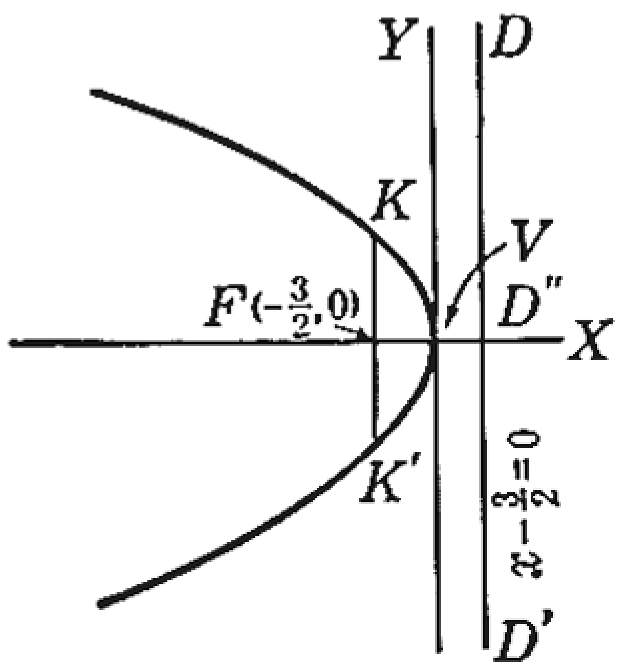


FIG. 102

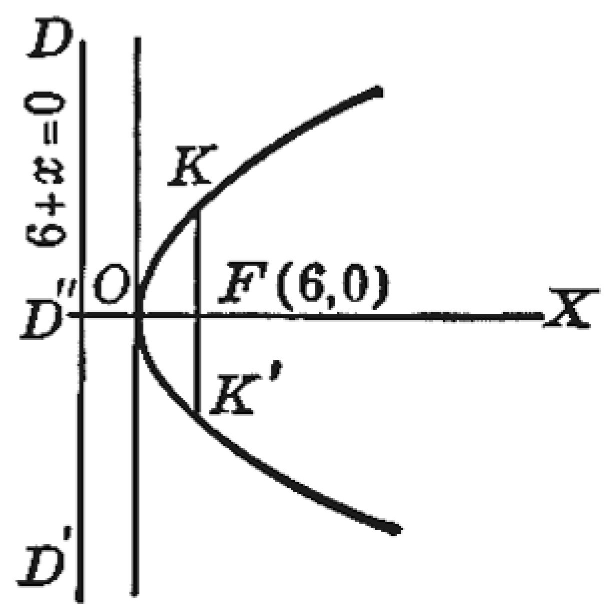


FIG. 103

EXAMPLE 2. Find the equation of a parabola, the coördinates of its focus, and the equation of its directrix, if the vertex is at the origin, the focus is on the x-axis to the right of the vertex, and the length of the latus rectum is 24.

Since the focus is to the right of the vertex, p is positive and equal to one-half of the length of the latus rectum, or 12. Since the coördinate axes are placed so that the equation of the parabola is in the standard form (3), the required equation of the curve is $y^2 = 24x$, the coördinates of the focus are (6, 0), and the equation of the directrix is $x + 6 = 0$ (Fig. 103).

Exercises

Draw each of the following parabolas. Find the coördinates of the focus, the equation of the directrix, and the length of the latus rectum.

1. $y^2 = 16x$. **2.** $y^2 + 2x = 0$. **3.** $x^2 = 10y$.
2 **4.** $x^2 = -3y$. **5.** $x^2 = 7y$. **6.** $y^2 = 11x$.
7. $2x^2 = 9y$. **8.** $5y^2 + 14x = 0$. **9.** $3x^2 + 5y = 0$.

Find the equation of the parabola having its vertex at the origin, given that:

10. Its focus is (5, 0). **11.** It directrix is $x + 7 = 0$.
12. Its focus is (0, 3). **13.** Its directrix is $y - 4 = 0$.

13 **14.** It has the x-axis as its principal axis and passes through (5, 2).

15. It has the y-axis as its principal axis and passes through (− 2, 7).

16. It has the x-axis as its principal axis and its directrix passes through (3, 6).

17. It has the x-axis as its principal axis and its focus lies on the line $3x - 5y = 15$.

18. It has the y-axis as principal axis, opens downward, and the length of its latus rectum is 18.

Find the points of intersection of the two parabolas.

19. $y^2 = 3x,\ x^2 = 3y.$ **20.** $y^2 = 54x,\ x^2 = 2y.$
21. $y^2 = -9x,\ 3x^2 = 8y.$ **22.** $y^2 = 8x,\ x^2 = y.$

23. Find the length of the chord of the parabola $y^2 = 12x$ that lies on the line $2x - 3y + 12 = 0$.

24. Find the equation of the line through the points on the parabola $y^2 = 10x$ whose ordinates are 2 and 5.

25. Find the equation of the circle that has the latus rectum of $y^2 = 2px$ as its diameter. Show that the center of the circle is the focus and that it touches the directrix.

26. The **focal radius** of a point on a parabola is its undirected distance from the focus. If the point $P_1(x_1, y_1)$ lies on the parabola $y^2 = 2px$, show that its focal radius is numerically equal to $x_1 + p/2$.

27. Using the results of Ex. 26, find the focal radii of the points on the parabola $y^2 = 20x$ for which:

(*a*) $x = 2$, (*b*) $x = 5$, (*c*) $y = 30$, (*d*) $x + y = 0$.

28. Find the points on the parabola $y^2 = 4x$ whose focal radii are (*a*) 5, (*b*) 37.

29. Find the points on the parabola $y^2 = 2px$ for which the focal radii are equal in length to the latus rectum.

30. The distance between the towers of Brooklyn Bridge is about 1600 feet and the lowest point of the cables is about 140 feet below these points of support. Assuming that the form of the cables is parabolic, take the lowest point of one of them as origin, the y-axis vertical, and find the equation of the parabola.

The Ellipse

183. The Standard Form of the Equation of the Ellipse. *An ellipse is the locus of a point the sum of whose undirected distances from two fixed points equals a constant.* The two fixed points are the **foci,** the midpoint of the segment joining them is the **center,** and the line through them is the **principal axis** of the ellipse.

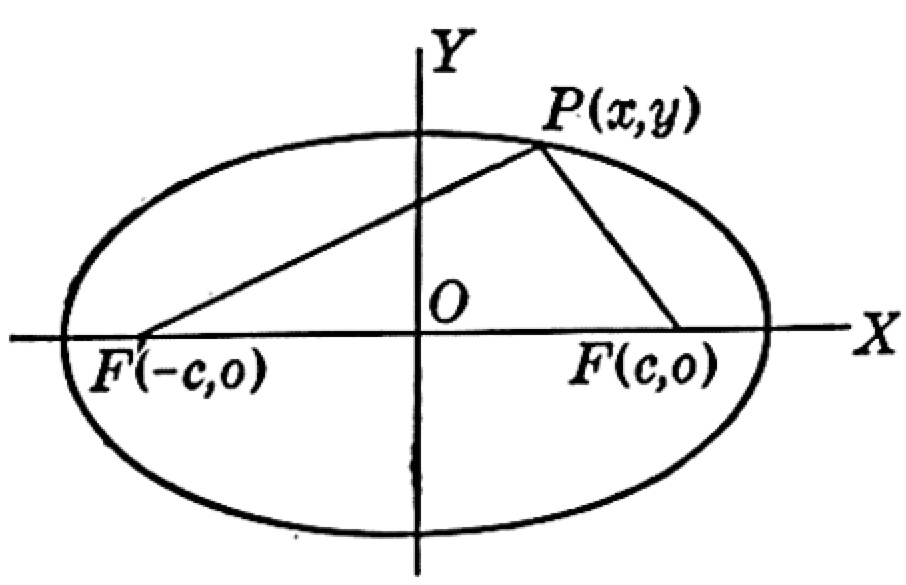


FIG. 104

To derive the equation of the ellipse in the standard form, we take the principal axis of the ellipse as x-axis and the center as the origin. Let F and F' (Fig. 104) be the foci and let $2c$ be the distance between them, so that the coördinates of F are $(c, 0)$ and of F' are $(-c, 0)$.

$a > c$

Let $P(x, y)$ be any point on the ellipse and let the sum of its distances from the foci be $2a$, so that

$$F'P + FP = 2a, \tag{5}$$

or

$$\sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2} = 2a.$$

If we transpose the second radical, square, and solve for the radical expression, we find that

$$\sqrt{(x - c)^2 + y^2} = a - \frac{c}{a}x. \tag{6}$$

By squaring again, and simplifying, we obtain

$$\frac{a^2 - c^2}{a^2}x^2 + y^2 = a^2 - c^2 \tag{7}$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1. \tag{8}$$

But $a > c$, since, in the triangle $F'PF$, the sum of the two sides $F'P + FP$, which equals $2a$ by (5), is greater than the third side $F'F$, which is equal to $2c$. Hence $a^2 - c^2$ is positive. We shall denote this positive number by b^2, that is

$$b^2 = a^2 - c^2. \tag{9}$$

If we substitute this value for $a^2 - c^2$ in (8), we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{10}$$

By reversing the steps in the foregoing discussion, it can be shown, conversely, that, if the coördinates of a point $P(x, y)$ satisfy equation (10), then $F'P + FP = 2a$, so that P lies on the ellipse.

Equation (10) is the standard form of the equation of the ellipse obtained by taking the principal axis of the ellipse as x-axis and the center as origin. If we take the principal axis as y-axis, the center remaining at the origin, we obtain, in a precisely similar way,

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1 \tag{11}$$

as the standard form of the equation when the foci are at $(0, c)$ and $(0, -c)$.

To distinguish, in numerical problems, between the cases in which the foci are on the x-axis (equation 10) or on the y-axis (equation 11) we notice that, because of (9), the number a, for the ellipse, cannot be less than b. It is equal to b only if $c = 0$, in which case the ellipse becomes a circle.

184. Discussion of the Equation. If we solve equation (10) for y and for x, we obtain

$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2} \quad \text{and} \quad x = \pm \frac{a}{b}\sqrt{b^2 - y^2} \tag{12}$$

respectively.

From the first of these equations, it follows that if $x^2 > a^2$, y is imaginary and, from the second, that if $y^2 > b^2$, then x is imaginary. There are thus no points on the ellipse outside of the rectangle formed by the lines $x = \pm a$ and $y = \pm b$.

To any value of x numerically less than a, there correspond two values of y, numerically equal but opposite in sign. Hence the curve is symmetric to the x-axis. By similar reasoning, from the second of equations (12), we find that it is also symmetric to the y-axis.

If $x = 0$, $y = \pm b$; and, as x increases numerically, the numerical value of y decreases and becomes zero when $x = \pm a$.

185. Definitions. The ellipse defined by equation (10) intersects the x-axis at the points $V(a, 0)$ and $V'(-a, 0)$ (Fig. 105). These points are

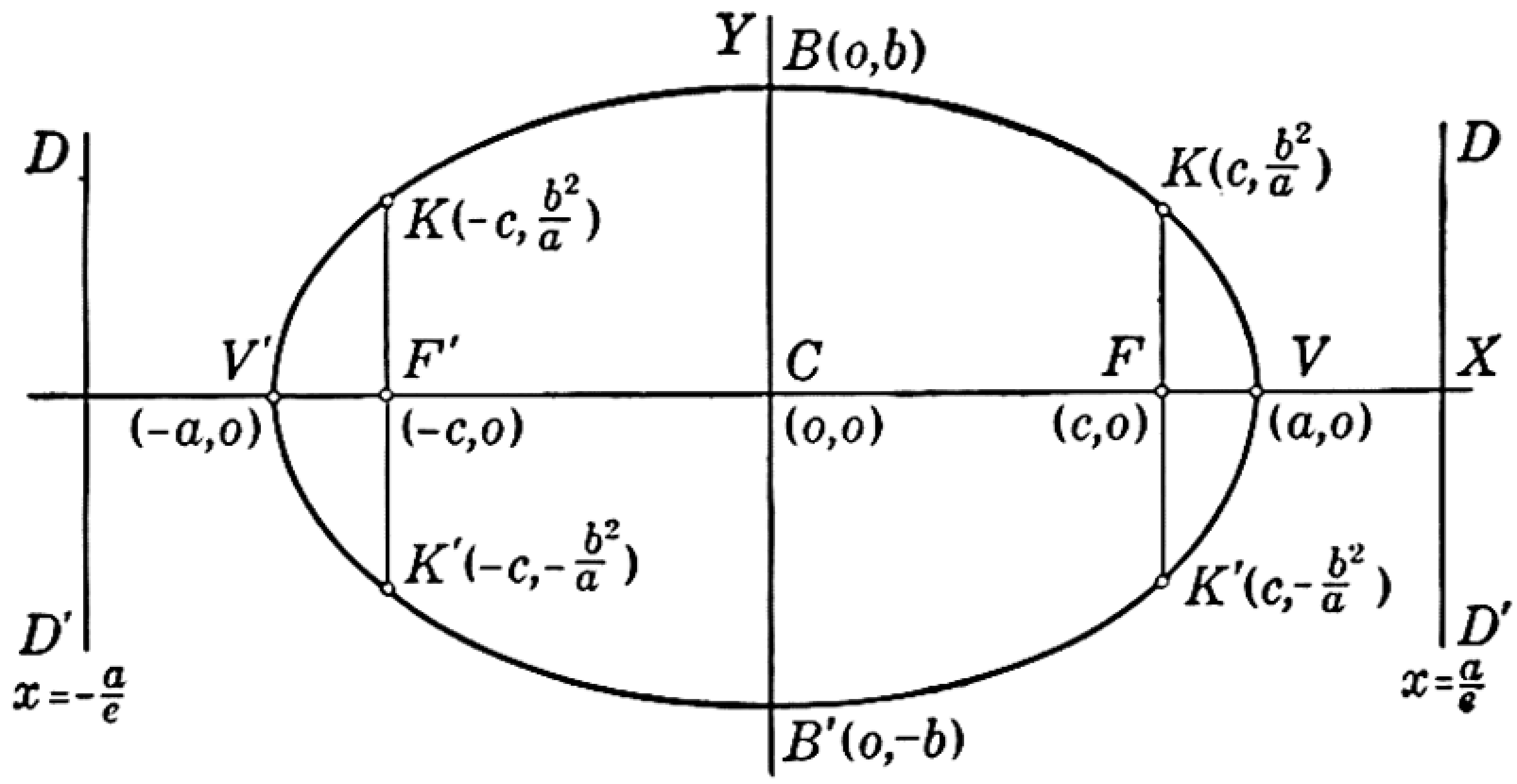


FIG. 105

called the **vertices** of the ellipse. The chord $V'V$ joining the vertices is the **major axis** of the ellipse. Its length is $2a$ and it is the longest chord that can be drawn in the ellipse. The chord of the ellipse on the y-axis is of length $2b$ and is called the **minor axis.** The numbers a and b are thus the lengths of the **semi-major axis** and the **semi-minor axis,** respectively.

Latus Rectum. The chord $K'K$, through either focus perpendicular to the major axis, is called the **latus rectum.** Its length is obviously twice the ordinate of K. To find this ordinate, put $x = \pm c$ in (10) and solve for y. We find, by the aid of (9), that

$$y = \pm \frac{b}{a}\sqrt{a^2 - c^2} = \pm \frac{b^2}{a}.$$

Hence, *the length of the latus rectum is* $2b^2/a$.

Eccentricity. The fraction c/a is called the **eccentricity** and is denoted by the letter e. We have, by (9),

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}.$$

Since, for an ellipse, c is always less than a, it follows that *the eccentricity of an ellipse is always less than unity.*

The shape of the ellipse (but not its size) is determined by its eccentricity. Thus, if $e = 0$, then $c = 0$, $b = a$, and the ellipse is a circle with its foci coincident at the center. As e increases from zero, the ellipse becomes more and more flattened. If we let e approach unity, holding a fixed, then c approaches a and b approaches zero so that the foci approach the vertices and the ellipse becomes very narrow.

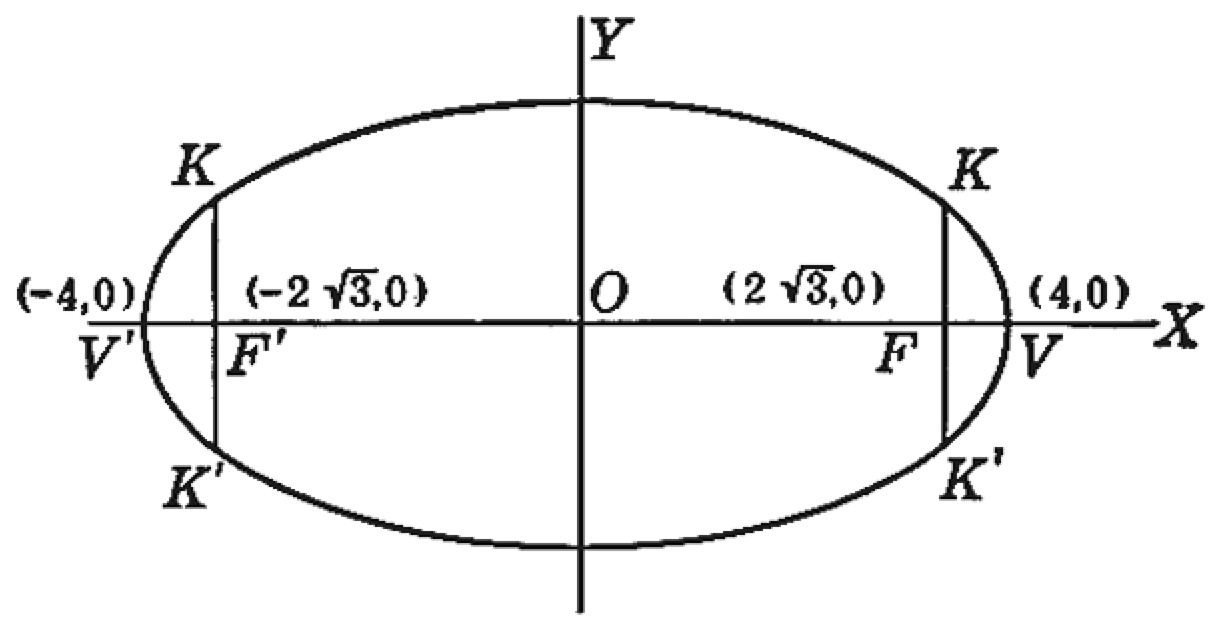


FIG. 106

Directrices. If $e \neq 0$, the two lines $x = \pm a/e$ (the lines $D'D$ in Fig. 105) are called the **directrices.** Their importance will appear in Art. 186.

EXAMPLE 1. The vertices of an ellipse are $(\pm 4, 0)$ and its eccentricity is $e = \sqrt{3}/2$. Find its equation, locate the foci, and find the length of the latus rectum.

Since the vertices $(\pm a, 0)$ are at $(\pm 4, 0)$, we have $a = 4$. Further, $c = ae = 4\sqrt{3}/2 = 2\sqrt{3}$. The coördinates of the foci are thus $(\pm 2\sqrt{3}, 0)$. Moreover, $b^2 = a^2 - c^2 = 16 - 12 = 4$; giving $b = 2$. The length of the latus rectum is found from its formula $2b^2/a$, to be 2.

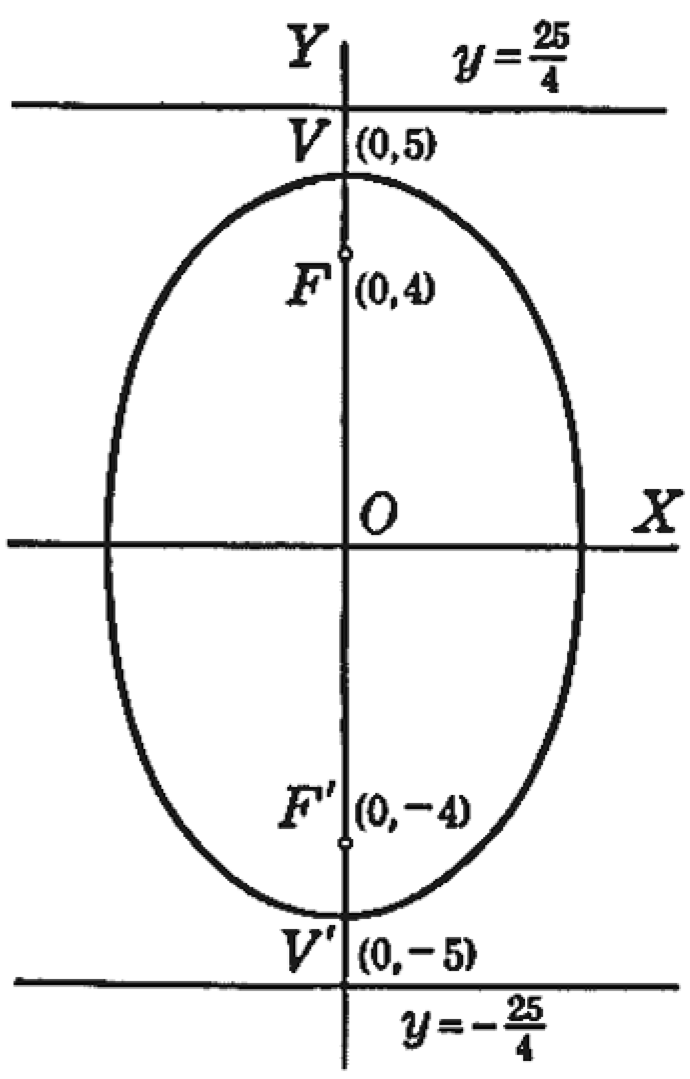


FIG. 107

Since the center of the ellipse is at the origin and the vertices are on the x-axis, the equation is in the standard form of equation (10). Putting $a = 4$ and $b = 2$ in (10), we have, as the required equation of the ellipse, $\frac{x^2}{16} + \frac{y^2}{4} = 1$.

EXAMPLE 2. The equation of an ellipse is $25x^2 + 9y^2 = 225$. Find the coördinates of the vertices and of the foci and the equations of the directrices.

If we write the given equation in the form

$$\frac{x^2}{9} + \frac{y^2}{25} = 1,$$

we find, by the last paragraph of Art. 183, that the foci are on the y-axis. We have

$$a = 5,\ b = 3,\ c = \sqrt{a^2 - b^2} = 4, \quad \text{and} \quad e = \frac{c}{a} = \frac{4}{5}.$$

The coördinates of the vertices are $(0, \pm 5)$ and of the foci are $(0, \pm 4)$. The equations of the directrices are $y = \pm \frac{25}{4}$.

Exercises

Find the vertices, foci, eccentricity, semi-axes, length of the latus rectum, and equations of the directrices of each of the following ellipses.

1. $16x^2 + 25y^2 = 400$.
2. $25x^2 + 169y^2 = 4225$.
3. $4x^2 + 9y^2 = 36$.
4. $4x^2 + 25y^2 = 100$.
5. $16x^2 + 9y^2 = 144$.
6. $9x^2 + y^2 = 9$.
7. $4x^2 + y^2 = 9$.
8. $25x^2 + 4y^2 = 25$.
9. $3x^2 + 5y^2 = 15$.
10. $5x^2 + 7y^2 = 4$.

Find the equations of the following ellipses.

11. Vertices $(\pm 4, 0)$, foci $(\pm 2, 0)$.
12. Vertices $(0, \pm 10)$, $e = \frac{4}{5}$.
13. Foci $(0, \pm 3)$, directrices $y = \pm 12$.
14. Vertices $(\pm 8, 0)$, length of latus rectum $\frac{49}{4}$.
15. Foci $(\pm 3, 0)$, ends of minor axes $(0, \pm 3)$.
16. Foci $(0, \pm 3\sqrt{3})$, length of latus rectum 3.

Find the equation, in one of the standard forms, of the ellipse passing through the given points.

17. (3, 4), (6, 2). **18.** (9, 2), (3, 6). **19.** (4, 2), (2, 5).

20. Find the points of intersection of the line $x + 2y = 5$ with the ellipse $x^2 + 16y^2 = 65$.

21. Find the points of intersection of the ellipse $2x^2 + 5y^2 = 22$ and the parabola $y^2 = 4x$.

22. The ends of the base of a triangle are $(\pm 8, 0)$. The sum of the lengths of the sides is 20. Find the locus of the vertex of the triangle.

23. Find the equation of the locus of the midpoints of the ordinates of the points on the circle $x^2 + y^2 = a^2$.

HINT. Let $P'(x', y')$ be a point on the given circle. Then $x'^2 + y'^2 = a^2$. The coördinates of $P(x, y)$, the midpoint of the ordinate of P', are $x = x'$, $y = y'/2$.

24. Find the equation of the locus of a point that divides the ordinates of the points on the circle $x^2 + y^2 = a^2$ in the ratio $b:a$.

186. A Second Definition of the Ellipse. We shall prove the following theorem: *The ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is the locus of a point for which the ratio of its undirected distance from the focus $F(c, 0)$ to its undirected distance from the directrix $x - a/e = 0$ equals e, the eccentricity.*

Let $P(x, y)$ be any point such that $\dfrac{PF}{PR} = e$ (Fig. 108). We shall show that P lies on the ellipse. We have

$$FP = ePR, \quad \text{or} \quad FP^2 = e^2PR^2. \tag{13}$$

But $$FP^2 = (x - c)^2 + y^2, \quad \text{and} \quad PR = \frac{a}{e} - x. \qquad \text{(Why?)}$$

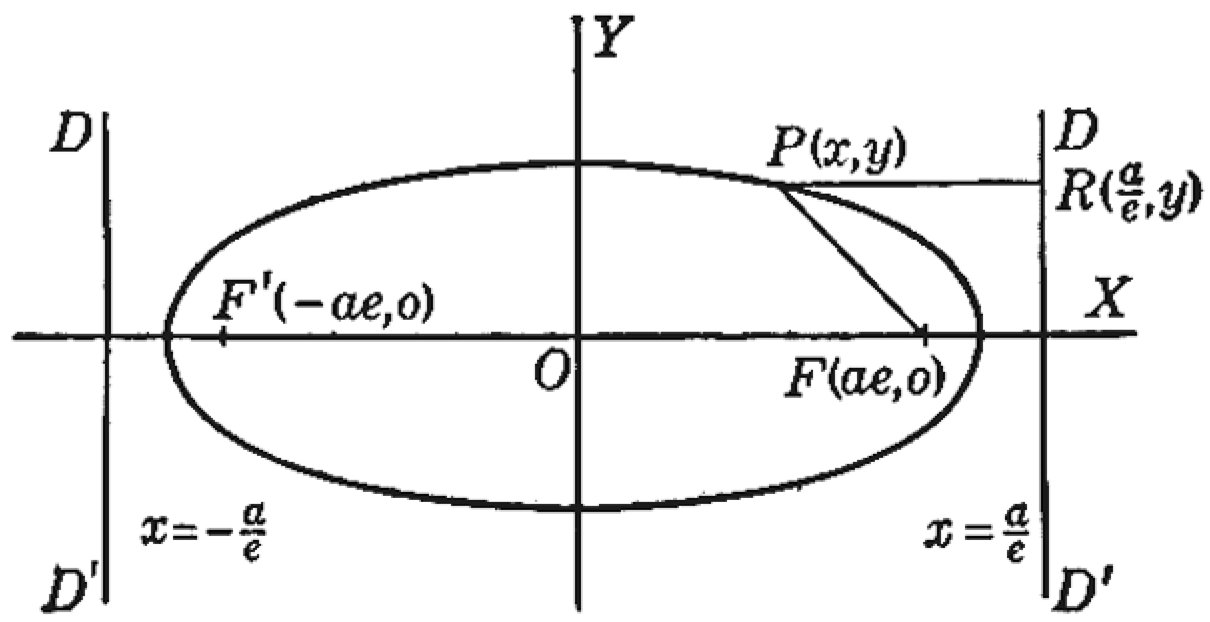


FIG. 108

On substituting these values in equation (13), we get

$$(x - c)^2 + y^2 = e^2\left(\frac{a}{e} - x\right)^2 = (a - ex)^2 = \left(a - \frac{c}{a}x\right)^2.$$

If we simplify this equation and multiply by a^2, we get

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2).$$

Since $a^2 - c^2 = b^2$, this equation reduces to

$$b^2x^2 + a^2y^2 = a^2b^2. \tag{14}$$

Hence, P lies on the ellipse.

Conversely, if P lies on the ellipse, its coördinates satisfy equation (14) and, by reversing the steps in the proof just given, we find that $FP = ePR$ so that P satisfies the conditions of the theorem.

From the symmetry of the figure with respect to the y-axis, it follows at once that the theorem of this article remains true if we replace the focus $F(c, 0)$ by $F'(-c, 0)$ and the directrix $x - a/e = 0$ by the directrix $x + a/e = 0$.

Because of the theorem of this article, we call the focus $F(c, 0)$ and the directrix $x - a/e = 0$ a *corresponding* focus and directrix. Similarly, $F'(-c, 0)$ and $x + a/e = 0$ are a corresponding focus and directrix.

Exercises

Derive the equations of the following ellipses using the theorem of this article. The given focus and directrix are corresponding focus and directrix.

1. $F(3, 0)$, $e = \frac{1}{2}$, directrix $x = 12$.
2. $F(0, 3)$, $e = \frac{1}{3}$, directrix $y = 27$.
3. $F(0, -4)$, $e = \frac{2}{5}$, directrix $y + 25 = 0$.
4. $F(-8, 0)$, $e = \frac{2}{3}$, directrix $x + 18 = 0$.

Find the equations of the following ellipses, given that their centers are at the origin.

5. Vertex $(4, 0)$, $e = \frac{1}{2}$.
6. Focus $(0, 4)$, $e = \frac{2}{3}$.
7. Directrix $y = -12$, $e = \frac{1}{3}$.
8. Vertex $(-6, 0)$, focus $(-2, 0)$.

9. Vertex $(-10, 0)$, directrix $x = -\frac{25}{2}$.

10. Focus $(5, 0)$, directrix $x = 10$.

11. Directrix $x = -8$, ends of minor axis $(0 \pm 2\sqrt{3})$.

12. Find the equation of the parabola with vertex at the origin that passes through the ends of the latus rectum to the right of the origin of the ellipse $5x^2 + 9y^2 = 45$.

13. The arch of a bridge is a semi-ellipse with major axis horizontal. The span is 40 feet and the top of the arch is 12 feet above the major axis. The roadway is horizontal and 15 feet above the major axis. Find, at 5-foot intervals, to three significant figures, the vertical distance from the arch to the roadway.

14. The undirected distances of a point P on the ellipse from the foci are called the **focal radii** of P. Show that the focal radii of $P(x, y)$ on the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ are $a - ex$ and $a + ex$.

HINT. Find the distances of $P(x, y)$ from the directrices and use the theorem of the present article.

Find the focal radii (Ex. 14) of the given point on the given ellipse.

15. $(10, 15)$, $3x^2 + 4y^2 = 1200$. **16.** $(4, \sqrt{21})$, $7x^2 + 16y^2 = 448$.

17. $(3, 2\sqrt{10})$, $5x^2 + 9y^2 = 405$. **18.** $(-6, -4)$, $x^2 + 2y^2 = 68$.

19. The earth's orbit is an ellipse with the sun at one of the foci. If the major semi-axis of the ellipse is 92.9 million miles and the eccentricity is 0.0168, find, to three significant figures, the greatest and least distance of the earth from the sun.

20. Verify the following construction for an ellipse. Fasten thumb tacks at the foci. Form a loop of thread of length $2a + 2c$, pass it around the thumb tacks, and draw it taut with the point of a pencil. Move the pencil around the foci, holding the thread constantly taut. Then the pencil point will describe an ellipse.

The Hyperbola

187. The Standard Form of the Equation of the Hyperbola. *A hyperbola is the locus of a point the difference of whose undirected distances from two fixed points equals a constant.* The two fixed points are the **foci,** the midpoint of the segment joining them is the **center,** and the line through them is the **principal axis** of the hyperbola.

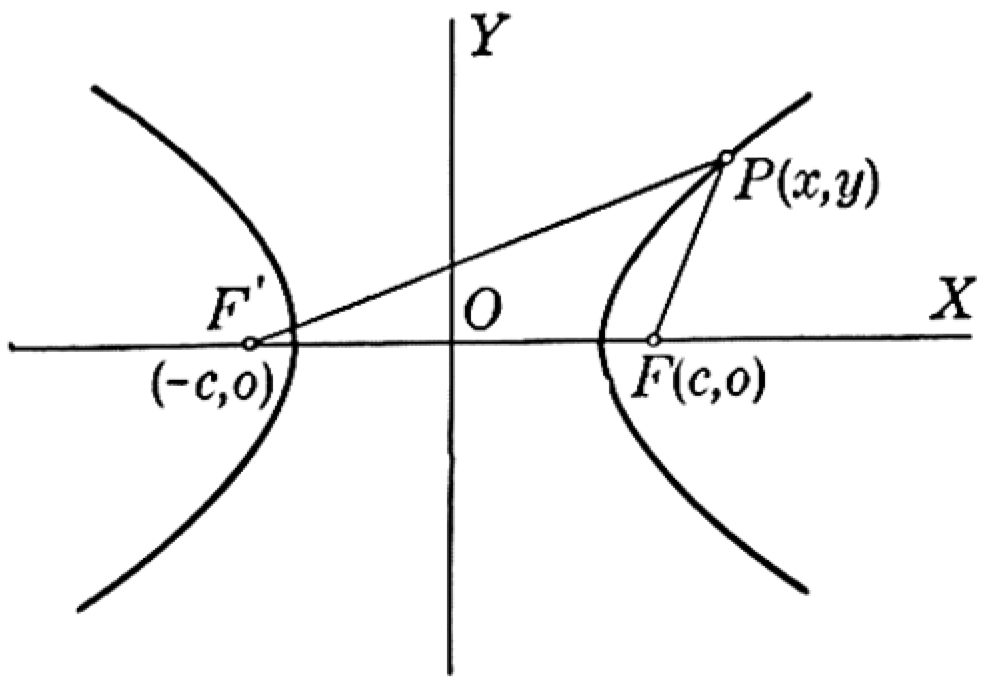


FIG. 109

The derivation of the standard form of the equation of the hyperbola parallels that of the ellipse. We take the principal axis as the x-axis and the center as origin. Let the distance between the foci be $2c$ so that the coördinates of F are $(c, 0)$ and of F' are $(-c, 0)$. Further, let the difference

of the distances of any point $P(x, y)$ on the hyperbola from F and F' be $2a$. Since the difference of two sides of the triangle $F'PF$ is less than the third side, $2a < 2c$, or $a < c$.

From the definition of the hyperbola, we have

$$F'P - FP = \pm 2a, \tag{15}$$

the positive sign holding for the points on the curve that lie to the right of the y-axis and the negative sign for the points to the left.

On substituting for $F'P$ and FP their values from the distance formula, we have

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a.$$

By transposing the second radical, squaring, and simplifying, we find that

$$cx - a^2 = \pm a\sqrt{(x-c)^2 + y^2}.$$

If we square again and collect terms, we obtain

$$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2). \tag{16}$$

We have seen that, for the hyperbola, $a < c$, hence we may put

$$b^2 = c^2 - a^2. \tag{17}$$

If we make this substitution in (16), that equation becomes

$$b^2x^2 - a^2y^2 = a^2b^2.$$

or

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{18}$$

By reversing the steps in the foregoing discussion, we find that, if the coördinates of $P(x, y)$ satisfy equation (18), then $F'P - FP = \pm 2a$, so that P lies on the hyperbola.

Equation (18) is the standard form of the equation of the hyperbola when the foci are taken at $(\pm c, 0)$ on the x-axis. When the axes are taken so that the foci are at $(0, \pm c)$ on the y-axis, we obtain similarly the equation of the curve in the standard form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1. \tag{19}$$

In the equation of a hyperbola, a may be less than, equal to, or greater than b. To determine, in a numerical problem, whether the foci are on the x-axis or on the y-axis, we first write the equation in the form (18) or (19), then notice whether the coefficient of x^2, or of y^2, is positive.

188. Discussion of the Equation. Asymptotes. If we solve equation (18) for y and for x, we obtain

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2} \quad \text{and} \quad x = \pm \frac{a}{b}\sqrt{y^2 + b^2} \tag{20}$$

respectively.

The curve is symmetric with respect to both coördinate axes. From the first of equations (20), we find that, if $x^2 < a^2$, y is imaginary. There are thus no points on the curve between the lines $x = -a$ and $x = a$. For every real value of y, however, there are two real values of x. These values of x are numerically smallest, and equal to $\pm a$, when $y = 0$. They increase indefinitely as y becomes numerically larger, so that the curve extends indefinitely far from both axes in each quadrant.

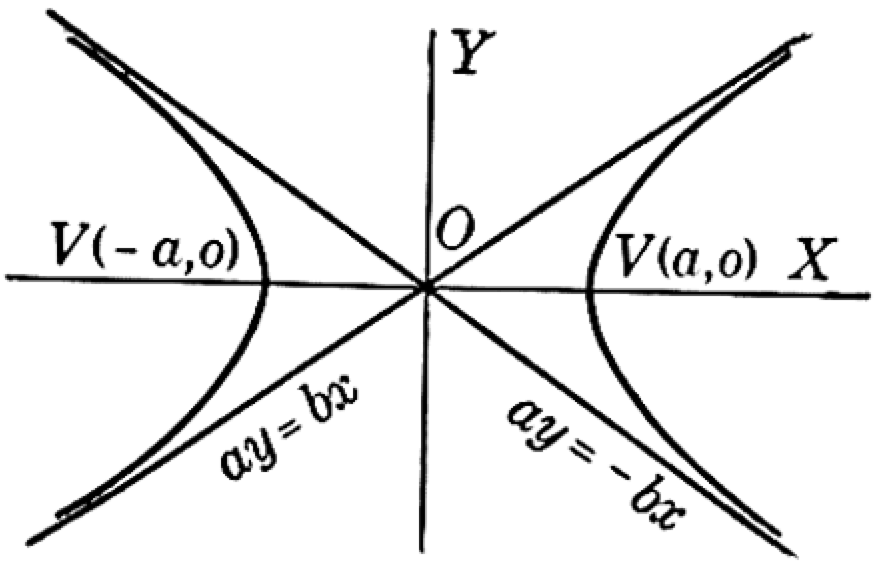


FIG. 110

To determine how the curve approaches infinity, we may write the first of equations (20) in the form

$$y = \pm \frac{b}{a} x \sqrt{1 - \frac{a^2}{x^2}}.$$

If x is numerically very large, the expression $\sqrt{1 - \frac{a^2}{x^2}}$ is very nearly unity and we obtain, as an *approximation* to the form of the curve a long way from the origin, the lines

$$y = \frac{bx}{a} \quad \text{and} \quad y = -\frac{bx}{a}. \tag{21}$$

These lines are called the **asymptotes** to the hyperbola (18).

It can be shown (see Art. 192, Ex. 21) that, if a point P_1 recedes along the curve indefinitely far from the origin, its distance from one of these asymptotes becomes indefinitely small.

189. Definitions. The intersections of the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ with the x-axis are found, by putting $y = 0$ in the equation, to be $(\pm a, 0)$. These points, V' and V (Fig. 111), are the **vertices.** The segment $V'V$, of length $2a$, that joins the vertices is the **transverse axis** of the hyperbola. Although the hyperbola does not intersect the y-axis (Why?), the segment from $B'(0, -b)$ to $B(0, b)$, of length $2b$, is called the **conjugate axis.**

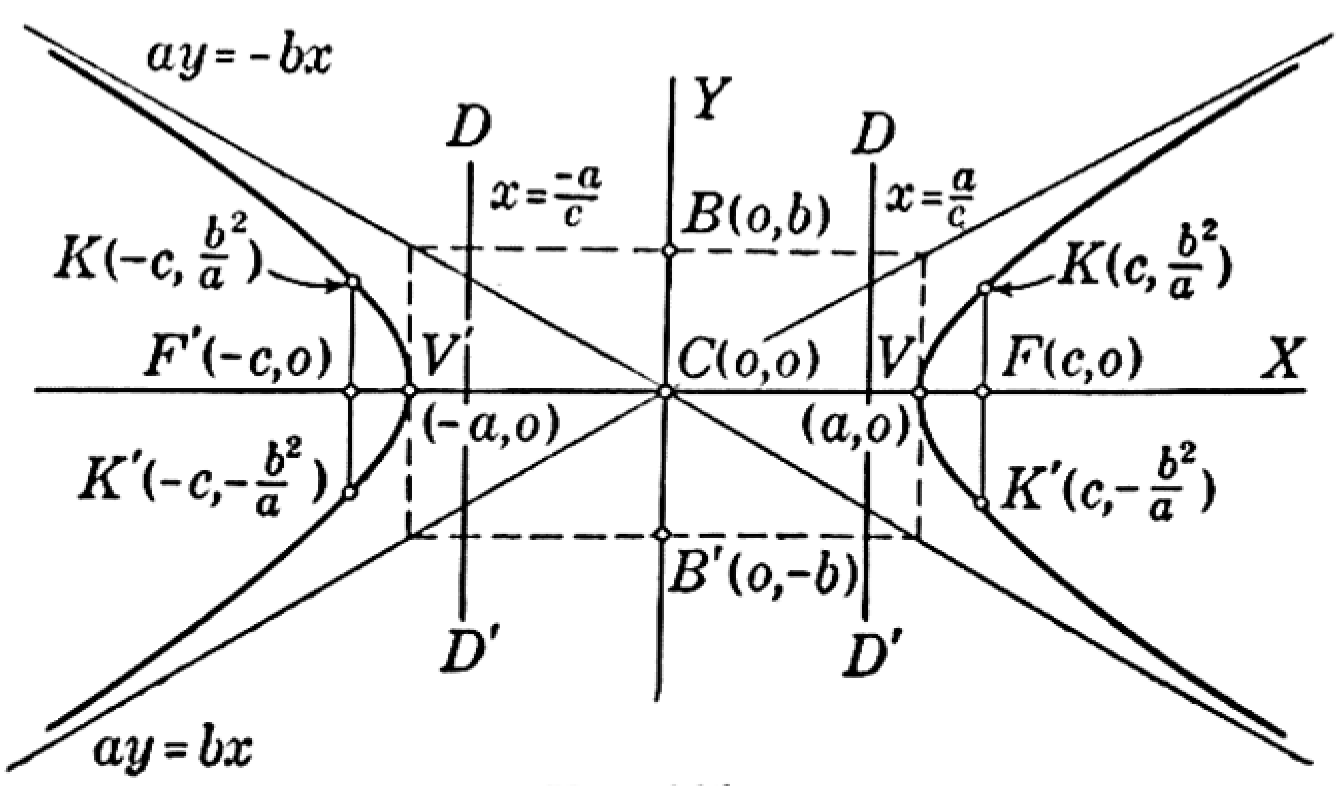


FIG. 111

The numbers a and b are thus the lengths of the **semi-transverse** and of the **semi-conjugate** axis, respectively.

The chord $K'K$ of the hyperbola through either focus perpendicular to the transverse axis is the **latus rectum.** Its length is found, as in Art. 185, to be $2b^2/a$.

The quotient c/a is denoted by e and is called the **eccentricity.** From (17), we have

$$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a}. \tag{22}$$

Since, for a hyperbola, a is always less than c, it follows that *the eccentricity of a hyperbola is always greater than unity.*

The lines $x = \frac{a}{e}$ and $x = -\frac{a}{e}$ are the **directrices.**

EXAMPLE 1. Determine the vertices, foci, eccentricity, and the equations of the asymptotes and directrices of the hyperbola $16x^2 - 9y^2 = 144$.

If we write the given equation in the standard form $\frac{x^2}{9} - \frac{y^2}{16} = 1$, we find that $a = 3$, $b = 4$, $c = \sqrt{a^2 + b^2} = 5$, and $e = c/a = 5/3$.

Since the transverse axis is on the x-axis, the vertices are $(\pm 3, 0)$ and the foci $(\pm 5, 0)$. The equations of the asymptotes are $y = \frac{4}{3}x$ and $y = -\frac{4}{3}x$. The equations of the directrices are $x = \frac{9}{5}$ and $x = -\frac{9}{5}$.

If one wishes to draw a hyperbola freehand, it is usually best to draw its asymptotes first, then to plot the vertices and to locate, from the equation, a few points on the curve. The hyperbola can then be drawn to pass through these points and to approach the asymptotes.

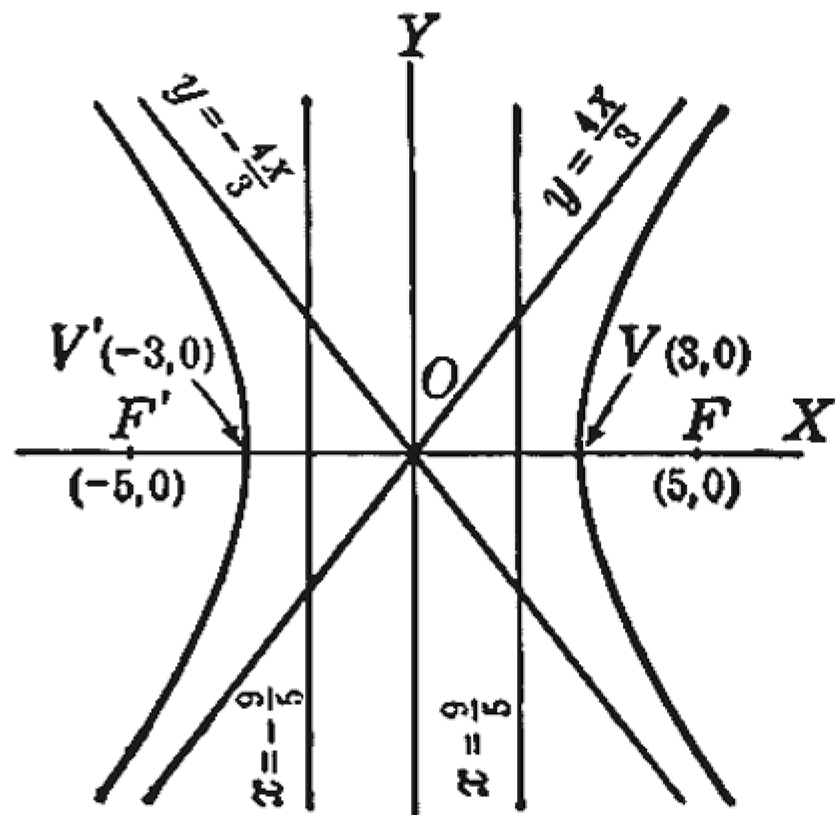


FIG. 112

EXAMPLE 2. The vertices of a hyperbola are $(0, \pm 4)$ and the equations of the directrices are $y = \pm \frac{8}{3}$. Find the eccentricity, the coördinates of the foci, the equation of the curve, and of its asymptotes.

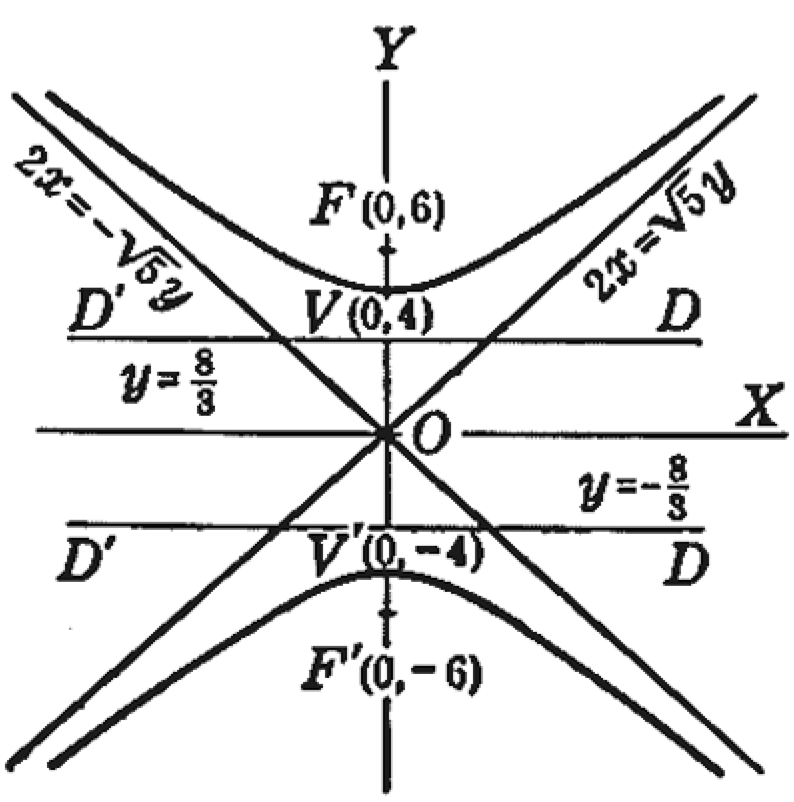


FIG. 113

We have $a = 4$. Moreover, the equations of the directrices are $y = a/e = 8/3$. Hence $e = 3/2$, $c = ae = 6$, and $b = \sqrt{c^2 - a^2} = \sqrt{20} = 2\sqrt{5}$.

The coördinates of the foci are, accordingly, $(0, \pm 6)$, the equations of the asymptotes are $x = \pm \frac{b}{a}y$, or $2x = \pm \sqrt{5}y$, and the equation of the curve is $\frac{y^2}{16} - \frac{x^2}{20} = 1$.

Exercises

Find the vertices, foci, eccentricity, length of the latus rectum, and equations of the asymptotes and of the directrices of the following hyperbolas.

1. $25x^2 - 144y^2 = 3600$.
2. $4x^2 - 25y^2 = 100$.
3. $y^2 - 4x^2 = 36$.
4. $x^2 - y^2 = 9$.
5. $81x^2 - 16y^2 = 36$.
6. $3x^2 - 2y^2 = 6$.
7. $25y^2 - 9x^2 = 1$.
8. $2x^2 - 5y^2 = 7$.

Find the equations of the hyperbolas that satisfy the following conditions.

9. Foci $(\pm 10, 0)$, $e = \frac{5}{4}$.

10. Vertices $(\pm 15, 0)$, asymptotes $15y = \pm 8x$.

11. Vertices $(0, \pm 3)$, directrices $5y = \pm 9$.

12. Latus rectum 3, vertices $(0, \pm 2)$.

13. Vertices $(\pm 3, 0)$, directrices $\sqrt{5}x = \pm 6$.

14. Asymptotes $3y = \pm 2x$, directrices $\sqrt{13}x = \pm 18$.

15. Ends of conjugate axes $(0, \pm 3)$, $e = 2$.

16. Asymptotes $y = \pm x$, passes through $(13, 5)$.

17. Asymptotes $3y = \pm 2x$, passes through $(6, 5)$.

18. Foci $(\pm 12, 0)$, length of latus rectum 20.

19. Find the equation of the hyperbola of eccentricity $\sqrt{2}$ having the same foci as the ellipse $9x^2 + 25y^2 = 225$.

20. Find the points of intersection of the hyperbolas $y^2 - 6x^2 = 1$ and $13x^2 - 2y^2 = 2$.

21. Find the equation of the locus of the center of a circle that passes through $(5, 0)$ and is tangent to the circle of radius 8 having its center at $(-5, 0)$.

HINT. If two circles are tangent externally (or internally), the distance between their centers equals the sum (or difference) of their radii.

22. Show that the distance from a focus of a hyperbola to an asymptote is numerically equal to b.

190. A Second Definition of a Hyperbola. *The hyperbola* $b^2x^2 - a^2y^2 = a^2b^2$ *is the locus of a point for which the ratio of its undirected distance from the focus* $F(c, 0)$ *to its undirected distance from the directrix* $x - a/e = 0$ *is equal to* e, *the eccentricity.*

The proof of this theorem parallels that given in Art. 186 for the ellipse and is left as an exercise for the student. From the symmetry of the figure, the theorem remains true if we replace the focus $F(c, 0)$ by $F'(-c, 0)$ and the directrix $x - a/e = 0$ by $x + a/e = 0$. The focus $F(c, 0)$ and the directrix $x - a/e = 0$ are called a *corresponding* focus and directrix as are also $F'(-c, 0)$ and $x + a/e = 0$.

191. Conjugate Hyperbolas. The two hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{and} \quad \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1, \tag{23}$$

are so related that *the transverse axis of each is the conjugate axis of the other*. For, the ends of the transverse axis of the first are $(a, 0)$ and $(-a, 0)$ and the ends of its conjugate axis are $(0, b)$ and $(0, -b)$. But these points are precisely the ends of the conjugate and transverse axis, respectively, of the second hyperbola.

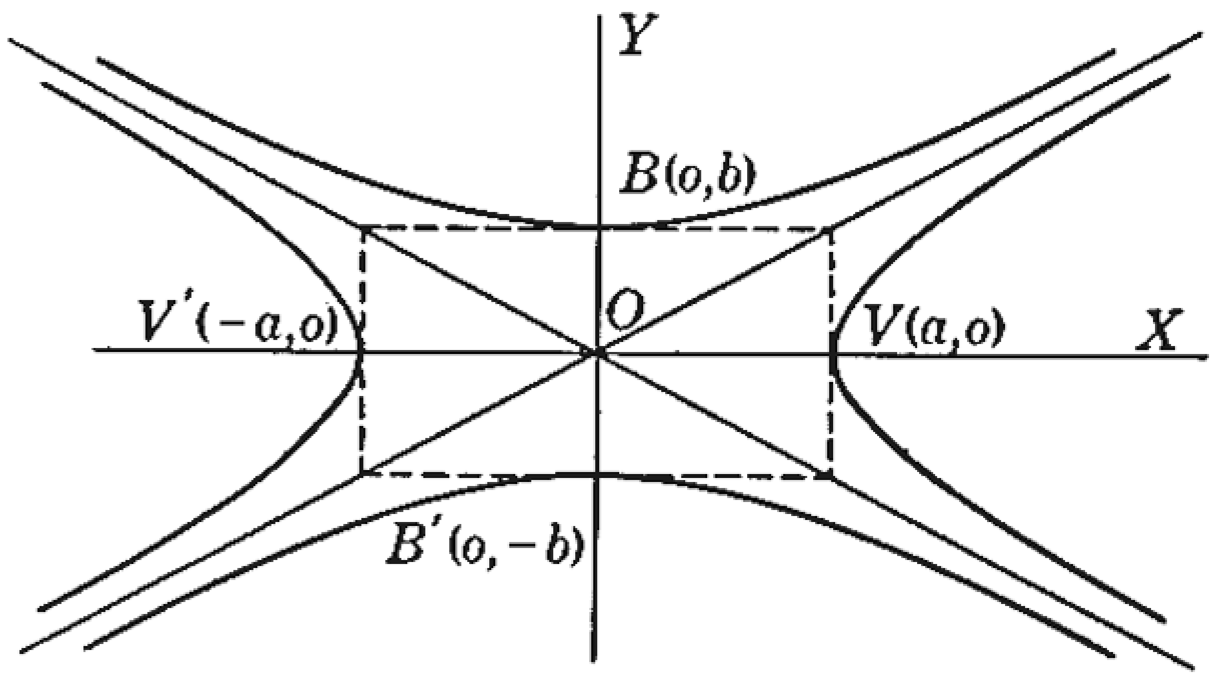


FIG. 114

Two hyperbolas, such as those defined by equations (23), which are so related that the transverse axis of each is coincident with the conjugate axis of the other are said to be a pair of **conjugate hyperbolas** and each of them is called the **conjugate hyperbola** of the other.

The asymptotes of the conjugate hyperbolas defined by equations (23) coincide since, in both cases, their equations are $ay = \pm bx$.

192. The Equilateral Hyperbola. If, in equation (18), we put $b = a$, that equation reduces to

$$x^2 - y^2 = a^2. \tag{24}$$

In this special case, the hyperbola is called an **equilateral** (or **rectangular**) **hyperbola.** It bears substantially the same relation to hyperbolas that the circle does to ellipses.

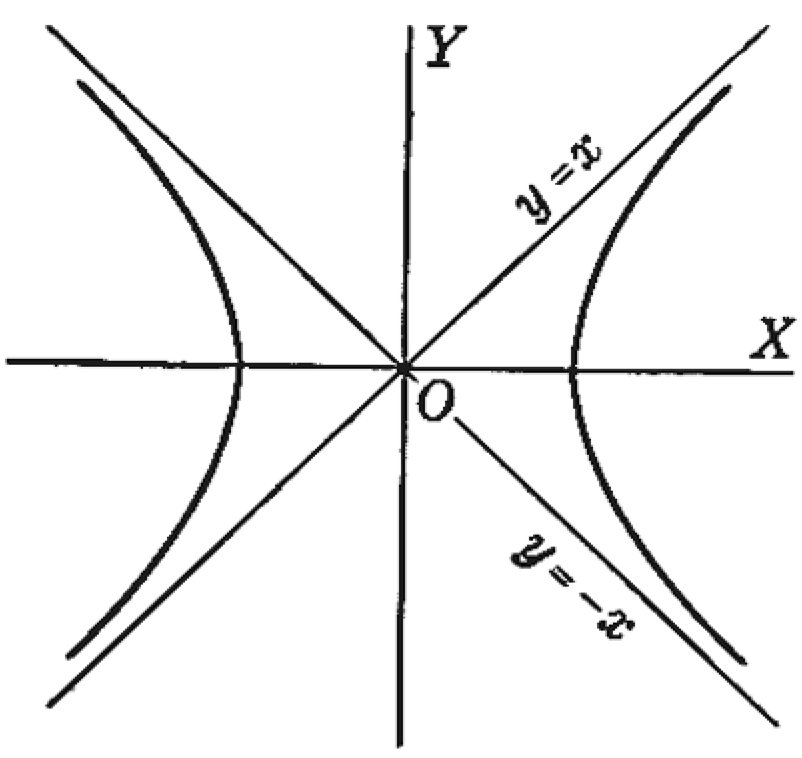


FIG. 115

The eccentricity of the equilateral hyperbola is found, by putting $b = a$ in equation (22), to be $e = \sqrt{2}$. Its asymptotes are the lines $y = x$ and $y = -x$ which are perpendicular to each other.

If we rotate the coördinate axes through an angle of $-45°$, so that the asymptotes of the hyperbola becomes the coördinate axes, we shall find (Art. 196, Ex. 9), that the equation of the equilateral hyperbola becomes

$$2xy = a^2. \tag{25}$$

In the applications, the equation of the equilateral hyperbola is often met with in this form.

Exercises

Using the second definition (Art. 190), find the equation of the hyperbola having the given point as focus, the given line as corresponding directrix, and the given eccentricity.

1. $(12, 0)$, $x = 3$, $e = 2$.
2. $(0, 5)$, $y = 1$, $e = \sqrt{5}$.
3. $(-9, 0)$, $x + 3 = 0$, $e = \sqrt{3}$.
4. $(0, -5)$, $y + 4 = 0$, $e = \sqrt{5}/2$.
5. $(0, -10)$, $5y + 32 = 0$, $e = \frac{5}{4}$.
6. $(6, 0)$, $3x = 8$, $e = \frac{3}{2}$.

Find the equation of the hyperbola conjugate to the given hyperbola. Find, for the given hyperbola and for its conjugate, the asymptotes, vertices, foci, and directrices.

7. $25x^2 - 144y^2 = 3600$.
8. $y^2 - 4x^2 = 20$.
9. $x^2 - 9y^2 = 4$.
10. $25x^2 - 4y^2 = 100$.
11. $11y^2 - 5x^2 = 55$.
12. $5y^2 - 11x^2 = 35$.

13. Show that the four foci of two conjugate hyperbolas lie on a circle.

14. If e and e' are the eccentricities of two conjugate hyperbolas, show that $e^2 + e'^2 = e^2e'^2$.

15. The undirected distances of a point P on a hyperbola from its foci are its **focal radii**. Show that the focal radii of a point $P(x, y)$ on the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ are numerically equal to $ex - a$ and $ex + a$.

Find the focal radii (Ex. 15) of the given point on the given hyperbola.

16. $(8, 2\sqrt{3})$, $x^2 - 4y^2 = 16$.
17. $(9, 4\sqrt{2})$, $4x^2 - 9y^2 = 36$.
18. $(4, -\sqrt{15})$, $5x^2 - 4y^2 = 20$.
19. $(2\sqrt{7}, 3\sqrt{2})$, $3x^2 - 4y^2 = 12$.

20. A focus of a hyperbola is (a, a), the corresponding directrix is $x + y - a = 0$, and its eccentricity is $\sqrt{2}$. Using the second definition of a hyperbola, show that it is an equilateral hyperbola referred to its asymptotes as coordinate axes.

21. If the point $P_1(x_1, y_1)$ lies on the hyperbola (18), so that $b^2x_1^2 - a^2y_1^2 = a^2b^2$, show that its distance d from the asymptote $bx - ay = 0$ is

$$d = \frac{bx_1 - ay_1}{-\sqrt{a^2 + b^2}} = \frac{a^2b^2}{-\sqrt{a^2 + b^2}(bx_1 + ay_1)}.$$

Hence show that, if P_1 recedes along the hyperbola indefinitely far from the origin in the first or third quadrants, its distance from the asymptote $bx - ay = 0$ becomes indefinitely small.

22. Show that the product of the distances of any point on a hyperbola from its asymptotes is a constant.

DERIVE IT MAY BE ON TEST

193. Standard Polar Equation of a Conic. By combining the definitions of Arts. 180, 186, and 190, we obtain the following single definition which holds equally for an ellipse, a parabola, or a hyperbola: *A conic is the locus of a point for which the ratio of its undirected distance from a fixed point (a focus) to its undirected distance from a fixed line (the corresponding directrix) equals a constant e, the eccentricity. The conic is*

$$\left(\begin{array}{l} \textit{an ellipse, if } e < 1, \\ \textit{a parabola, if } e = 1, \\ \textit{a hyperbola, if } e > 1. \end{array}\right)$$

We shall derive the standard polar equations of a conic from this definition.

Take the focus as origin, the principal axis of the conic as the polar axis, and let the polar axis be directed away from the given directrix (Fig. 116). Let D'' be the foot of the perpendicular from the focus O to the directrix and denote the length of the directed segment $D''O$ by p.

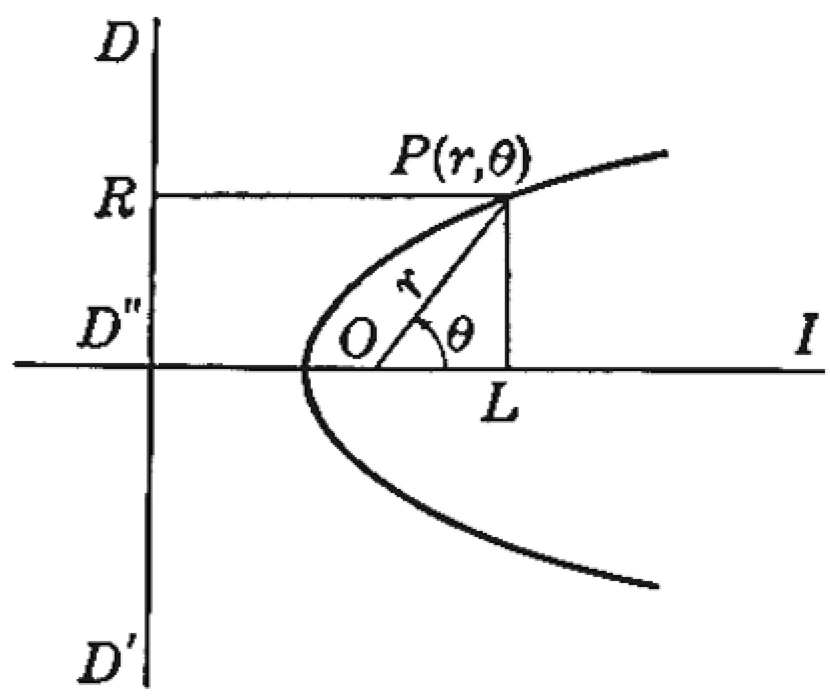


FIG. 116

Let $P(r, \theta)$ * be any point on the conic and let R and L be the feet of the perpendiculars from P to the directrix and to the polar axis, respectively. From the definition of the conic, we have

$$OP = e \cdot RP.$$

But
$$OP = r$$

and
$$RP = D''L = D''O + OL = p + r \cos \theta.$$

Hence
$$r = e(p + r \cos \theta) = ep + er \cos \theta.$$

If we solve the last of these equations for r, we obtain

$$r = \frac{ep}{1 - e \cos \theta}. \tag{26}$$

Conversely, if the coördinates of a point P satisfy (26), we find, by reversing the steps in the above proof, that $OP = e \cdot RP$ so that P lies on the given conic.

In deriving (26), we supposed that the polar axis was directed away from the directrix. If it is directed toward the directrix, we find in a similar way that

$$r = \frac{ep}{1 + e \cos \theta}. \tag{27}$$

Finally, if we take the polar axis parallel to the directrix, the origin remaining at the focus, we obtain, as the required equation of the conic, either

$$r = \frac{ep}{1 - e \sin \theta}, \quad \text{or} \quad r = \frac{ep}{1 + e \sin \theta}, \tag{28}$$

the first equation holding if the polar axis lies *above* the given directrix and the second, if it lies *below* it.

Equations (26), (27), and (28) are the **standard polar equations** of a conic.

* The following proof supposes (1) that the polar coördinates of P have been chosen so that r is positive and (2) that P and the origin lie on the same side of the directrix. Supposition (1) can always be made but (2) fails for one branch of the hyperbola. In that case, however, we are led to the same final equation if we suppose the coördinates of P chosen so that r is negative.

EXAMPLE. Reduce the equation $r = \dfrac{6}{2 - \cos\theta}$ to the standard form. Locate the vertices, the center, and the ends of the latus rectum that passes through the origin. Find the values of e, a, and b.

To reduce the equation to the standard form, we must make the first term in the denominator unity by dividing each term in numerator and denominator by 2. We thus obtain $r = \dfrac{3}{1 - \frac{1}{2}\cos\theta}$. By comparing this equation with (26), we find that $e = \frac{1}{2}$. The curve is thus an ellipse.

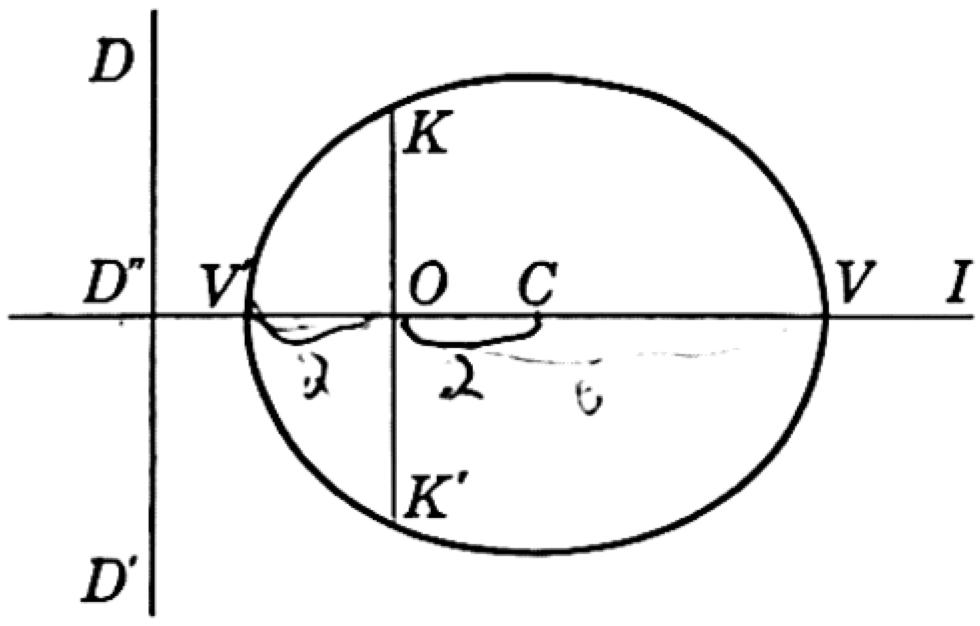


FIG. 117

The vertices of the ellipse are found, as the intersections of the principal axis with the curve, to be (6, 0°) and (2, 180°). Since the center lies midway between the vertices, its coördinates are (2, 0°). The distance from the center to either vertex is a. Hence $a = 4$.

The ends of the latus rectum are the intersections of the 90°-axis with the curve. By putting $\theta = \pm 90°$ in the equation, we find the coördinates of these points to be (3, 90°) and (3, − 90°). The length of the latus rectum is the distance between these points, which is 6. By Art. 185, the length of the latus rectum is $2b^2/a$. Since $a = 4$, we have $2b^2/a = 2b^2/4 = 6$. Hence, $b = \sqrt{12} = 2\sqrt{3}$.

Exercises

Find the eccentricity, the length of the latus rectum, the coördinates of the vertex or vertices, and draw the curve, given:

1. $r = \dfrac{8}{4 - 3\cos\theta}$. **2.** $r = \dfrac{5}{1 - \cos\theta}$. **3.** $r = \dfrac{12}{2 - 3\cos\theta}$.

4. $r = \dfrac{7}{1 + \cos\theta}$. **5.** $r = \dfrac{9}{3 + 5\cos\theta}$. **6.** $r = \dfrac{10}{5 + 2\cos\theta}$.

7. $r = \dfrac{15}{3 - \sin\theta}$. **8.** $r = \dfrac{8}{1 - 3\sin\theta}$. **9.** $r = \dfrac{5}{2 - 2\sin\theta}$.

10. $r = \dfrac{6}{1 + \sin\theta}$. **11.** $r = \dfrac{7}{2 + 5\sin\theta}$. **12.** $r = \dfrac{11}{5 + 3\sin\theta}$.

Write the polar equation of the conic with the origin at a focus, given:

13. Vertex (3, 180°), $e = 1$. **14.** Vertices (24, 0°), (24/5, 180°).

15. Vertices (2, 90°), (− 14, − 90°). **16.** Vertex (3, 90°), center (1, − 90°).

17. Center (8, 180°), $a = 10$. **18.** Directrix $r \sin\theta + 3 = 0$, $e = 1$.

19. Ends of latus rectum (5, 0°), (5, 180°), corresponding directrix $r\sin\theta + 10 = 0$.

20. If the conic is a parabola, show that the standard polar equations (26) and (27) may be reduced to $r\sin^2\theta/2 = p/2$ and $r\cos^2\theta/2 = p/2$.

Chapter 24

Transformation of Coördinates

194. Changing the Coördinate Axes. It frequently happens that the solution of a problem in analytic geometry can be simplified by the use of a different pair of coördinate axes from the one employed in the statement of the problem. The process of changing from one pair of coördinate axes to another is called a **transformation of coördinates.**

If the new axes are parallel, respectively, to the old ones, and if they have the same positive directions, the transformation is called a **translation of axes.** If the origin remains unchanged, and the new axes are obtained by revolving the old ones about the origin through a certain angle, then the transformation is a **rotation of axes.**

195. Translation of Axes. Let OX and OY (Fig. 118) be the original axes and let $O'X'$ and $O'Y'$ be the new ones, parallel, respectively, to the old and having the same positive directions. Let the coördinates of O', referred to OX and OY, be (h, k).

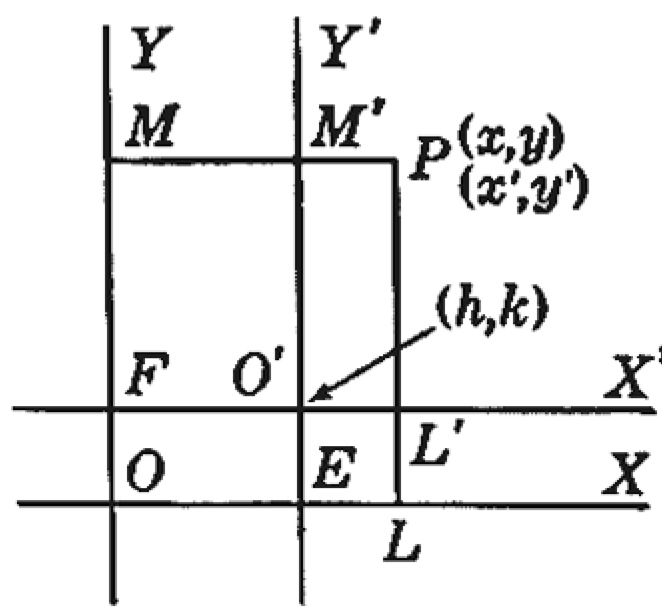


FIG. 118

Let P be any given point in the plane and let its coördinates, referred to the old axes, be (x, y) and, referred to the new ones, be (x', y'). It is required to find the values of x and y in terms of x' and y'.

Let L and L' be the feet of the perpendiculars from P on OX and $O'X'$, respectively, and M and M' the feet of the perpendiculars on OY and $O'Y'$. We have

$$x = OL = OE + EL = OE + O'L' = h + x'$$

$$y = OM = OF + FM = OF + O'M' = k + y'.$$

Hence *the formulas for a translation of axes are:*

$$x = x' + h \qquad y = y' + k \tag{1}$$

wherein (h, k) are the old coördinates of the new origin.

Equations (1) are said to define a *translation of the origin to the point* (h, k).

EXAMPLE 1. Find the equation of the conic $4x^2 - y^2 + 16x - 2y + 19 = 0$ when the origin is translated to the point $(-2, -1)$.

We have from (1), since $h = -2$ and $k = -1$,

$$x = x' - 2, \qquad y = y' - 1.$$

If we substitute these values of x and y in the given equation, we obtain

$$4(x'-2)^2-(y'-1)^2+16(x'-2)-2(y'-1)+19=0.$$

By expanding and simplifying this equation, we find, as the equation of the given conic referred to the new axes,

$$y'^2-4x'^2=4.$$

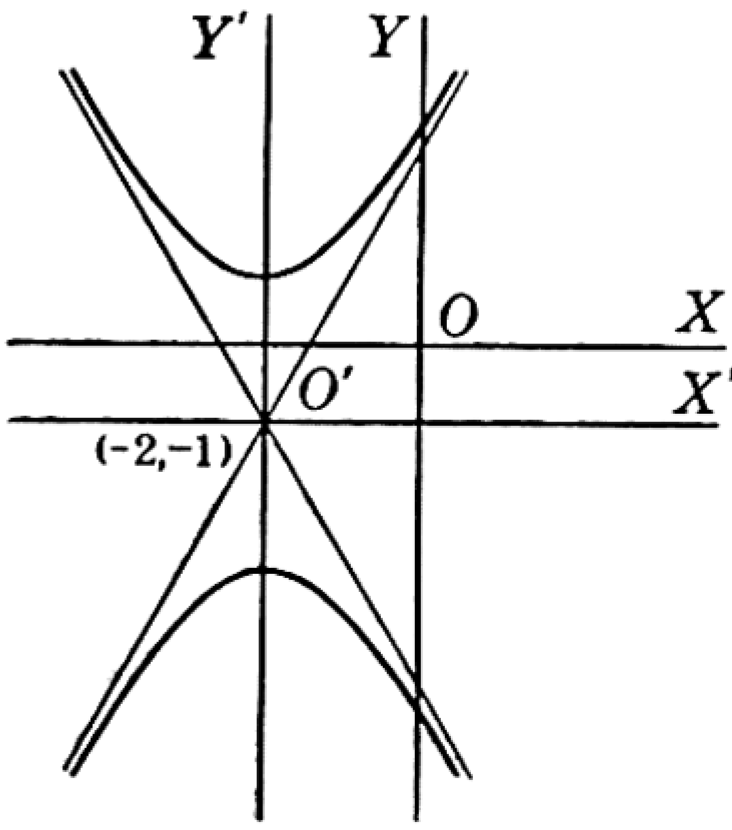


FIG. 119

This is the standard equation of a hyperbola with its center at the new origin and its transverse axis on the y'-axis. If we draw, with reference to the new axes, the curve defined by this last equation, the resulting locus will also be the graph of the original equation referred to the old axes (Fig. 119).

EXAMPLE 2. Find a translation of axes that will transform the equation $9x^2+4y^2+18x-24y+9=0$ into one in which the coefficients of the first degree terms are zero.

First solution. If we substitute the values of x and y from (1) in the given equation and collect the coefficients of the various powers of x' and y', we have

$$9x'^2+4y'^2+(18h+18)x'+(8k-24)y'+9h^2+4k^2+18h-24k+9=0.$$

Equating to zero the coefficients of x' and y' gives

$$18h+18=0 \quad \text{and} \quad 8k-24=0$$

so that $h=-1$ and $k=3$.

On substituting these values of h and k in the transformed equation, we obtain

$$9x'^2+4y'^2-36=0.$$

The curve is an ellipse which has its center at the new origin, its major axis on the y'-axis, and semi-axes $a=3$ and $b=2$.

Second solution. By collecting the terms in x and in y, and factoring out the coefficients of x^2 and y^2, respectively, we may write the given equation in the form

$$9(x^2+2x\qquad)+4(y^2-6y\qquad)=-9.$$

We can complete the square inside the first parentheses by adding 1 and, inside the second, by adding 9. Because of the coefficients outside these parentheses, by inserting these numbers we add 9 and 36, respectively, to the first member. To preserve the equality, we must add the same numbers to the second member. We then have

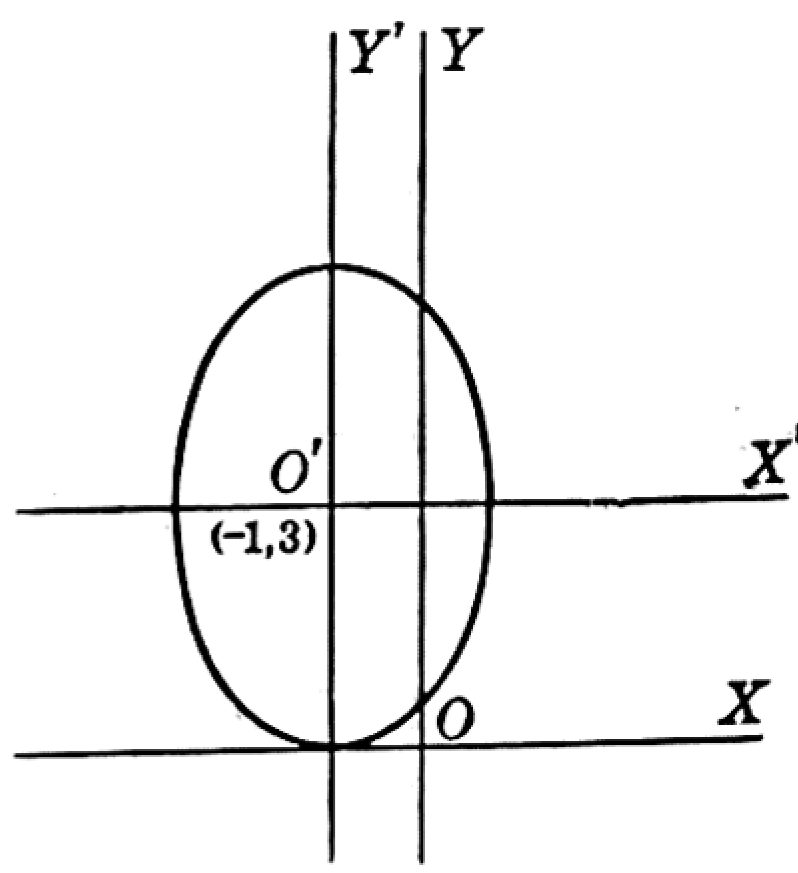


FIG. 120

$$9(x^2+2x+1)+4(y^2-6y+9)=-9+9+36$$

or $$9(x+1)^2+4(y-3)^2=36.$$

If we now translate the origin by putting $x + 1 = x'$, $y - 3 = y'$, that is

$$x = x' - 1, \qquad y = y' + 3,$$

we obtain as the required transformed equation

$$9x'^2 + 4y'^2 = 36.$$

This method of solving the given problem is shorter than the first one but it calls for more skill in algebraic manipulation. Also, it does not apply when one wishes to simplify a second degree equation containing a term in xy. Such equations should be simplified by the first method.

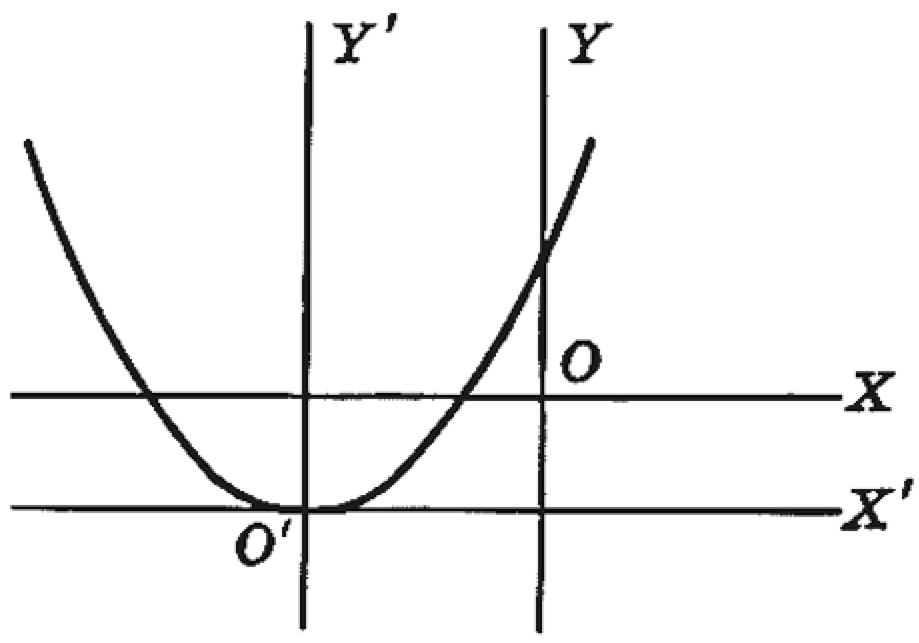


FIG. 121

EXAMPLE 3. By a translation of axes, reduce the equation $3x^2 + 12x - 5y + 17 = 0$ to a standard form.

Since the equation contains no term in y^2, we cannot remove the first degree term in y. Instead, we shall look for a translation which will remove the first degree term in x and the constant term.

By completing the square of the terms in x, we may write the given equation in the form

$$3(x^2 + 4x + 4) - 5y + 17 - 12 = 0,$$

or

$$3(x + 2)^2 = 5(y - 1).$$

By putting $x + 2 = x'$, $y - 1 = y'$, we reduce this equation to the form

$$3x'^2 = 5y'.$$

The curve is a parabola (Fig. 121) with its vertex at the new origin $(-2, 1)$.

Exercises

1. Find the new coördinates of the points (4, 1), (2, 8), (− 6, 9), (− 3, 6), and (− 1, 5) when the origin is translated to (− 3, 5).

2. After the origin has been translated to (4, − 2), the new coördinates of certain points are (5, 4), (6, − 8), (− 5, 2), (− 3, − 1), and (− 2, 5). Find the old coördinates.

Transform the following equation by taking the origin at the point indicated.

3. $6x - 5y + 9 = 0$, $(-4, -3)$.
4. $x^2 + y^2 - 8x + 4y + 8 = 0$, $(4, -2)$.
5. $4x^2 + 25y^2 - 40x + 150y + 225 = 0$, $(5, -3)$.
6. $25x^2 - 9y^2 + 100x - 72y - 269 = 0$, $(-2, -4)$.
7. $2x + y^2 - 4y - 4 = 0$ $(4, 2)$. Find, also, from the figure, the roots of $y^2 - 4y - 4 = 0$ to one decimal place.
8. $y = 3x^2 + 6x + 8$, $(-1, 5)$. Find, also, from the figure, the least value

the function $3x^2 + 6x + 8$ can have and the value of x for which it takes this least value.

Reduce the equations of the following conics to the standard form by translation of axes.

9. $9x^2 + 16y^2 - 54x + 32y - 47 = 0.$

10. $4x^2 + y^2 + 24x - 4y - 24 = 0.$

11. $3x^2 - y^2 - 24x - 20y + 11 = 0.$

12. $7y^2 - 2x^2 + 20x + 42y - 1 = 0.$

13. $5x^2 + 40x - 2y + 66 = 0.$

14. $3x + 7y^2 + 28y + 19 = 0.$

15. $y = ax^2 + bx + c$. Show, also, that the coördinates, referred to the original axes, of the vertex are $\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$.

196. Rotation of Axes. Let OX and OY be the old axes, OX' and OY', the new ones, and denote the angle XOX' by ϕ (Fig. 122).

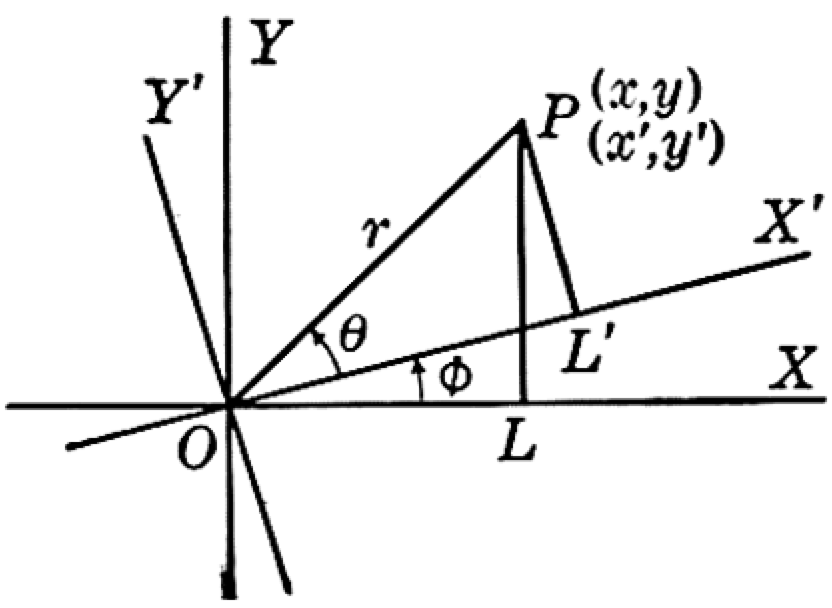


FIG. 122

Let P be any given point in the plane. Denote its coördinates, referred to the old axes, by (x, y) and, referred to the new ones, by (x', y'). It is required to find the values of x and y in terms of x' and y'.

Draw OP and let $OP = r$ and the angle $X'OP = \theta$. Drop perpendiculars from P to OX and OX' and denote the feet of these perpendiculars by L and L' respectively. From the definition of the sine and cosine of an angle, we have

$$x = OL = r\cos(\theta + \phi), \qquad y = LP = r\sin(\theta + \phi),$$

and

$$x' = OL' = r\cos\theta, \qquad y' = L'P = r\sin\theta.$$

From the formulas for the cosine and the sine of the sum of two angles, we now have

$$\begin{aligned} x &= r\cos(\theta + \phi) = r\cos\theta\cos\phi - r\sin\theta\sin\phi \\ &= x'\cos\phi - y'\sin\phi \end{aligned}$$

and

$$\begin{aligned} y &= r\sin(\theta + \phi) = r\sin\theta\cos\phi + r\cos\theta\sin\phi \\ &= x'\sin\phi + y'\cos\phi. \end{aligned}$$

Hence the required formulas for the *rotation of the axes through an angle ϕ are*

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{x'\cos\phi - y'\sin\phi} \\ \boldsymbol{y} &= \boldsymbol{x'\sin\phi + y'\cos\phi}. \end{aligned} \tag{2}$$

If we solve equations (2) for x' and y', and simplify, we obtain

$$\begin{aligned} x' &= x\cos\phi + y\sin\phi \\ y' &= -x\sin\phi + y\cos\phi. \end{aligned} \tag{3}$$

EXAMPLE. Find the equation of the parabola $x^2 - 2xy + y^2 - 2ax - 2ay + a^2 = 0$ when the axes are rotated through 45° and reduce the resulting equation by a translation of axes to the standard form.

Since $\sin 45° = \cos 45° = 1/\sqrt{2}$, equations (2) become, for this rotation,

$$x = \frac{x' - y'}{\sqrt{2}}, \quad y = \frac{x' + y'}{\sqrt{2}}.$$

On substituting these values of x and y in the given equation of the parabola, we obtain

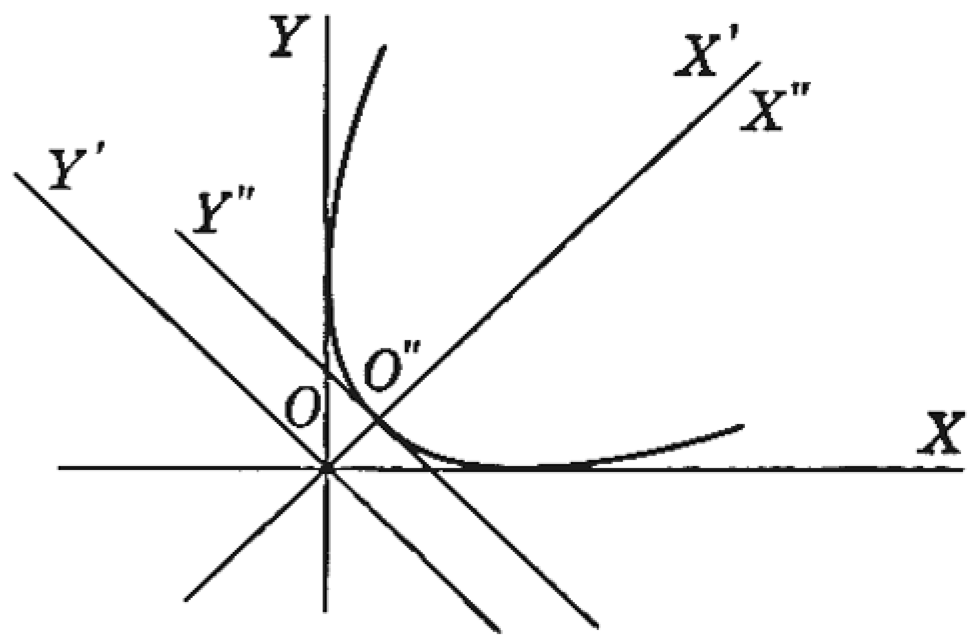


FIG. 123

$$\frac{(x' - y')^2}{2} - (x' - y')(x' + y') + \frac{(x' + y')^2}{2} - \sqrt{2}a(x' - y') - \sqrt{2}a(x' + y') + a^2 = 0.$$

On expanding and simplifying this equation, we find that it reduces to

$$2y'^2 - 2\sqrt{2}ax' + a^2 = 0.$$

If we now translate the origin to the point $(\sqrt{2}a/4, 0)$, the resulting equation may be reduced to the standard form $y''^2 = \sqrt{2}ax''$ of the equation of a parabola.

Exercises

1. Find the new coördinates of the following points when the axes are rotated through 45°: $(3, -5)$, $(-4, -2)$, $(7, 7)$, $(0, 12)$, $(-4, 4\sqrt{3})$.

2. Solve Ex. 1 when the angle of rotation is 120°.

3. Find the old coördinates, given that the new coördinates, after a rotation of 60°, are: $(3 + \sqrt{3}, -3\sqrt{3} + 1)$, $(6, 0)$, $(5, -5\sqrt{3})$, $(6, 4)$.

Find the transformed equation when the axes are rotated through the angle indicated.

4. $x - \sqrt{3}y = 5$, 30°.

5. $7x + 24y + 200 = 0$, $\tan^{-1} \frac{24}{7}$.

6. $3x^2 - 8xy + 3y^2 = 28$, 45°.

7. $7x^2 - 6\sqrt{3}xy + 13y^2 = 16$, 30°.

8. $7x^2 + 2\sqrt{3}xy + 9y^2 = 22$, 60°.

9. $x^2 - y^2 = a^2$, $-45°$.

10. $4x^2 + 24xy - 3y^2 = 60$, $\tan^{-1} \frac{3}{4}$.

11. $2x^2 + 4xy - y^2 = 6$, $\tan^{-1} \frac{1}{2}$.

12. $4x^2 - 12xy + 9y^2 = 6$, $\tan^{-1} \frac{2}{3}$.

13. $3x^2 - 24xy - 4y^2 + 30x - 16y + 18 = 0$. First rotate through the angle $\tan^{-1} \frac{4}{3}$, then simplify further by a suitable translation.

14. Using the definition of Art. 183, find the equation of the ellipse whose foci are $(2, 1)$ and $(-2, -1)$ and for which $a = 3$. Then rotate the axes so that the line joining the foci is the new x-axis.

15. Using the definition of Art. 190, find the equation of the hyperbola of eccentricity 2 having $(2, 2)$ as a focus and $x + y - 1 = 0$ as corresponding directrix. Then rotate the axes so that the directrix is perpendicular to the new x-axis.

Chapter 25

Conics with Equations Not in Standard Form

197. The General Equation of Second Degree. In Chapter 23, we studied the parabola, ellipse, and hyperbola, taking the axes, in each case, in such a position that the equation of the curve was in the standard form. Each of the equations derived in that chapter was a special case of the **general equation of second degree**; that is, of the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \tag{1}$$

wherein A, B, and C are not all zero.

In this chapter, we shall show that *the locus, if it exists, of any equation of second degree is a conic section;* that is, it is a parabola, ellipse, hyperbola, or two straight lines. We shall further show how, if the curve is a parabola, ellipse, or hyperbola, the equation of the curve may be reduced to its standard form. If $B = 0$, we shall see that this reduction may be effected by a translation of axes but, if $B \neq 0$, a rotation of axes will be necessary.

198. Conics with Principal Axis Parallel to a Coördinate Axis. If, in equation (1), $B = 0$, that equation takes the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0. \tag{2}$$

We first simplify this equation by the methods given in Art. 195. Except in certain special cases (to be stated presently) equation (2) may first be written in one of the following forms:

$$(y-k)^2 = 2p(x-h) \qquad (x-h)^2 = 2p(y-k) \tag{3}$$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, \qquad \frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1, \tag{4}$$

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1, \qquad \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1. \tag{5}$$

In any one of these cases, if we translate the origin to the point (h, k) by putting

$$x = x' + h, \qquad y = y' + k,$$

the equation reduces to the standard form of the equation of a parabola, ellipse, or hyperbola. It follows that: *if equation* (2) *reduces to one of the forms* (3), *the curve is a parabola; if it reduces to* (4), *it is an ellipse; and, if it reduces to* (5) *it is a hyperbola. If it is a parabola, its vertex, and if it is an ellipse or a hyperbola, its center, is at the point* (h, k). *In every case, its principal axis is parallel to one of the coördinate axes.*

In the following special cases, equation (2) cannot be reduced to one

of the forms (3), (4), or (5). If $A = D = 0$, so that x does not appear in the equation, (2) reduces to

$$Cy^2 + Ey + F = 0 \tag{6}$$

and the locus, if it exists, reduces to the two lines parallel to the x-axis defined by solving equation (6) for y. Similarly, if $C = E = 0$, so that y does not appear, the locus, if it exists, is two lines parallel to the y-axis.

Again, equation (2) may reduce, on completing the squares, to

$$A(x - h)^2 + C(y - k)^2 = 0.$$

If A and C agree in sign, there is only one point, (h, k), on the graph and the equation is said to define a **point ellipse.** If A and C are opposite in sign, the *graph consists of two lines intersecting at* (h, k); namely,

$$\sqrt{A}(x - h) + \sqrt{-C}(y - k) = 0, \quad \text{and} \quad \sqrt{A}(x - h) - \sqrt{-C}(y - k) = 0.$$

Finally, if equation (2) reduces to the form

$$A(x - h)^2 + C(y - k)^2 + F' = 0,$$

where A, C, *and* F' *all agree in sign,* there are no points on the locus and the equation is said to define an **imaginary ellipse.**

Exercises

Write the equation of each of the following parabolas in one of the forms of equations (3). Find the coördinates of the vertex and focus, the equation of the directrix, and draw the curve.

1. $y^2 - 12x + 10y + 37 = 0.$
2. $2y^2 - 5x + 8y - 12 = 0.$
3. $x^2 + 4x - 8y - 4 = 0.$
4. $3x^2 - 6x + 8y + 19 = 0.$

Write each of the following equations in one of the forms (4) or (5). Find the coördinates of the center, vertices, and foci, and draw the curve.

5. $9x^2 + 25y^2 - 54x - 100y - 44 = 0.$
6. $x^2 - y^2 + 4x - 6y - 21 = 0.$
7. $9x^2 - 16y^2 + 90x + 32y + 65 = 0.$
8. $2x^2 + y^2 - 8x - 6y + 8 = 0.$
9. $5y^2 - 4x^2 + 32x + 50y + 41 = 0.$
10. $9x^2 + 5y^2 - 18x + 10y - 31 = 0.$
11. $4x^2 + 9y^2 - 12x + 18y - 18 = 0.$
12. $7x^2 - 2y^2 + 28x + 4y + 40 = 0.$

Describe each of the following loci and draw the graph if it exists.

13. $3x^2 + 5y^2 - 12x - 30y + 57 = 0.$
14. $4x^2 - 9y^2 - 16x + 18y + 7 = 0.$
15. $5x^2 + 7y^2 - 10x + 14y + 20 = 0.$
16. $6x^2 - x - 15 = 0.$

199. Conics Satisfying Given Conditions. It is sometimes necessary to set up the equation of a conic having its principal axis parallel to one of the coördinate axes and satisfying a sufficient number of additional conditions to fix its position. If the curve is a parabola, we must find the coördinates of the vertex, which we take as (h, k), and the value of p, which is twice the directed distance from the vertex to the focus, and substitute these values of h, k, and p in one of equations (3). Similarly, if the curve is an ellipse or a hyperbola, we locate the center, (h, k), find the values of a and b, and substitute these values in one of equations (4) or (5).

EXAMPLE 1. Find the equation of a parabola with focus (5, 2) and directrix $x = 1$.

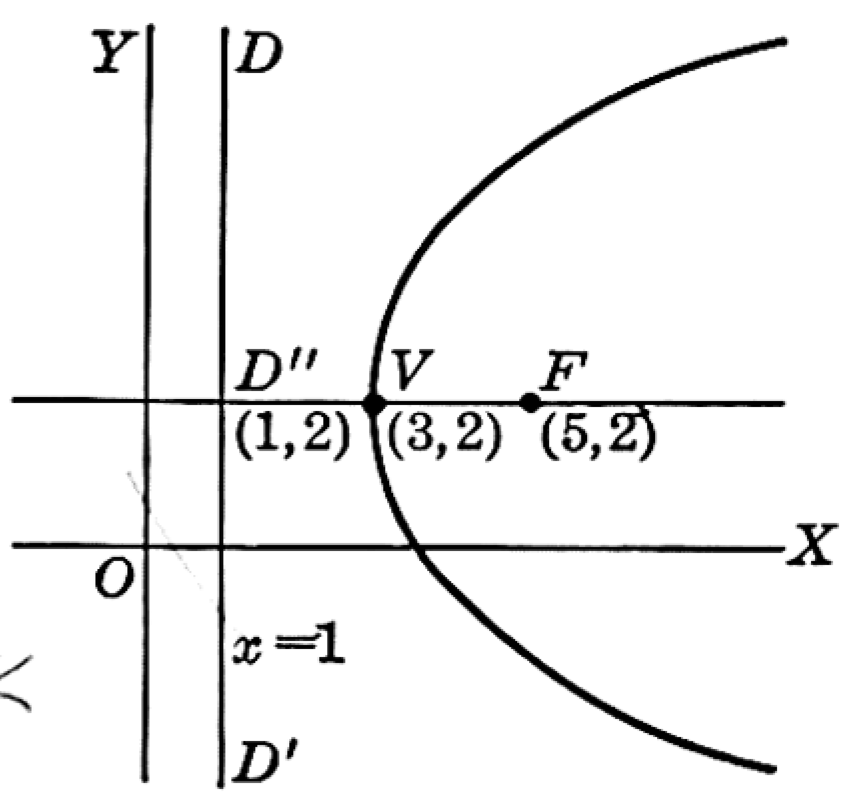


FIG. 124

Since $p = D''F$ (Fig. 124) is the directed distance from the directrix to the focus, we have $p = 4$. Further, since the vertex V is the midpoint of the segment $D''F$, its coördinates are (3, 2). Since the principal axis is parallel to the x-axis, we substitute $p = 4$, $h = 3$, and $k = 2$ in the first of equations (3). The result is

$$(y - 2)^2 = 8(x - 3).$$

This is the required equation of the parabola.

EXAMPLE 2. Find the equation of a hyperbola given that its vertices are $(-1, -1)$ and $(-1, 7)$ and that the line $2x - y + 5 = 0$ is an asymptote.

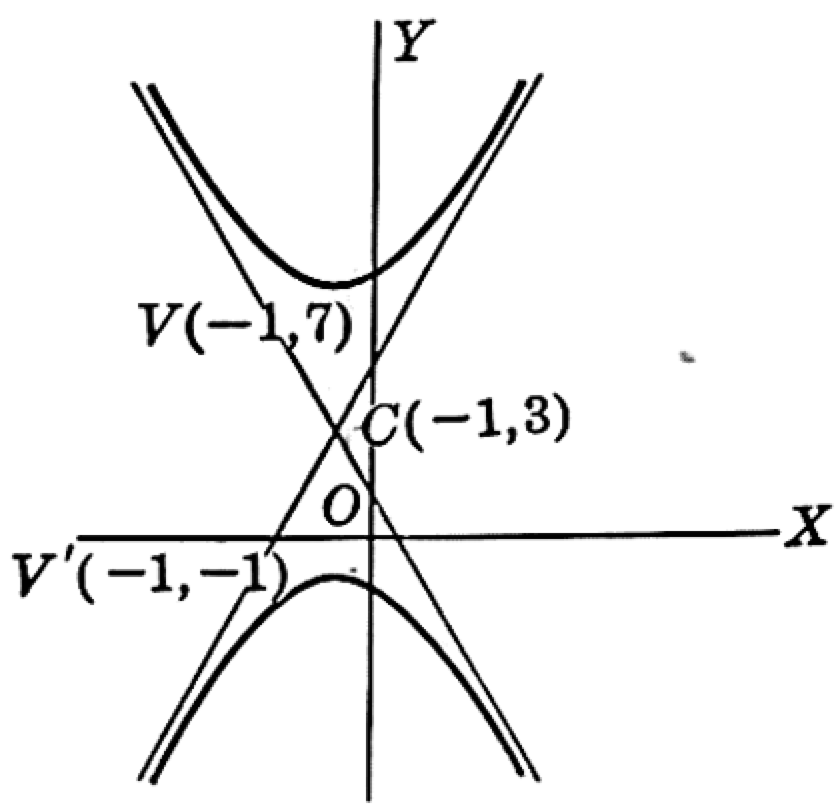


FIG. 125

The distance between the vertices is 8. Since this distance is $2a$, we have $a = 4$. Since the center lies midway between the vertices, its coördinates are, by Art. 150, equations (8), $(-1, 3)$.

Since the given asymptote passes through the center, its equation may be written in the form $2(x + 1) = (y - 3)$. But, in terms of a and b, the equation of the asymptotes are $a(x + 1) = \pm b(y - 3)$. Hence, $a/b = 2/1$, or, since $a = 4$, $b = 2$.

The required equation is, from equations (5), $(y - 3)^2 - 4(x + 1)^2 = 16.$

Exercises

Find the equations of the following conics.

1. Parabola, vertex $(-3, -1)$, focus $(2, -1)$.
2. Parabola, vertex $(5, 1)$, focus $(1, 1)$.
3. Parabola, vertex $(3, -7)$, directrix $y + 4 = 0$.

4. Parabola, focus (4, 2), directrix $y = 6$.
5. Ellipse, center (3, 1), vertex (3, − 2), $e = \frac{1}{3}$.
6. Ellipse, vertices (4, 2) and (− 2, 2), $b = 1$.
7. Ellipse, foci (5, 3) and (5, − 1), $e = \frac{4}{5}$.
8. Ellipse, center (2, 1), directrix $y + 7 = 0$, $c = 2$.
9. Hyperbola, foci (4, 7) and (4, − 3), $b = 3$.
10. Hyperbola, center (3, 1), directrix $x = 5$, $e = 2$.
11. Hyperbola, foci (− 3, 8) and (− 3, 2), ends of conjugate axis (− 5, 5) and (− 1, 5).
12. Hyperbola, vertices (1, − 1) and (− 5, − 1), asymptotes $2x + 3y + 7 = 0$ and $2x - 3y + 1 = 0$.

200. Simplification by Rotation. Removal of the *xy*-term. If, in the general equation of a conic,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \tag{7}$$

the coefficient $B \neq 0$, it is always possible, by rotating the axes through a suitable angle, to reduce the equation to one in which the coefficient of the $x'y'$-term is equal to zero.

Let the axes be rotated through an angle ϕ; that is, replace x and y, in equation (7), by

$$x = x' \cos \phi - y' \sin \phi, \qquad y = x' \sin \phi + y' \cos \phi. \tag{8}$$

After this substitution has been effected, equation (7) takes the form

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0, \tag{9}$$

wherein

$$\begin{aligned} A' &= A \cos^2 \phi + B \sin \phi \cos \phi + C \sin^2 \phi, \\ B' &= 2(C - A) \sin \phi \cos \phi + B(\cos^2 \phi - \sin^2 \phi), \\ C' &= A \sin^2 \phi - B \sin \phi \cos \phi + C \cos^2 \phi, \\ D' &= D \cos \phi + E \sin \phi, \\ E' &= E \cos \phi - D \sin \phi, \quad \text{and} \quad F' = F. \end{aligned}$$

The condition that B', the coefficient of the $x'y'$-term, vanishes is that the angle of rotation, ϕ, is chosen so that

$$B' = 2(C - A) \sin \phi \cos \phi + B(\cos^2 \phi - \sin^2 \phi) = 0. \tag{10}$$

By the formulas for the sine and cosine of twice an angle, we have

$$2 \sin \phi \cos \phi = \sin 2\phi, \qquad \cos^2 \phi - \sin^2 \phi = \cos 2\phi.$$

With the aid of these equations, we reduce equation (10) to

$$(C - A) \sin 2\phi + B \cos 2\phi = 0.$$

If $A \neq C$, we may reduce this equation further to

$$\tan 2\phi = \frac{B}{A - C}. \tag{11}$$

If $A = C$ and $B \neq 0$, it reduces to

$$\cos 2\phi = 0. \tag{11'}$$

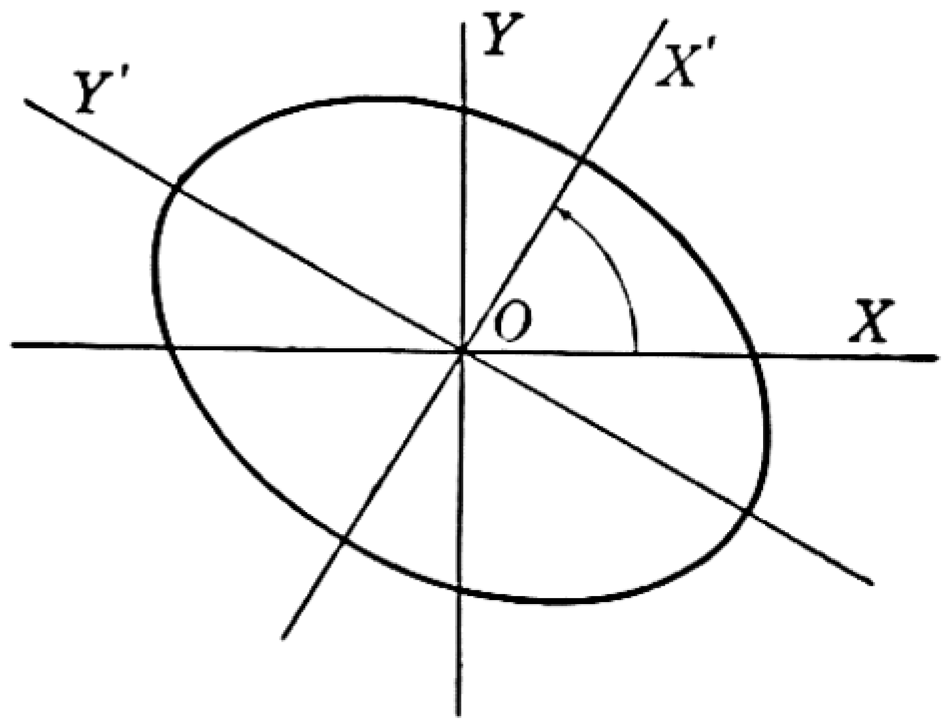


FIG. 126

EXAMPLE. Remove the xy-term from $5x^2 + 4xy + 8y^2 = 9$ by a rotation of axes.

By (11), we must rotate the axes through an angle ϕ such that

$$\tan 2\phi = \frac{4}{5-8} = -\frac{4}{3}.$$

By trigonometry, we have * $\cos 2\phi = -\frac{3}{5}$. Further, from the half-angle formulas, we have

$$\sin \phi = \sqrt{\frac{1-\cos 2\phi}{2}} = \sqrt{\frac{1+\frac{3}{5}}{2}} = \frac{2}{\sqrt{5}}$$

$$\cos \phi = \sqrt{\frac{1+\cos 2\phi}{2}} = \sqrt{\frac{1-\frac{3}{5}}{2}} = \frac{1}{\sqrt{5}}.$$

On substituting these values of $\sin \phi$ and $\cos \phi$ in (8), we obtain, as the formulas for the required rotation of axes,

$$x = \frac{x' - 2y'}{\sqrt{5}}, \quad y = \frac{2x' + y'}{\sqrt{5}}.$$

If, in the given equation, we replace x and y by these expressions, and simplify, we find, as the required equation,

$$9x'^2 + 4y'^2 = 9.$$

This is the equation of an ellipse having the y'-axis as principal axis and with semi-axes of lengths $\frac{3}{2}$ and 1.

Exercises

Remove the xy-term by a rotation of axes. Draw both sets of axes and the curve.

* In determining 2ϕ by means of equation (11) or (11'), we shall suppose, throughout, that $0 \leq 2\phi < 180°$. Under this assumption, $\cos 2\phi$ will always agree in sign with $\tan 2\phi$ and, since $0 \leq \phi < 90°$, $\sin \phi$ and $\cos \phi$ will always be positive or zero.

1. $3x^2 + 24xy - 4y^2 = 48.$
2. $3x^2 + 12xy - 2y^2 = 65.$
3. $19x^2 + 6xy + 11y^2 = 50.$
4. $4x^2 - 4xy + y^2 = 45.$
5. $3x^2 + 5xy - 9y^2 = 40.$
6. $7x^2 - 8xy + 7y^2 = 21.$
7. $5x^2 + 15xy - 3y^2 = 11.$
8. $50x^2 - 8xy + 35y^2 = 102.$
9. $16x^2 - 24xy + 9y^2 + 12x + 16y = 0.$

201. Reduction of Numerical Equations to the Standard Form. If the second degree terms in the given equation do not form a perfect square; that is, if $B^2 - 4AC \neq 0$, we shall first translate the origin to the point (h, k), then determine h and k, as in the following example 1, so that the coefficients of x' and y' are zero. We then rotate the axes so as to remove the $x'y'$-term.

If the second degree terms do form a perfect square; that is, if $B^2 - 4AC = 0$, it is usually not possible to make the coefficients of x' and y' both zero by a translation of axes. In this case, accordingly, we shall first rotate the axes and then complete the reduction to the standard form by a translation, as in the following example 2.

EXAMPLE 1. Simplify $6x^2 + 24xy - y^2 - 12x + 26y + 11 = 0$.

Since $B^2 - 4AC = 24^2 - 4 \cdot 6(-1) \neq 0$, we first translate the origin to the point (h, k) by putting $x = x' + h$, $y = y' + k$. We have, after collecting the coefficients,

$$6x'^2 + 24x'y' - y'^2 + (12h + 24k - 12)x' + (24h - 2k + 26)y' + 6h^2 + 24hk - k^2 - 12h + 26k + 11 = 0.$$

If we equate to zero the coefficients of x' and y' in this equation, we have

$$12h + 24k - 12 = 0, \qquad 24h - 2k + 26 = 0.$$

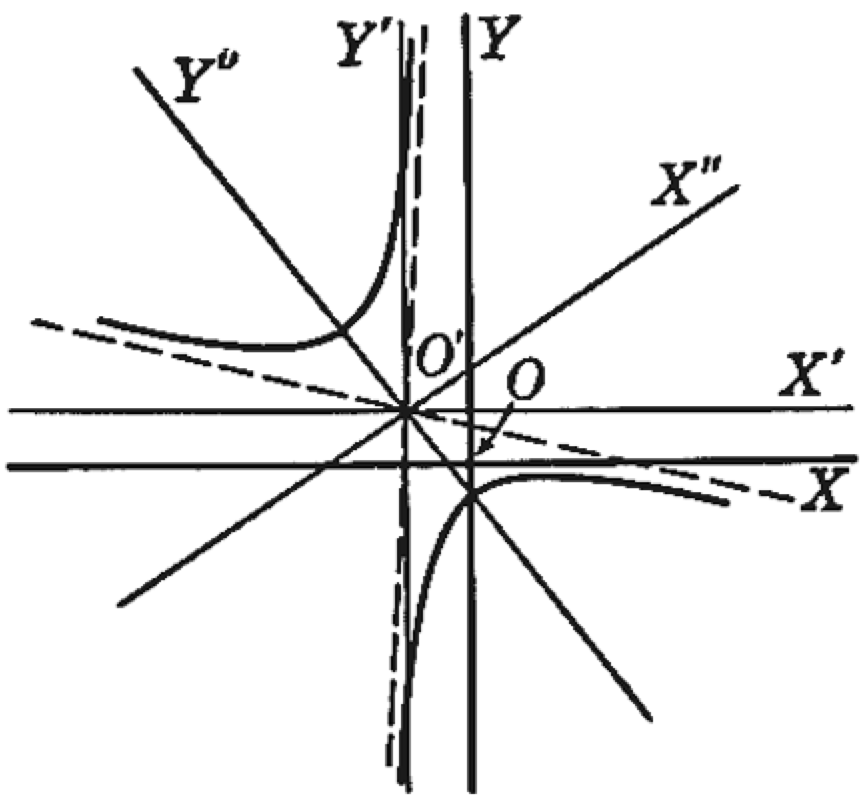


FIG. 127

By solving these equations as simultaneous, we find $h = -1$, $k = 1$. If we now substitute these values of h and k in the above equation, we obtain

$$6x'^2 + 24x'y' - y'^2 + 30 = 0.$$

To remove the xy-term, we rotate the axes through an angle ϕ such that $\tan 2\phi = \frac{24}{7}$. Then $\cos 2\phi = \frac{7}{25}$, $\sin \phi = \frac{3}{5}$, $\cos \phi = \frac{4}{5}$, and the equations of the rotation are

$$x' = \frac{4x'' - 3y''}{5}, \quad y' = \frac{3x'' + 4y''}{5}.$$

On making these substitutions, and simplifying, we obtain, as the required equation of the conic,

$$15x''^2 - 10y''^2 + 30 = 0, \text{ or } \frac{y''^2}{3} - \frac{x''^2}{2} = 1.$$

The curve is a hyperbola with center at the new origin and transverse axis on the y''-axis.

EXAMPLE 2. Simplify $144x^2 - 120xy + 25y^2 - 118x + 190y - 81 = 0$.

Since $B^2 - 4AC = 14400 - 14400 = 0$, we first remove the xy-term by a rotation of axes.

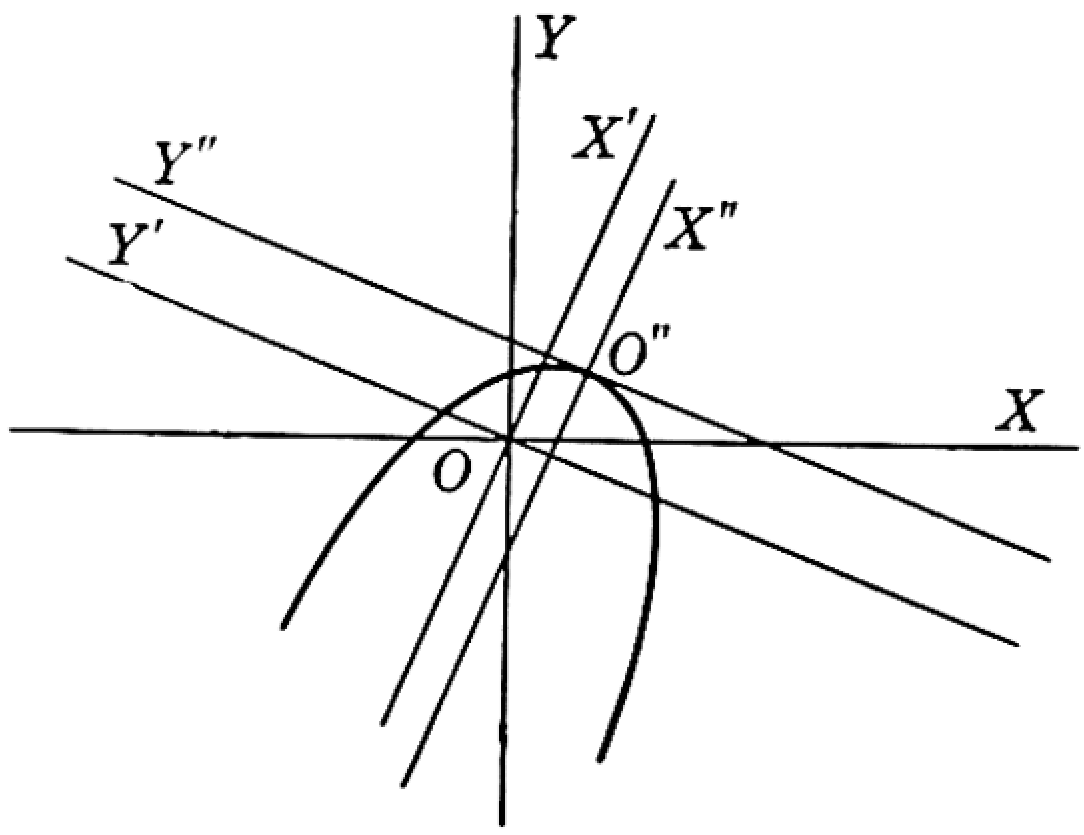


FIG. 128

To determine ϕ, we put $\tan 2\phi = -\frac{120}{119}$. Hence $\cos 2\phi = -\frac{119}{169}$, $\sin \phi = \frac{12}{13}$, $\cos \phi = \frac{5}{13}$, and the equations of the required rotation are

$$x = \frac{5x' - 12y'}{13}, \quad y = \frac{12x' + 5y'}{13}.$$

If we substitute these values of x and y in the given equation, and simplify, we obtain

$$169y'^2 + 130x' + 182y' - 81 = 0.$$

We may write this equation in the form

$$169(y' + \tfrac{7}{13})^2 + 130(x' - 1) = 0.$$

Hence, if we put $x' = x'' + 1$, $y' = y'' - \frac{7}{13}$, the above equation becomes

$$169y''^2 + 130x'' = 0, \quad \text{or} \quad 13y''^2 + 10x'' = 0.$$

The curve is a parabola with vertex at the new origin and principal axis coinciding with the x''-axis.

Exercises

Simplify the following equations, draw all the axes, and draw the curve if it exists.

1. $9x^2 + 4xy + 6y^2 + 10x - 20y + 5 = 0$.
2. $3x^2 - 8xy - 12y^2 - 30x - 64y = 0$.
3. $2x^2 + 12xy + 7y^2 + 36x + 42y + 43 = 0$.
4. $4x^2 + 4xy + y^2 - 18x + 26y + 64 = 0$.
5. $4x^2 - 3xy + 4y^2 - 30x + 25y + 63 = 0$.
6. $31x^2 - 6xy + 39y^2 + 136x - 168y + 184 = 0$.
7. $50x^2 - 40xy + 8y^2 + 86x - 17y + 11 = 0$.
8. $5x^2 - 4xy + 2y^2 + 2x - 8y + 13 = 0$.
9. $3x^2 + 10xy + 3y^2 + 46x + 50y + 143 = 0$.
10. $13x^2 - 32xy - 47y^2 + 12x + 252y - 315 = 0$.
11. $16x^2 - 24xy + 9y^2 - 38x - 34y - 15 = 0$.
12. $26x^2 + 12xy - 9y^2 + 48x + 18y + 10 = 0$.
13. $12xy - 20x + 9y^2 - 27 + 4x^2 - 30y = 0$.
14. $18x^2 + 20x + 3y^2 - 4y + 8xy + 14 = 0$.

15. Using the values of A', B', and C' in terms of $\sin \phi$ and $\cos \phi$ given in Art. 200, show that, if equation (7) is transformed into (9) by a rotation of

axes through any angle ϕ whatever, then $A' + C' = A + C$ and $B'^2 - 4A'C' = B^2 - 4AC$.

NOTE. Because of these equations, the quantities $A + C$ and $B^2 - 4AC$ are said to be **invariants** under rotation of axes.

16. Using the last equation of Ex. 15 with ϕ chosen so that $B' = 0$, show that, if the first member of equation (7) does not break up into two linear factors, then the curve is

$$\begin{aligned}
&\textit{an ellipse}, &&\textit{if } B^2 - 4AC < 0,\\
&\textit{a parabola}, &&\textit{if } B^2 - 4AC = 0,\\
&\textit{a hyperbola}, &&\textit{if } B^2 - 4AC > 0.
\end{aligned}$$

Find the equation of the conic through the five given points.

17. $(0, 0)$, $(5, 0)$, $(0, -3)$, $(1, 1)$, $(-2, 2)$.
18. $(5, 1)$, $(1, -2)$, $(-1, 1)$, $(1, 7)$, $(5, 4)$.
19. $(1, 1)$, $(2, 3)$, $(3, -1)$, $(-3, 2)$, $(-2, -1)$.
20. $(4, 1)$, $(2, 2)$, $(3, -2)$, $(4, -1)$, $(1, -3)$.

21. A line through the fixed point $P_1(x_1, y_1)$ intersects the coördinate axes at the points A and B. Find the equation of the locus of the midpoint of the segment AB as the line rotates around P_1.

22. The ends of the base of a triangle are $(0, 0)$ and $(a, 0)$. Find the equation of the locus of the vertex, given that the sum of the slopes of the sides is b.

23. The ends of the base of a triangle are $(-a, 0)$ and $(a, 0)$. The vertex moves along the line $y - b = 0$. Find the equation of the locus of the point of intersection of the altitudes.

24. Find the equation of the locus of a point, given that the square of its distance from the origin equals the sum of the squares of its distances from the lines $ax + by - ab = 0$ and $bx + ay - ab = 0$.

Chapter 26

Tangents and Normals; Differentiation and Integration

202. Definitions. Before we attempt to find the equation of the tangent line to a curve at a point on it, we must set up a working definition of a tangent line. The following definition is the one customarily employed in calculus, and throughout advanced mathematics.

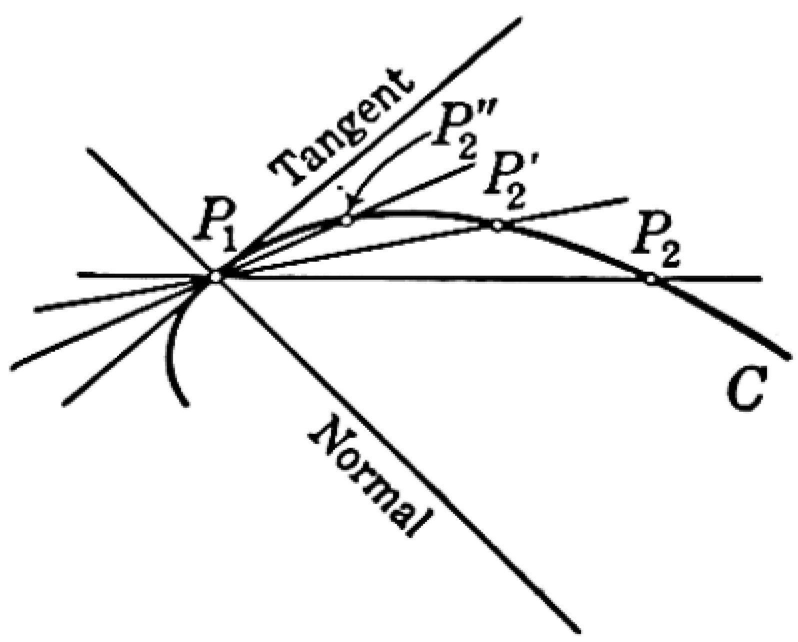


FIG. 129

Let P_1 be a given point on a given curve C. It is required to define the tangent line to C at P_1.

Let P_2 be another point on C and draw the secant line P_1P_2. If we now hold P_1 fixed and let P_2 move along C and approach P_1, the secant line P_1P_2 will turn around P_1. *The limiting position of the line P_1P_2, as P_2 approaches P_1 as a limit, along the curve, is the* **tangent line** *to C at P_1.*

The line through P_1 perpendicular to the tangent is the **normal line** to C at P_1.

In the following sections, we shall derive the equations of the tangent and normal lines to a number of curves according to the foregoing definitions. The principles we shall employ are those of differential calculus, and the discussion should be thought of as preparatory to that subject.

203. Tangent and Normal to a Parabola. Let $P_1(x_1, y_1)$ be a given point on the parabola $y^2 = 2px$. It is required to find the equations of the tangent and normal lines to this parabola at P_1.

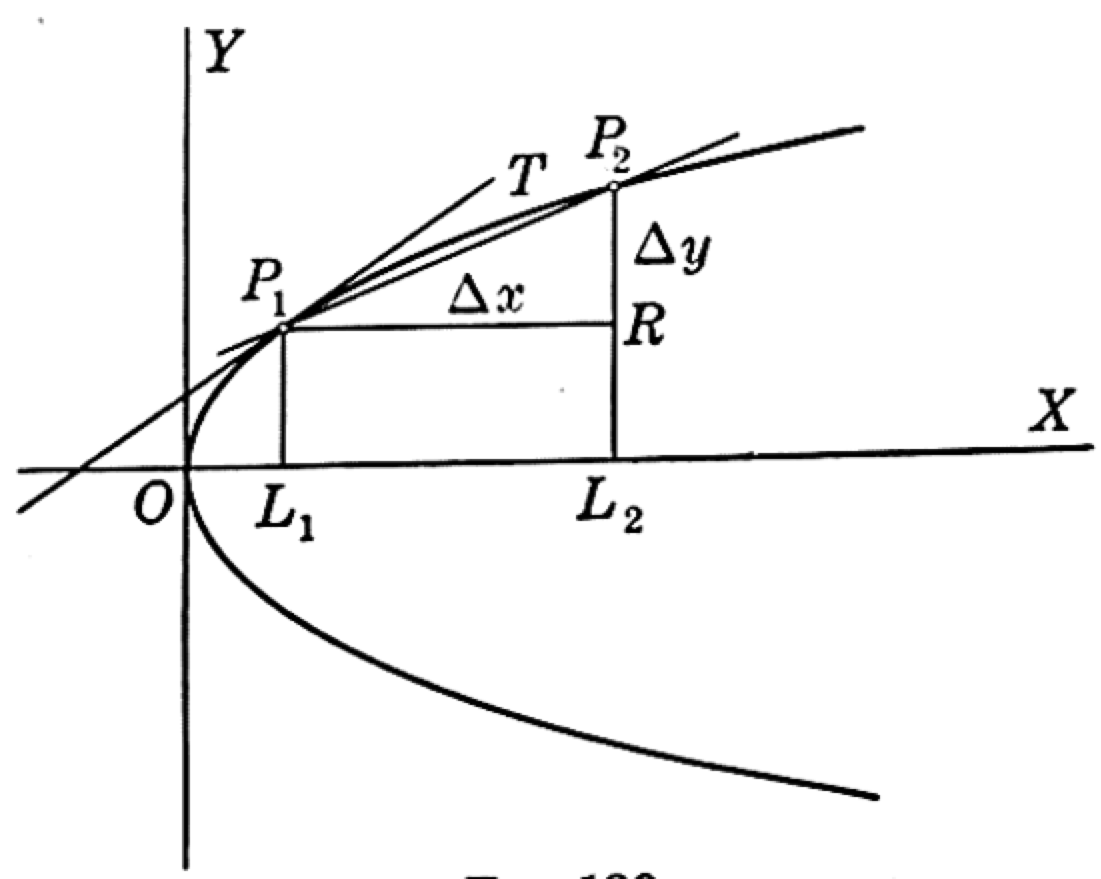


FIG. 130

Since P_1 lies on the required tangent line P_1T, we can find the equation of this line if we can find its slope m, for we can then substitute this value of m in the point-slope form $y - y_1 = m(x - x_1)$ of the equation of a line through $P_1(x_1, y_1)$.

To find the slope m of the tangent line, we must, by the definition of Art. 202, first find the slope of the secant line P_1P_2 and then find the limiting value of this slope as P_2 approaches P_1 along the curve.

Denote the coördinates of P_2 by $(x_1 + \Delta x, y_1 + \Delta y)$.* From the figure, the slope, m', of the secant line P_1P_2 is seen to be

$$m' = \frac{y_1 + \Delta y - y_1}{x_1 + \Delta x - x_1} = \frac{\Delta y}{\Delta x}. \tag{1}$$

The limit of this value of m' as P_2 approaches P_1 along the curve; that is, as Δy and Δx approach zero, is the required slope of the tangent line.†

Since $P_1(x_1, y_1)$ lies on the given parabola, we have

$$y_1^2 = 2px_1, \tag{2}$$

and, since $P(x_1 + \Delta x, y_1 + \Delta y)$ also lies on the parabola,

$$(y_1 + \Delta y)^2 = 2p(x_1 + \Delta x),$$

or

$$y_1^2 + 2y_1\Delta y + (\Delta y)^2 = 2px_1 + 2p\Delta x. \tag{3}$$

Subtracting equation (2) from (3) gives

$$2y_1\Delta y + (\Delta y)^2 = 2p\Delta x, \quad \text{or} \quad \Delta y(2y_1 + \Delta y) = 2p\Delta x.$$

By dividing this equation through by Δx, solving for $\Delta y/\Delta x$, and substituting in equation (1), we have

$$m' = \frac{\Delta y}{\Delta x} = \frac{2p}{2y_1 + \Delta y}. \tag{4}$$

Now let P_2 approach P_1; that is, let Δx and Δy approach zero. From (4), we see that the value of m' approaches p/y_1. But this limit which m' approaches is m, the slope of the tangent line at P_1. Hence, the slope of the tangent line to the parabola at P_1 is

$$m = \frac{p}{y_1}.$$

On substituting this value of the slope m in the point-slope form, $y - y_1 = m(x - x_1)$, of the equation of a line, we obtain, as the required equation of the tangent line to the parabola at P_1,

$$y - y_1 = \frac{p}{y_1}(x - x_1), \quad \text{or} \quad y_1y - y_1^2 = px - px_1. \tag{5}$$

This equation may be simplified. In the second of equations (5), replace y_1^2 by its value from (2) and simplify. We have

$$y_1y = px + px_1. \tag{6}$$

* The symbol Δx is read "delta x." It means simply the difference between the abscissas of P_1 and P_2; that is, it is the directed length P_1R in Figure 130. Similarly, Δy is read "delta y." It is the difference RP_2 between the ordinates of P_1 and P_2.

† The limiting value, as P_2 approaches P_1 along the curve, of the fraction $\frac{\Delta y}{\Delta x}$ is denoted by $\frac{dy}{dx}$ and is called "the derivative of y with respect to x." (See Art. 208.)

This is the equation of the tangent line to the parabola $y^2 = 2px$ at the point $P_1(x_1, y_1)$ on the curve.

The slope of the normal line at $P_1(x_1, y_1)$ is the negative reciprocal of the slope of the tangent at that point, or $-y_1/p$. Hence

$$y - y_1 = \frac{-y_1}{p}(x - x_1),$$

or

$$p(y - y_1) + y_1(x - x_1) = 0, \tag{7}$$

is the equation of the normal line to the parabola at $P_1(x_1, y_1)$.

204. Tangent and Normal to the Ellipse. To find the tangent and normal lines to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ at a point $P_1(x_1, y_1)$ on it, we follow the same line of reasoning that we used for the parabola.

Let $P_2(x_1 + \Delta x, y_1 + \Delta y)$ be any point other than P_1 on the ellipse Figure 131. Then we have

$$b^2(x_1 + \Delta x)^2 + a^2(y_1 + \Delta y)^2 = a^2b^2,$$

or

$$b^2x_1^2 + 2b^2x_1\Delta x + b^2(\Delta x)^2 + a^2y_1^2 + 2a^2y_1\Delta y + a^2(\Delta y)^2 = a^2b^2.$$

We have also, since $P_1(x_1, y_1)$ lies on the ellipse,

$$b^2x_1^2 + a^2y_1^2 = a^2b^2. \tag{8}$$

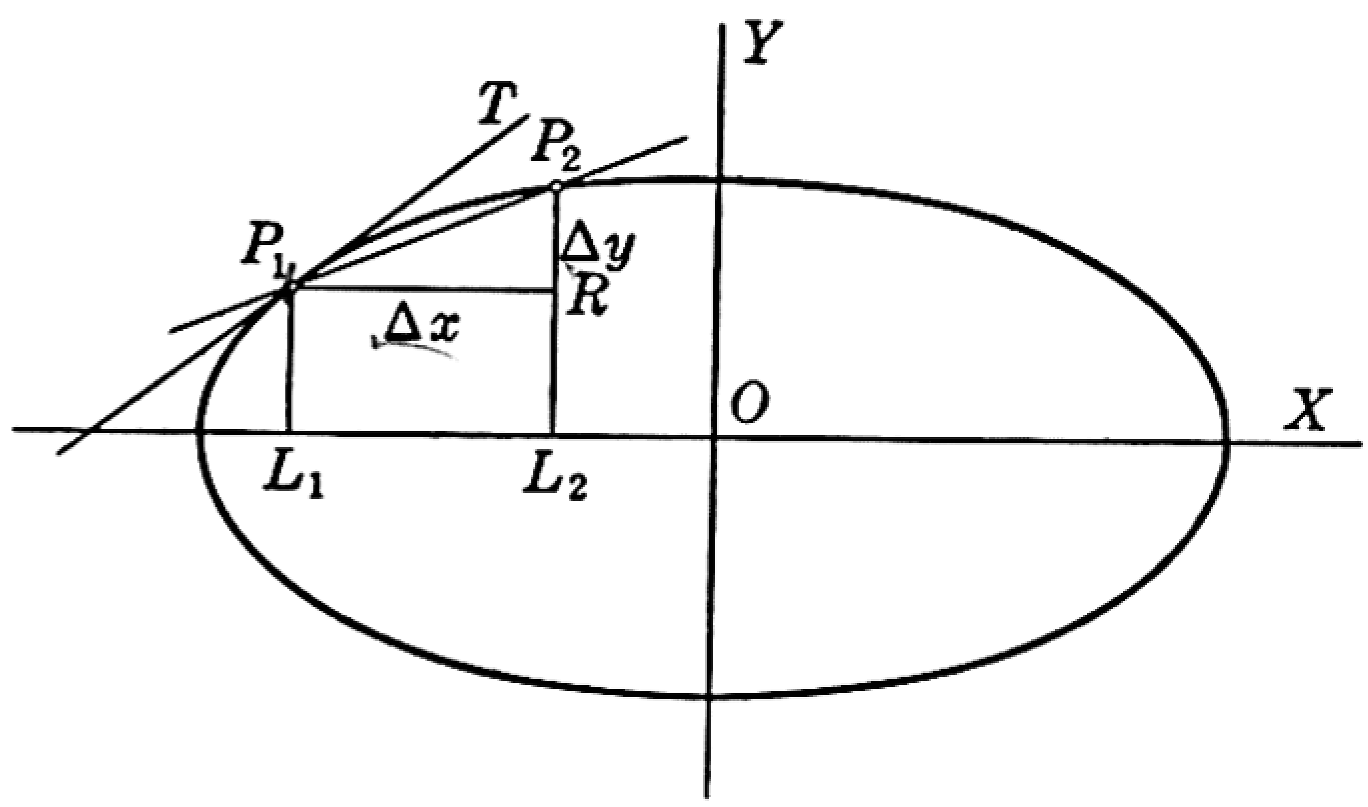


FIG. 131

By subtraction, we obtain from the last two equations,

$$2b^2x_1\Delta x + b^2(\Delta x)^2 + 2a^2y_1\Delta y + a^2(\Delta y)^2 = 0,$$

or

$$m' = \frac{\Delta y}{\Delta x} = -\frac{2b^2x_1 + b^2\Delta x}{2a^2y_1 + a^2\Delta y}.$$

As P_2 approaches P_1, Δx and Δy approach zero and m' approaches $-b^2x_1/a^2y_1$, which is the slope, m, of the tangent at $P_1(x_1, y_1)$. The equation of this tangent is, accordingly,

$$y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1).$$

To simplify this equation, we multiply by a^2y_1, and rearrange, giving

$$b^2x_1x + a^2y_1y = b^2x_1^2 + a^2y_1^2.$$

On substituting for the second member its value from (8), we have

$$b^2x_1x + a^2y_1y = a^2b^2, \tag{9}$$

which is the equation of the tangent line to the ellipse at $P_1(x_1, y_1)$.

Since the normal line is perpendicular to the tangent, its slope is a^2y_1/b^2x_1, and its equation reduces to

$$b^2x_1(y - y_1) = a^2y_1(x - x_1). \tag{10}$$

This is the equation of the normal line to the ellipse at $P_1(x_1, y_1)$.

205. Tangent and Normal to the Hyperbola. If the point $P_1(x_1, y_1)$ lies on the hyperbola

$$b^2x^2 - a^2y^2 = a^2b^2, \tag{11}$$

we find that

$$b^2x_1x - a^2y_1y = a^2b^2 \tag{12}$$

is the equation of the tangent line and

$$b^2x_1(y - y_1) + a^2y_1(x - x_1) = 0 \tag{13}$$

is the equation of the normal line to the hyperbola at P_1.

The derivation of these equations, which differs but little from that given in the preceding article for the ellipse, is left as an exercise for the student.

206. Tangents and Normals to Other Curves. The method we have just been using to find the equations of the tangent and normal lines to a conic at a point on it can be applied equally well to curves whose equations are not of second degree.

EXAMPLE. Find the equations of the tangent and normal lines to the semi-cubical parabola $ay^2 = x^3$ at a point $P_1(x_1, y_1)$ on the curve.

Let $P_2(x_1 + \Delta x, y_1 + \Delta y)$ be a second point on the curve. Then, since P_1 and P_2 both lie on the curve,

$$ay_1^2 = x_1^3, \quad \text{and} \quad a(y_1 + \Delta y)^2 = (x_1 + \Delta x)^3.$$

By subtracting the first equation from the second and solving for $\Delta y/\Delta x$, we get

$$m' = \frac{\Delta y}{\Delta x} = \frac{3x_1^2 + 3x_1\Delta x + (\Delta x)^2}{2ay_1 + a\Delta y}.$$

As P_2 approaches P_1, Δy and Δx approach zero and we find, as the slope of the tangent line at P_1, $m = 3x_1^2/2ay_1$. It follows that

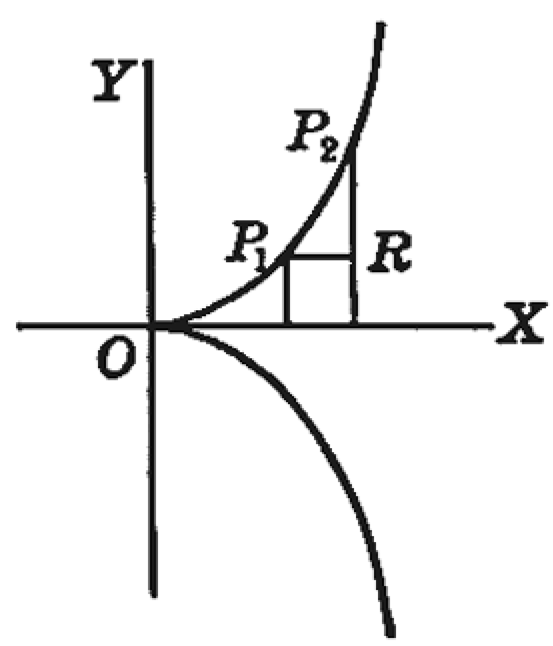


FIG. 132

$$2ay_1(y - y_1) = 3x_1^2(x - x_1),$$

and

$$3x_1^2(y - y_1) + 2ay_1(x - x_1) = 0,$$

are, respectively, the equations of the tangent and the normal lines to the curve at the point $P_1(x_1, y_1)$.

Exercises

Find the equations of the tangent and normal lines to the given curve at the point indicated.

1. $16x^2 + 25y^2 = 800$, $(5, -4)$.
2. $y^2 = 3x$, $(12, 6)$.
3. $12x^2 - 5y^2 = 3$, $(2, -3)$.
4. $x^2 + y^2 - 6x + 4y - 12 = 0$, $(6, 2)$.
5. $x^2 + 6y - 7 = 0$, $(5, -3)$.
6. $xy = 18$, $(-3, -6)$.
7. $3y = 2x^2 + 5x - 9$, $(-3, -2)$.
8. $2x = 4y^2 - 3y + 2$, $(6, 2)$.
9. $y = x^3$, $(2, 8)$.
10. $5y = 2x^3 - 3x^2 - 7$, $(3, 4)$.
11. $y^2 = 6x^3 - 6x$, $(2, 6)$.
12. $x^3 + y^3 = 19$, $(3, -2)$.
13. $x^3 + y^3 = 9xy$, $(2, 4)$.
14. $x^2y + 4y = 40$, $(4, 2)$.
15. $(y - k)^2 = 2p(x - h)$, (x_1, y_1).
16. $b^2(x - h)^2 + a^2(y - k)^2 = a^2b^2$, (x_1, y_1).
17. $b^2(x - h)^2 - a^2(y - k)^2 = a^2b^2$, (x_1, y_1).
18. $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, (x_1, y_1).

19. Show that the inclination of the tangent at P_1 to the parabola $y^2 = 2px$ is one-half the inclination of the line through the focus and the point P_1.

NOTE. It follows from this theorem that, if the surface of a mirror is formed by revolving the parabola around its principal axis, and if a source of light is placed at the focus, then the rays striking the mirror will be reflected parallel to the principal axis.

20. A tangent to an ellipse or hyperbola bisects one pair of vertical angles formed by the lines through the foci and the point of tangency.

HINT. Let F' and F be the foci and P_1 the point of tangency. By geometry, in the triangle $F'P_1F$, the bisectors of the internal (or external) angles at P_1 divide the side opposite internally (or externally) into segments proportional to the adjacent sides. The sides $F'P_1$ and FP_1 are the focal radii of P_1 (Art. 186, Ex. 14 and Art. 192, Ex. 15).

207. Tangents Having a Given Slope. Let there be given, not the point of tangency, but the slope, m, of a tangent line to a conic. It is required to find the equations of the lines tangent to the conic having the given slope m.

(a) *The parabola.* To find the line of slope m that is tangent to the parabola

$$y^2 = 2px,$$

we consider, first, a secant line

$$y = mx + k \tag{14}$$

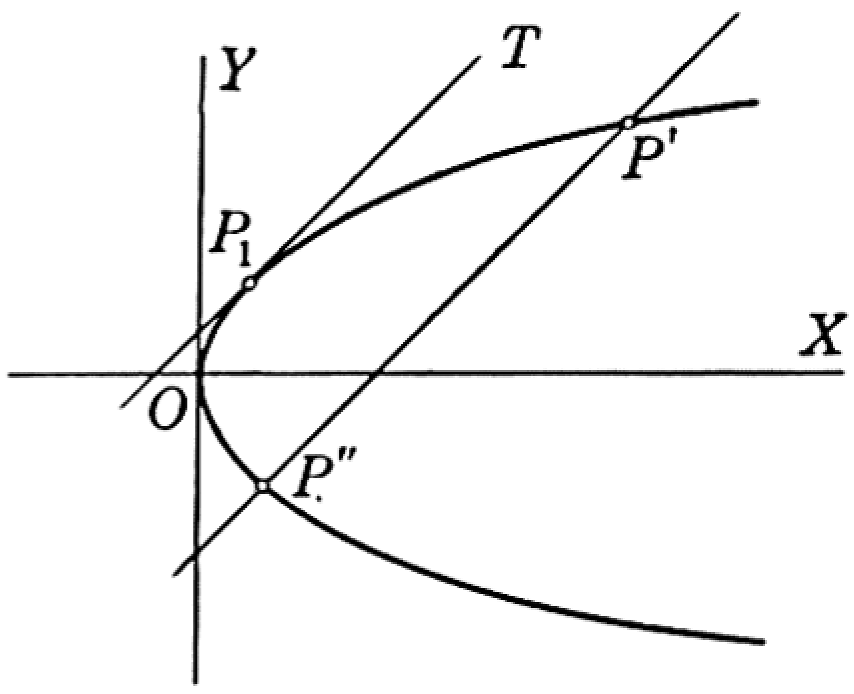


FIG. 133

that has the given slope m and that meets the parabola in two points P' and P'' (Fig. 133). If, by holding m constant and varying k, we move this line parallel to its original position until it becomes tangent to the parabola, its intersections P' and P'' will move into coincidence at the point of tangency P_1. To make the line (14) tangent to the parabola,

we must thus impose on k the condition that the two intersections of this line with the parabola coincide.

Substitute the value of y from (14) in the equation of the parabola. The roots of the resulting equation

$$(mx + k)^2 = 2px,$$

or

$$m^2x^2 + 2(mk - p)x + k^2 = 0, \tag{15}$$

are the abscissas of the points of intersection of the line (14) with the parabola. (Why?)

The condition that the two intersections coincide is, consequently, that the two roots of equation (15) are equal. This condition is (Art. 61)

$$4(mk - p)^2 - 4m^2k^2 = 0, \quad \text{or} \quad k = p/2m.$$

On substituting this value of k in equation (14), we find that

$$y = mx + \frac{p}{2m} \tag{16}$$

is the equation of the line of slope m tangent to the parabola $y^2 = 2px$.

(*b*) *The ellipse.* To find the lines of slope m that are tangent to the ellipse

$$b^2x^2 + a^2y^2 = a^2b^2,$$

we proceed as we did for the parabola.

The abscissas of the two intersections of the line $y = mx + k$ with the given ellipse are the roots of the equation

$$b^2x^2 + a^2(mx + k)^2 = a^2b^2,$$

or

$$(b^2 + a^2m^2)x^2 + 2a^2mkx + a^2(k^2 - b^2) = 0. \tag{17}$$

If we impose the condition that the two roots of equation (17) are equal, we obtain

$$4a^4m^2k^2 - 4a^2(b^2 + a^2m^2)(k^2 - b^2) = 0,$$

so that

$$k = \pm \sqrt{a^2m^2 + b^2}.$$

If we substitute these values of k in the equation $y = mx + k$ of the line, we have

$$y = mx \pm \sqrt{a^2m^2 + b^2} \tag{18}$$

as the equations of the lines of slope m that are tangent to the given ellipse. There are two such lines, as is shown in Figure 134.

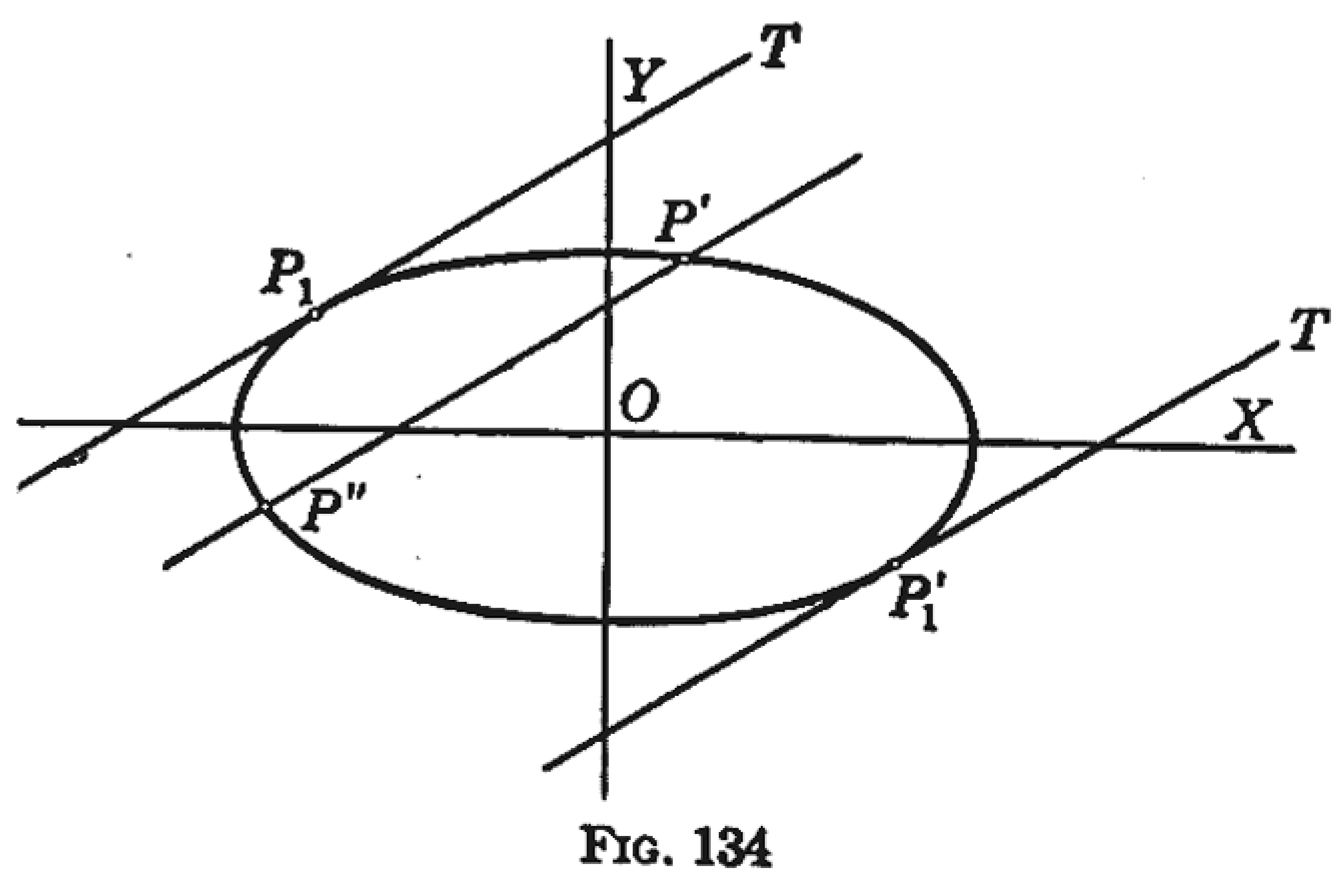


FIG. 134

(c) *The hyperbola.* For the hyperbola

$$b^2x^2 - a^2y^2 = a^2b^2,$$

we find in the same way that the equations of the tangent lines of slope m are

$$y = mx \pm \sqrt{a^2m^2 - b^2}. \tag{19}$$

The proof is left as an exercise for the student.

Exercises

1. Find the line of slope 3 that is tangent to the parabola $y^2 = 24x$.

2. Find two lines tangent to the hyperbola $x^2 - 5y^2 = 20$ that are (a) parallel and (b) perpendicular to the line $y = 2x + 8$.

3. Find two lines parallel to the line through $(4, -1)$ and $(-2, 1)$ that are tangent to the ellipse $2x^2 + 7y^2 = 14$.

4. Find the lines through the point $(1, 3)$ that are tangent to the ellipse $4x^2 + 9y^2 = 36$.

5. Find the normal line of slope m to the parabola $y^2 = 2px$.

HINT. Find the point of tangency of the tangent line of slope $-1/m$. Through this point, find the line of slope m.

6. Show that the product of the distances to the foci from a tangent line to an ellipse or to a hyperbola is a constant.

7. Show that the locus of the point of intersection of two mutually perpendicular tangents to a parabola is the directrix.

8. Find the locus of the point of intersection of two mutually perpendicular tangents to (a) an ellipse, (b) a hyperbola.

In the following exercises find the lines of slope m tangent to the given curve.

9. The parabola $x^2 = 2py$.

10. The rectangular hyperbola $2xy = a^2$.

11. The parabola $(y - k)^2 = 2p(x - h)$.

12. The ellipse $b^2(x - h)^2 + a^2(y - k)^2 = a^2b^2$.

13. The hyperbola $b^2(x - h)^2 - a^2(y - k)^2 = a^2b^2$.

208. The Derivative. We saw in Arts. 202–206 that, if (x, y) and $(x + \Delta x, y + \Delta y)$ are two points both of which lie on a given curve, then the limit of $\dfrac{\Delta y}{\Delta x}$ as Δx approaches zero is the slope of the tangent to the curve at the point (x, y). This limit we denote by the symbol $\dfrac{dy}{dx}$, that is, we put

$$\operatorname*{limit}_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}. \tag{20}$$

This limit is called *the derivative of y with respect to x.* It is also frequently referred to, especially in the applications of mathematics, as, "the rate of change of y with respect to x." Because this limit occurs frequently in many of the applications of mathematics, we shall now take up a discussion of some of its properties.

In physics, for example, the rate of change of the position of a moving body with respect to the time is called its velocity, as is illustrated by the following example.

EXAMPLE. The distance in feet that a body falls from rest in t seconds under the attraction of gravity is given by the formula $s = \frac{1}{2}gt^2$. Find its velocity at the end of t seconds.

Let s be the distance it has fallen at the end of t seconds and $s + \Delta s$ the distance it has fallen at the end of $t + \Delta t$ seconds. Then

$$s = \tfrac{1}{2}gt^2 \quad \text{and} \quad s + \Delta s = \tfrac{1}{2}g(t + \Delta t)^2,$$

so that

$$\Delta s = \tfrac{1}{2}g(t + \Delta t)^2 - \tfrac{1}{2}gt^2 = gt\Delta t + \tfrac{1}{2}g(\Delta t)^2. \tag{21}$$

If we divide the distance Δs by the interval of time Δt during which it falls through this distance, we obtain an *average* velocity for the motion of the body across this interval. From (21), we have for this average velocity

$$\frac{\Delta s}{\Delta t} = \frac{gt\Delta t + \frac{1}{2}g(\Delta t)^2}{\Delta t} = gt + \frac{1}{2}g\Delta t.$$

To find the velocity at the time t, we take the limit of this average velocity as Δt approaches zero. We have

$$v = \frac{ds}{dt} = \underset{\Delta t \to 0}{\text{limit}} \; (gt + \tfrac{1}{2}g\Delta t) = gt.$$

The required velocity is thus the derivative with respect to t of the distance the body has fallen at the end of t seconds.

209. Formulas. To save the labor of computing the limit every time we need to find the derivative of a function, we shall now set up some formulas which will enable us, in many cases, to write down at once the expression for the derivative without going through the steps of the limiting process.

Let $u = f(x)$ and $v = g(x)$ be two given functions and let c and n be constants. Then

$$\frac{dc}{dx} = 0. \tag{I}$$

$$\frac{d(u + v)}{dx} = \frac{du}{dx} + \frac{dv}{dx} \tag{II}$$

$$\frac{d(u \cdot v)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \tag{III}$$

$$\frac{d(c \cdot v)}{dx} = c\frac{dv}{dx} \tag{IV}$$

Single Value
be continuous for $x \neq 0$

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \qquad \text{V}$$

$$\frac{d\left(\frac{c}{v}\right)}{dx} = \frac{-c\frac{dv}{dx}}{v^2} \qquad \text{VI}$$

$$\frac{d(x^n)}{dx} = nx^{n-1} \qquad \text{VII}$$

To prove formula I, let $y = c$, then $y + \Delta y = c$, so that $\Delta y = 0$. Then

$$\frac{\Delta y}{\Delta x} = \frac{0}{\Delta x} = 0.$$

Since $\frac{\Delta y}{\Delta x}$ is constantly zero, its limit is also zero.

To prove II, let $y = u + v = f(x) + g(x)$. Then

$$y + \Delta y = f(x + \Delta x) + g(x + \Delta x) = u + \Delta u + v + \Delta v,$$

where $\quad u + \Delta u = f(x + \Delta x) \quad$ and $\quad v + \Delta v = g(x + \Delta x).$

Then $\quad \Delta y = u + \Delta u + v + \Delta v - (u + v) = \Delta u + \Delta v.$

On dividing by Δx, we have

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}.$$

By taking the limit of each of these expressions as Δx approaches zero, we find that $\frac{dy}{dx} = \frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$, which is formula II.

To prove III, we put $y = u \cdot v = f(x) \cdot g(x)$. Then

$$\begin{aligned}\Delta y &= f(x + \Delta x) \cdot g(x + \Delta x) - f(x) \cdot g(x) = (u + \Delta u)(v + \Delta v) - uv \\ &= u\Delta v + v\Delta u + \Delta u\Delta v.\end{aligned}$$

Hence, $$\frac{\Delta y}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}.$$

If we now let Δx, and hence Δy, Δu, and Δv, approach zero, we have formula III. To prove formula IV, we put $u = c$ in formula III. Then, since $\frac{du}{dx} = \frac{dc}{dx} = 0$, formula III now reduces to formula IV.

To prove V, put $y = \frac{u}{v} = \frac{f(x)}{g(x)}$. Then

$$\Delta y = \frac{f(x + \Delta x)}{g(x + \Delta x)} - \frac{f(x)}{g(x)} = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{v\Delta u - u\Delta v}{v^2 + v\Delta v}.$$

Divide by Δx.

$$\frac{\Delta y}{\Delta x} = \frac{v\frac{\Delta u}{\Delta x} - u\frac{\Delta v}{\Delta x}}{v^2 + v\Delta v}.$$

If we now let Δx, Δy, Δu, and Δv approach zero, we have formula V. Formula VI follows at once by putting $u = c$, $\frac{du}{dx} = 0$ in formula V.

We shall prove formula VII for the case in which n is a positive integer. The proof that the formula is true when n is not a positive integer will be found in the textbooks devoted to calculus.

Let $y = x^n$, where n is a positive integer. With the aid of the binomial formula (Art. 237) which we shall derive in Chapter 30, we find that

$$y + \Delta y = (x + \Delta x)^n = x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{1 \cdot 2}x^{n-2}(\Delta x)^2 \cdots + (\Delta x)^n,$$

where the dots indicate a sequence of terms each of which contains Δx to the third or a higher power. Then

$$\frac{\Delta y}{\Delta x} = nx^{n-1} + \frac{n(n-1)}{1 \cdot 2}x^{n-2}\Delta x \cdots + (\Delta x)^{n-1}.$$

If we now let Δx approach zero, we obtain in the limit,

$$\frac{dy}{dx} = nx^{n-1}.$$

Example 1. Find $\frac{ay}{dx}$, given $y = 3x^4 - 5x^2 + \frac{2}{x^3}$.

We have

$$\frac{dy}{dx} = \frac{d}{dx}\left(3x^4 - 5x^2 + \frac{2}{x^3}\right) = \frac{d}{dx}(3x^4) + \frac{d}{dx}(-5x^2) + \frac{d}{dx}\left(\frac{2}{x^3}\right)$$

$$= 3\frac{dx^4}{dx} - 5\frac{dx^2}{dx} + 2\frac{d}{dx}\left(\frac{1}{x^3}\right) = 12x^3 - 10x - \frac{6}{x^4}.$$

Example 2. Differentiate: $y = \frac{3x^2 - x - 9}{2x + 1}$.

Use formula V.

$$\frac{dy}{dx} = \frac{(2x+1)\frac{d}{dx}(3x^2 - x - 9) - (3x^2 - x - 9)\frac{d}{dx}(2x+1)}{(2x+1)^2}$$

$$= \frac{(2x+1)(6x-1) - (3x^2 - x - 9)2}{(2x+1)^2}$$

$$= \frac{6x^2 + 6x + 17}{(2x+1)^2}.$$

In problems involving differentiation, it is usually best to replace radicals, if they appear, by fractional exponents. Formula VII may then be applied.

Example 3. Differentiate: $y = 6\sqrt[3]{x^7} + 5\sqrt{x^3} - 2\sqrt[4]{x^5}$.

$$\frac{dy}{dx} = \frac{d}{dx}(6x^{\frac{7}{3}} + 5x^{\frac{3}{2}} - 2x^{\frac{5}{4}}) = \frac{d}{dx}(6x^{\frac{7}{3}}) + \frac{d}{dx}(5x^{\frac{3}{2}}) - \frac{d}{dx}(2x^{\frac{5}{4}})$$

$$= 6\frac{dx^{\frac{7}{3}}}{dx} + 5\frac{dx^{\frac{3}{2}}}{dx} - 2\frac{dx^{\frac{5}{4}}}{dx} = 14x^{\frac{4}{3}} + \frac{15}{2}x^{\frac{1}{2}} - \frac{5}{2}x^{\frac{1}{4}}$$

$$= 14\sqrt[3]{x^4} + \tfrac{15}{2}\sqrt{x} - \tfrac{5}{2}\sqrt[4]{x}.$$

Exercises

Find the derivative of y with respect to x.

1. $y = 3x^2 - 7x + 4$.

2. $y = 8 + 5x - 7x^2$.

3. $y = 7x^3 + 2x^2 - 9$.

4. $y = x^8 - 3x^5 - 2x$.

5. $y = 8x^2 - \dfrac{2}{x} + \dfrac{3}{x^2}\cdot$

6. $y = 7 + \dfrac{3}{x^2} - \dfrac{5}{x^3}\cdot$

7. $y = x^2(2x^3 + 1)$.

8. $y = (3x + 1)(2x - 5)$.

9. $y = \dfrac{2x - 3}{6x + 5}\cdot$

10. $y = \dfrac{x}{x^2 - 1}\cdot$

11. $y = \dfrac{x^2 + 2}{2x - 3}\cdot$

12. $y = \dfrac{4x^3 + 7x^2}{8x^2 + 5}\cdot$

13. $y = 4x^{\frac{1}{2}} - 10x^{\frac{5}{3}}$.

14. $y = 5\sqrt{x} + 3\sqrt[3]{x^5}$.

15. $y = \dfrac{\sqrt{x}}{\sqrt{x} + 1}\cdot$

16. $y = \dfrac{1 + \sqrt[3]{x}}{\sqrt{x} - 5}\cdot$

17. $y = a_0x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5$.

18. A projectile is fired vertically upward. Its height y, in feet, is given by the formula $y = 576t - 16t^2$. Find its velocity (the derivative of y with respect to t) as a function of the time. What is its velocity at the end of 10 seconds?

19. The projectile in Ex. 18 has reached its greatest height when its velocity is zero. How high does it go?

20. Find the rate of change of the area of a square with respect to its side.

21. The radius of a vertical cylindrical water tank is 6 feet. Water is running in at the rate of 8 cubic feet per second. How fast is the depth of the water in the tank increasing?

HINT. Express V, the volume of water in the tank, in terms of h, the depth. The rate at which water is flowing into the tank is $\dfrac{dV}{dt} = 8$. Find $\dfrac{dh}{dt}\cdot$

22. The top of a ladder 26 feet long rests against a vertical wall. If the bottom of the ladder is being pulled horizontally directly away from the wall at the rate of 4 feet per second, how fast is the top descending when the bottom is 10 feet from the wall?

23. An electric light is suspended 24 feet directly above a horizontal walk. A man 6 feet tall is going along the walk at the rate of 3 feet per second. How fast is the end of his shadow moving?

24. A stone dropped into a pool of water sends out a series of concentric ripples. If the radius of the outer ripple is increasing at the rate of 2 feet per second, how fast is the area of the outer ripple increasing when its radius is 3 feet?

210. Derivative of a Function of a Function. Let $y = f(u)$ and $u = g(x)$. It is required to find the derivative of y with respect to x. We have

$$u + \Delta u = g(x + \Delta x) \quad \text{and} \quad y + \Delta y = f(u + \Delta u).$$

If $\Delta u \neq 0$, we have
$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}.$$

If Δx, Δu, and Δy all approach zero, we now have, in the limit,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}. \tag{22}$$

EXAMPLE. Find $\frac{dy}{dx}$, given $y = \sqrt{x^2 - 6x + 11}$.

Let $u = x^2 - 6x + 11$, then $y = \sqrt{u} = u^{\frac{1}{2}}$.

From formulas (22), VII, II, and I, we have

$$\frac{dy}{dx} = \frac{du^{\frac{1}{2}}}{du} \cdot \frac{d}{dx}(x^2 - 6x + 11) = \frac{1}{2\sqrt{u}}(2x - 6) = \frac{x - 3}{\sqrt{x^2 - 6x + 11}}.$$

Exercises

Find $\frac{dy}{dx}$.

1. $y = \sqrt{2x^3 - 5x^2 - 1}$.
2. $y = \sqrt{\frac{2x - 1}{3x + 4}}$.
3. $y = \sqrt{x^2 + 1} - \frac{1}{\sqrt{x^2 + 1}} + 3$.
4. $y = \sqrt{5x + 7} - \sqrt[3]{3 - x - 2x^2}$.
5. $y = \sqrt[4]{a^4 + x^4}$.
6. $y = \sqrt[3]{x^3 - 6\sqrt{x}}$.

211. Derivatives of Transcendental Functions. It is proved in the textbooks on calculus that, *provided the angle x is measured in radians*,

$$\frac{d}{dx}\sin x = \cos x \quad \text{and} \quad \frac{d}{dx}\cos x = -\sin x; \tag{23}$$

and also that, if $e = 2.71828^+$ is the base of the natural system of logarithms (Art. 86), then

$$\frac{d}{dx}e^x = e^x \quad \text{and} \quad \frac{d}{dx}\log_e x = \frac{1}{x}. \tag{24}$$

Exercises

Find $\frac{dy}{dx}$, using equations (23) and (24).

1. $y = \tan x = \frac{\sin x}{\cos x}$.
2. $y = \sec x = \frac{1}{\cos x}$.
3. $y = \cot x$.
4. $y = \csc x$.
5. $y = \sin\left(\frac{\pi}{2} - x\right)$.
6. $y = \frac{\sin 2x}{1 + \cos 2x}$.

7. $y = e^{kx}$.
8. $y = a^x = e^{x \log_e a}$.
9. $y = e^{-x^2}$.
10. $y = \log_e \sec x$.
11. $y = \log_e (\sec x + \tan x)$.
12. $y = e^{\sin x}$.
13. $y = \sin x^2$.
14. $y = \log_e \sqrt{x^2 + 2x + 5}$.
15. $y = \cos e^x$.

212. Successive Differentiation. Let $y = f(x)$. Then the derivative of y with respect to x is also a function of x which we shall denote by $f'(x)$, so that

$$\frac{dy}{dx} = f'(x).$$

We shall speak of $\frac{dy}{dx}$ [or $f'(x)$] as the *first derivative* of y with respect to x.

The derivative of the first derivative is called the *second derivative* of y with respect to x. We shall denote $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ by $\frac{d^2y}{dx^2}$ and $\frac{d}{dx}f'(x)$ by $f''(x)$, so that

$$\frac{d^2y}{dx^2} = f''(x).$$

If we differentiate the second derivative, we obtain, in a similar way, the *third derivative*, and so on.

Exercises

Find the first, second, and third derivatives of y with respect to x.

1. $y = 5x^2 - 2x - 3$.
2. $y = x^3 + 4x^2 - 2x - 7$.
3. $y = 2x^2 + \frac{3}{x}$.
4. $y = x^{\frac{1}{2}} - 2x^{\frac{1}{3}}$.
5. $y = \frac{2x + 5}{x + 1}$.
6. $y = \sqrt{x^2 + x + 1}$.

213. Maxima and Minima. We saw in Arts. 202–206 that, if (x_1, y_1) is a point on a curve, then the value of $\frac{dy}{dx}$ at the point (x_1, y_1) is the slope of the tangent to the curve at that point. It follows that, if the value of the derivative at the point (x_1, y_1) is positive, then the slope of the tangent at that point is positive and y is increasing as x increases; similarly, if the derivative is negative, y is decreasing as x increases. Finally, if the derivative is zero at (x_1, y_1), the tangent at that point is horizontal. We wish now to consider in more detail this last case in which the derivative is zero at (x_1, y_1).

Suppose first that, for points on the curve for which x is slightly less than x_1, the derivative is positive and that, for points for which x is a little larger than x_1, the derivative is negative. Then, as the value of x increases to x_1, the corresponding value of y increases to y_1; as x increases beyond x_1, the value of y decreases. The origin, in Figure 135, illustrates this case. Such a point is called a **maximum** point on the curve.

If, now, the derivative is negative if x is slightly less than x_1 and positive if x is slightly greater, then, as x increases through the value x_1, the value of y first decreases to y_1, then increases again [as at the point $(4, -32)$ in Figure 135]. Such a point is a **minimum** point on the curve.

It should be observed that, in the neighborhood of a maximum point, the first derivative is decreasing from positive to negative. Hence, its derivative, that is, the second derivative of the given function, is *negative* (or zero) at a *maximum* point. Similarly, in the neighborhood of a minimum point, the first derivative is increasing from negative to positive. Hence the second derivative is *positive* (or zero) at a *minimum* point.

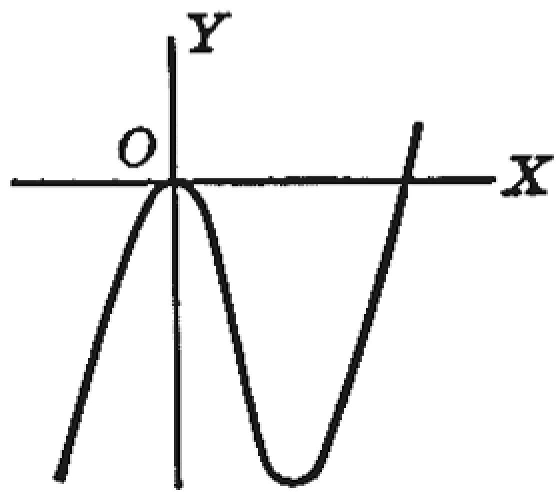


FIG. 135

EXAMPLE. Given $y = x^3 - 6x^2$, find the maximum and minimum points.

We have

$$y = x^3 - 6x^2, \quad \frac{dy}{dx} = 3x^2 - 12x, \quad \frac{d^2y}{dx^2} = 6x - 12.$$

Equate to zero the first derivative; $3x^2 - 12x = 0$, giving $x = 0$ or $x = 4$. If $x = 0$, $\frac{d^2y}{dx^2} = -12$ and, if $x = 4$, $\frac{d^2y}{dx^2} = 12$. Hence $x = 0$ is the abscissa of a maximum, and $x = 4$ of a minimum, point. By putting these values of x in the equation of the curve, we find that $(0, 0)$ is the maximum and $(4, -32)$ is the minimum point (Fig. 135).

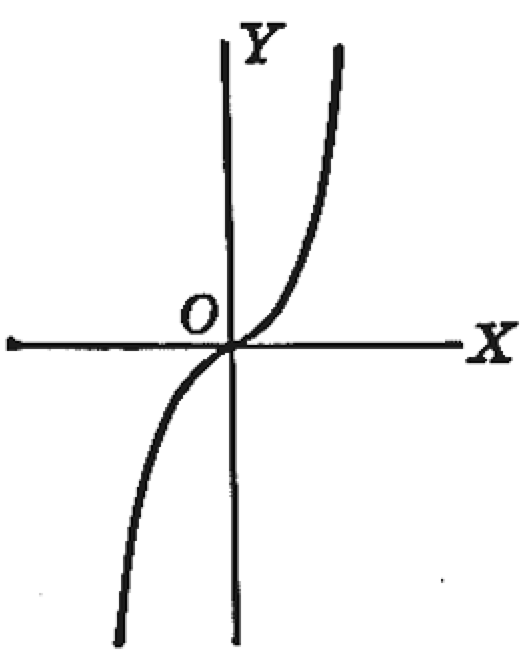


FIG. 136

It should further be observed that the first derivative may become zero without changing sign. The point determined by such a value of x is called a *point of inflection* and is neither a maximum nor a minimum.

Thus, if $y = x^3$, then $\frac{dy}{dx} = 3x^2$ and $\frac{d^2y}{dx^2} = 6x$. The first derivative is zero at $x = 0$ but it does not change sign as x increases from negative to positive values. The second derivative is also zero for this value of x. The origin is a point of inflection on this curve (Fig. 136).

Exercises

Find the maximum and minimum points on the following curves.

1. $y = 3x^2 - 12x + 3$.
2. $y = x^3 - 12x$.
3. $y = x^4 - 8x^2$.
4. $y = 2x^3 - 15x^2 + 36x$.
5. $y = x + \frac{1}{x}$.
6. $y = x + \frac{4}{x^2}$.
7. $y = \frac{a^3}{a^2 + x^2}$.
8. $y = \frac{a^2x}{a^2 + x^2}$.

9. Find the dimensions of a rectangle of minimum perimeter that has an area of 64 square feet.

10. A plumber wishes to make an open gutter of maximum cross-section, having a horizontal base and vertical sides, out of a strip of tin 24 inches wide. Find the width and the depth of the gutter.

11. A merchant estimates that he can sell 50 hats per month at a price which will net him a profit of $1 per hat and that his monthly sales will increase by 5 hats for each decrease of one cent profit per hat. What profit per hat will net him the greatest monthly profit?

12. A tin can in the shape of a right circular cylinder is to have a capacity of 24 cubic inches. Find the radius of the base and the altitude if its total surface is a minimum.

214. The Indefinite Integral. The process of finding a function $F(x)$ whose derivative is a given function $f(x)$ is called **integration.** The function $F(x)$ obtained as the result of the integration process is called an **integral** of $f(x)$; that is, if

$$\frac{d}{dx}F(x) = f(x)$$

then $F(x)$ is an integral of $f(x)$.

EXAMPLE. Find an integral of the function x^2.

From formulas IV and VII of Art. 209, we have

$$\frac{d}{dx}\left(\frac{x^3}{3}\right) = \frac{1}{3}\frac{d}{dx}x^3 = \frac{3x^2}{3} = x^2.$$

Hence, $x^3/3$ is an integral of x^2.

Further, since the derivative of a constant is zero, the function $x^3/3 + C$, where C is any constant, is an integral of x^2. For

$$\frac{d}{dx}\left(\frac{x^3}{3} + C\right) = \frac{d}{dx}\left(\frac{x^3}{3}\right) + \frac{dC}{dx} = x^2 + 0 = x^2.$$

We shall, accordingly, say that $x^3/3 + C$ is the required integral.

As we saw in the foregoing example, if $F(x)$ is an integral of $f(x)$, then $F(x) + C$, where C is any constant, is also an integral. For if

$$\frac{d}{dx}F(x) = f(x), \quad \text{then} \quad \frac{d}{dx}[F(x) + C] = \frac{d}{dx}F(x) + \frac{dC}{dx} = f(x).$$

Because of the fact that we can add to the integral any constant we please, we say that the expression $F(x) + C$ is the **indefinite integral** of $f(x)$.

It is customary to denote the indefinite integral of the function $f(x)$ by the symbol

$$\int f(x)dx + C.$$

From the formulas of Arts. 209 and 211, we obtain at once the following formulas for integration.

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx + C$$

$$\int k\, f(x)dx = k\int f(x)dx + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \qquad \text{provided } n \neq -1$$

$$\int \frac{1}{x} dx = \log_e x + C$$

$$\int e^x dx = e^x + C$$

$$\int \sin x\, dx = -\cos x + C, \quad \text{provided } x \text{ is in radians}$$

$$\int \cos x\, dx = \sin x + C, \qquad \text{provided } x \text{ is in radians}$$

To verify these formulas, differentiate the second members and show that the results are equal to the quantities to be integrated in the first members.

EXAMPLE 1. Find the integral $\int (x^{\frac{2}{3}} - 3x^{-\frac{1}{2}} + 10x^4)dx.$

$$\int (x^{\frac{2}{3}} - 3x^{-\frac{1}{2}} + 10x^4)dx = \int x^{\frac{2}{3}}dx - 3\int x^{-\frac{1}{2}}dx + 10\int x^4 dx$$
$$= \tfrac{3}{5}x^{\frac{5}{3}} - 6x^{\frac{1}{2}} + 2x^5 + C.$$

EXAMPLE 2. Find the integral $\int e^{mx}dx.$

Let $u = mx$, so that $\dfrac{du}{dx} = m$. Then $du = \dfrac{du}{dx}\, dx = mdx.$

Then $\int e^{mx}dx = \dfrac{1}{m}\int e^{mx}mdx = \dfrac{1}{m}\int e^u du = \dfrac{1}{m}e^u + C = \dfrac{1}{m}e^{mx} + C.$

EXAMPLE 3. Find the integral $\int (2x + 1) \cos (x^2 + x + 3)\, dx.$

Let $u = x + x + 3$, $\dfrac{du}{dx} = 2x + 1$, so that $du = (2x + 1)\, dx.$

$$\int \cos (x^2 + x + 3) \cdot (2x + 1)dx = \int \cos u\, du = \sin u + C = \sin (x^2 + x + 3) + C.$$

Exercises

Find the indicated integrals.

1. $\int (3x^2 + 12x + 7)dx.$

2. $\int (2x^3 + 5x^2 + 3x - 4)dx.$

3. $\int \left(\sqrt{x^3} + 2\right) dx.$

4. $\int (x^2 + 4x^7)dx.$

5. $\int (2 - x)^2 dx.$

6. $\int \left(\frac{1}{\sqrt{x}} + 3\sqrt{x}\right) dx.$

7. $\int (\sqrt{x} + 3)^2 dx.$

8. $\int (x^{\frac{1}{3}} - x^{-\frac{1}{3}})^3\, dx.$

9. $\int \frac{2x}{(x^2 + 1)^2}\, dx.$

10. $\int \frac{2x + 3}{x^2 + 3x + 2}\, dx.$

HINT. Let $u = x^2 + 1$. Then $\frac{du}{dx} = 2x$ and the integral becomes $\int \frac{du}{u^2}$.

11. $\int \cos\,(2x - 5) dx.$

12. $\int 3e^{5x} dx.$

13. $\int xe^{-x^2} dx.$

14. $\int \cot x\, dx = \int \frac{\cos x}{\sin x}\, dx.$

15. Find the equation of the curve, given that $\frac{dy}{dx} = 6x - 1$. Fix the value of the constant by imposing the condition that the curve goes through the point (2, 3).

215. The Definite Integral. Let $F(x)$ be any one of the indefinite integrals of $f(x)$. Then the expression,

$$F(b) - F(a),$$

is called *the definite integral of* $f(x)$ *from* a *to* b.* The reason for this apparently rather arbitrary definition will appear in the following article.

It is customary to denote the definite integral by the symbol $\int_a^b f(x)dx$; that is, by definition,

$$\int_a^b f(x)dx = F(b) - F(a).$$

It should be observed that, if we add to the given indefinite integral $F(x)$ any constant we please, we obtain the same value for the definite integral. For

$$[F(b) + C] - [F(a) + C] = F(b) - F(a).$$

Exercises

1–10. Find the definite integrals from 1 to 3 in each of the exercises 1–10 in Art. 214.

216. The Area under a Curve. Let us consider a portion of the curve $y = f(x)$ that lies entirely above the x-axis. We shall suppose further that

* For certain special cases in which this definition fails, the student is referred to more advanced textbooks.

every vertical line in the interval from $x = a$ to $x = b$ $(a < b)$ intersects this portion of the curve in one and only one point (Fig. 137). The area bounded by the given portion of the curve, the x-axis, and the ordinates of the points on the curve whose abscissas are a and b is called the *area under the curve from a to b*. We wish to find this area.

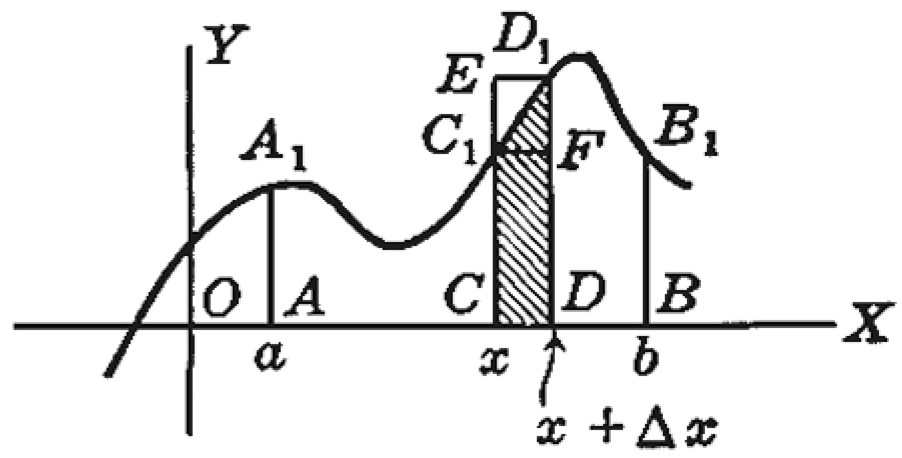


FIG. 137

Consider, first, the area under the curve from a to some number x less than b. Since the magnitude of this area depends on x, we shall denote it by the function $A(x)$. Then the area under the curve from a to $x + \Delta x$ is $A(x + \Delta x)$. It follows that the shaded area in Figure 137 is

$$\Delta A = A(x + \Delta x) - A(x).$$

From the figure, this shaded area is seen to be greater than the area of the rectangle $CDFC_1$ and less than that of the rectangle CDD_1E; that is

$$\text{area } CDFC_1 < \Delta A < \text{area } CDD_1E.$$

These rectangles are both of width Δx and of altitudes $f(x)$ and $f(x + \Delta x)$, respectively. Hence

$$f(x)\Delta x < \Delta A < f(x + \Delta x)\Delta x \quad \text{or} \quad f(x) < \frac{\Delta A}{\Delta x} < f(x + \Delta x).$$

As Δx approaches zero, $f(x + \Delta x)$ approaches $f(x)$. It follows that $\frac{\Delta A}{\Delta x}$, which lies between $f(x)$ and $f(x + \Delta x)$, also approaches $f(x)$. But the limit of $\frac{\Delta A}{\Delta x}$ as Δx approaches zero is $\frac{dA}{dx}$; that is,

$$\frac{dA}{dx} = f(x).$$

It follows that $A(x)$ is an indefinite integral of $f(x)$, so that

$$A(x) = F(x) + C.$$

To find the value of the constant, observe that $A(a) = 0$ since the width of the area is zero when $x = a$. Hence,

$$0 = F(a) + C, \quad C = -F(a), \quad \text{and} \quad A(x) = F(x) - F(a).$$

To find the area under the curve from a to b, we now put $x = b$, giving

$$\boldsymbol{A(b) = \text{required area} = F(b) - F(a) = \int_a^b f(x)dx.} \qquad (25)$$

We have supposed that the portion of the curve under consideration lies above the x-axis. If it lies below, this formula will give, if $a < b$, a negative number numerically equal to the area.

EXAMPLE. Find the area under the curve $y = 3x^2 - 4x + 2$ from $x = 1$ to $x = 4$.

We have the indefinite integral $F(x) = \int (3x^2 - 4x + 2)dx = x^3 - 2x^2 + 2x.$ From equation (25), the required area is now found to be

$$(64 - 32 + 8) - (1 - 2 + 2) = 40 - 1 = 39.$$

Exercises

Find the area under the curve:

1. $y = 2x - 1$ from 2 to 5. Verify your result by elementary geometry.
2. $y = 6x^2 + 1$ from -2 to 3.
3. $y = \sqrt{x}$ from 4 to 25.
4. $y = x^3 + x + 1$ from 0 to 2.
5. $y = (2x + 5)^2$ from -2 to 1.
6. $y = 2x^{-\frac{1}{3}}$ from 1 to 8.
7. $y = \sin x$ from 0 to π.
8. $y = 6e^{2x}$ from -1 to 1.

Chapter 27

The Graph of an Equation

217. Introduction. In this chapter, we shall consider the following problem: given an equation in the coördinates of a point, find (at least approximately) the locus (or graph) formed by the points whose coördinates satisfy the equation.

The process of determining the form of the curve by plotting points on it can often be shortened by noticing certain properties of its equation that serve to indicate the general form of the curve. By observing these properties, we are frequently able to draw the curve without plotting as many points as would otherwise be necessary. We intend now to point out a number of important properties which are often found in the equations of the curves most frequently met with, which can be determined quite readily from the equation, and which are especially helpful in drawing the graph. The process of determining which of these properties hold for a given equation is called the *discussion of the equation.*

A plane curve is **algebraic** if it can be defined by equating to zero a polynomial in which x and y appear only with positive, integral exponents; otherwise, it is **transcendental.** Lines and conics, for example, are algebraic curves as also are the loci of such equations as $x^3 + y^3 = a^3$ and $x^2y^2 = x^2 + y^2$. The curves defined by such equations as $y = \tan x$ or $y \log x = x^2$ are transcendental.

I. Algebraic Curves

218. Discussion of the Equation.

(*a*) *Symmetries.* A curve is symmetric with respect to a given line or to a given point if, when $P(x, y)$ is any point on the curve, then its symmetric point with respect to the given line or the given point also lies on the curve (Art. 41).

The point symmetric to $P(x, y)$ with respect to:

(1) the x-axis is $(x, -y)$ (2) the y-axis is $(-x, y)$
(3) the line $y = x$ is (y, x) (4) the origin is $(-x, -y)$.

Hence, an algebraic curve is symmetric to:

(1) the x-axis if y enters its equation only to even powers;
(2) the y-axis if x enters its equation only to even powers;
(3) the line $y = x$, if its equation remains unchanged when x and y are interchanged in it;

(4) the origin, if the equation remains unchanged when x and y are replaced by $-x$ and $-y$, respectively.

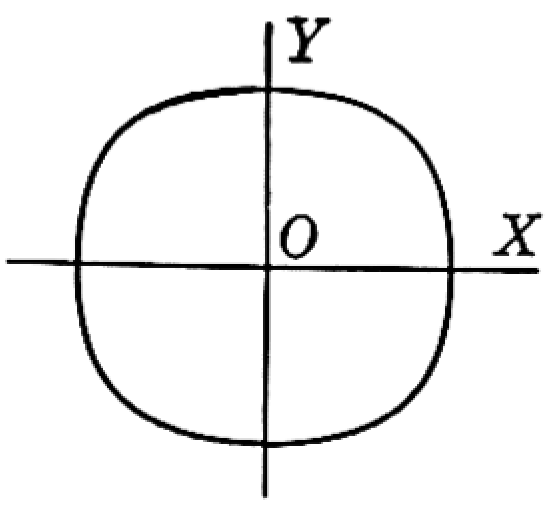


FIG. 138

The curve $x^4 + y^4 = a^4$ (Fig. 138), for example, exhibits all of the above-mentioned symmetries. It is therefore symmetric with respect to the x-axis, the y-axis, the line $y = x$, and the origin.

(*b*) *Intercepts*. The intercepts on the x-axis are found by putting $y = 0$ in the equation of the curve and solving for x; and, on the y-axis, by putting $x = 0$ and solving for y. The graph must pass through every point determined in this way and it does not meet either axis in any other point.

Thus, the curve $x^4 + y^4 = a^4$, shown in Figure 138, meets the x-axis at $(\pm a, 0)$, and the y-axis at $(0, \pm a)$. It has no other point in common with either axis.

(*c*) *Tangent Lines at the Origin*. If the given curve passes through the origin, we may find its approximate form near that point, since x and y are small, by considering only their lowest powers that appear in the equation, and neglecting all higher powers; that is, *to find the tangent lines to the curve at the origin, equate to zero those terms in its equation for which the sum of the exponents of x and y has the lowest value.*

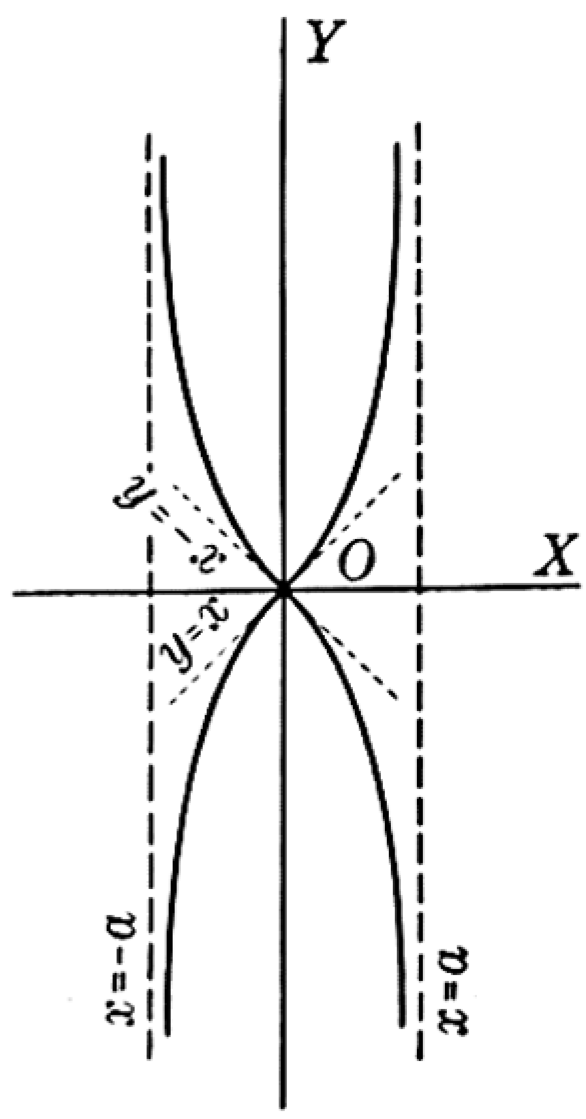


FIG. 139

Thus, for the curve $x^2y^2 + a^2x^2 - a^2y^2 = 0$, the sum of the exponents of x and y in the first term is 4 and in each of the remaining two terms is 2. The tangent lines at the origin are found, by equating these last two terms to zero, to be $x^2 - y^2 = 0$, or $x + y = 0$ and $x - y = 0$.

(*d*) *Horizontal and Vertical Asymptotes*. When we were studying the hyperbola, we noticed in Art. 188 that the curve recedes toward infinity, in any one of the quadrants, in such a way that it approaches a fixed line which we called an asymptote to the hyperbola.

Since many other curves extend out indefinitely far in a similar way, we make the following definition: *If a branch of a curve extends toward infinity in such a way that it approaches indefinitely near to a fixed line, this line is called an asymptote to the curve.*

If an algebraic curve has vertical or horizontal asymptotes, these lines can be determined from the equation of the curve as in the following example.

EXAMPLE 1. Find the vertical and the horizontal asymptotes to the curve $xy^2 - a^2y - b^2x = 0$.

In this equation, if we assign to x a fixed value $x_1 \neq 0$, the resulting quadratic equation in y has two roots which are the ordinates of the two intersections of the vertical line $x = x_1$ with the curve.

If we now let x_1 approach indefinitely near to zero, one of the two corresponding values of y increases indefinitely in numerical value and the corresponding point on the curve recedes toward infinity in such a way that it approaches indefinitely near to the line $x = 0$, that is, to the y-axis (Fig. 140). This line is, accordingly, an asymptote.

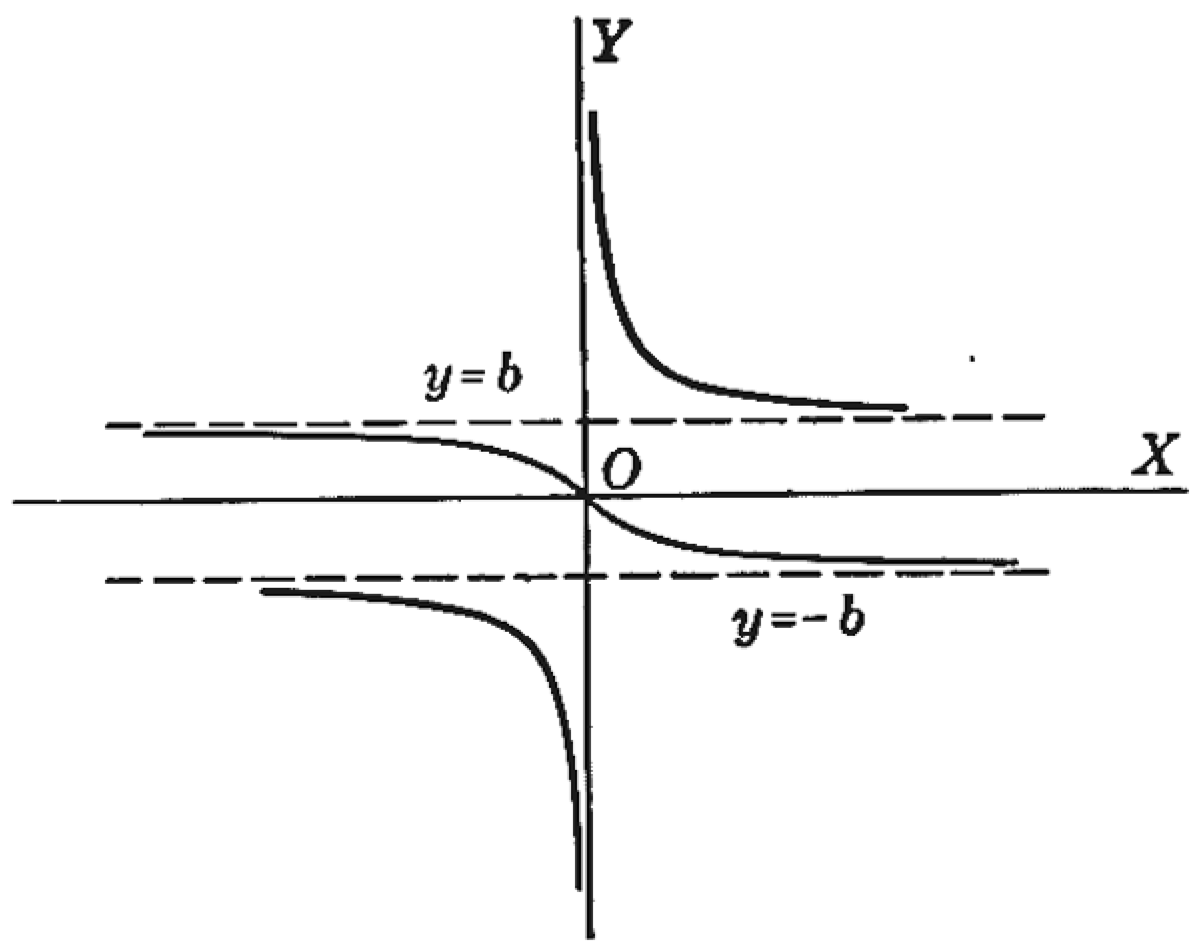


FIG. 140

Similarly, if we arrange the given equation in powers of x,

$$(y^2 - b^2)x - a^2y = 0,$$

and assign to y any value $y_1 \neq \pm b$, the root of the resulting equation in x is the abscissa of the single intersection of the line $y = y_1$ with the curve. If y_1 is now made to approach $+b$ or $-b$, the numerical value of x will increase indefinitely and the corresponding point on the curve will approach the asymptote $y = b$ or $y = -b$.

By applying reasoning similar to the foregoing to any given algebraic curve, we deduce the following general rule for finding the vertical and horizontal asymptotes: *To find the vertical asymptotes to an algebraic curve, equate to zero the real, linear factors of the coefficient of the highest power of y in the equation. To find the horizontal asymptotes, equate to zero the real, linear factors of the coefficient of the highest power of x in the equation.*

If the coefficient of the highest power of y (or of x) in the given equation is a constant, or if its linear factors are all imaginary, there are no vertical (or no horizontal) asymptotes.

Thus, the curve $y^4 + x^2y^2 + x^2 + 1 = 0$ has no vertical asymptotes since the coefficient of the highest power of y is unity. It has no horizontal asymptotes since the factors of $y^2 + 1$, the coefficient of the highest power of x, are imaginary.

(*e*) *Excluded Intervals.* If, when the given equation is solved for y, square roots occur in the second member, the values of x throughout certain intervals may cause the quantity under the radical sign to be negative and thus make y imaginary. Since no point can be plotted if either of its coördinates is imaginary, any such interval should be excluded from consideration in drawing the graph.

In the same way, when we solve for x, certain intervals may be found

on the y-axis for which the values of x are imaginary. These intervals must also be excluded when we draw the graph.

If, for example, we solve the equation $x^2y^2 - a^2x^2 - a^2y^2 = 0$ for y, we find that $y = \dfrac{\pm ax}{\sqrt{x^2 - a^2}}$, which shows that, for all values of x between $x = -a$ and $x = a$, except $x = 0$, y is imaginary. There are, accordingly, no points on the curve except the point (0, 0) within this interval (Fig. 141).

If we now solve the above equation for x, we obtain $x = \dfrac{\pm ay}{\sqrt{y^2 - a^2}}$. It follows that, for all values of y between $-a$ and a, except $y = 0$, x is imaginary. This interval on the y-axis should also be excluded in drawing the graph.

A point, such as the origin in this example, that lies on the curve, but which has no other points on the curve in its neighborhood, is called a *conjugate* (or *isolated*) *point* on the curve.

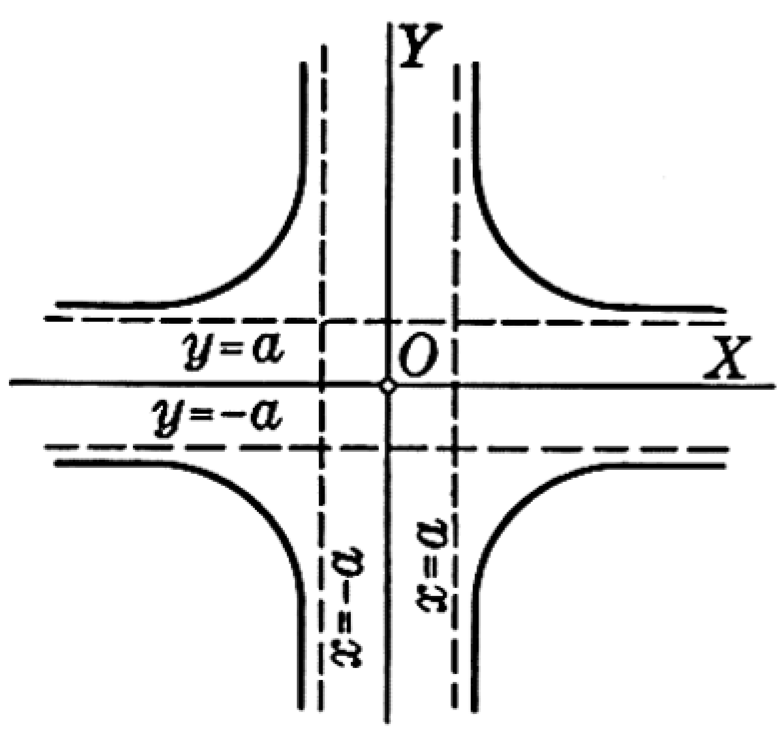


FIG. 141

(*f*) *The Slope of the Tangent Line. Maxima and Minima.* We saw in Chapter 26 that the slope of the tangent line at a point on the curve equals the value of the derivative at that point. Since the slope of the tangent line fixes the direction of that line (Art. 151) and thus the direction of the curve at the point of tangency, it follows that:

(1) In any region in which the slope of the tangent line (the value of the derivative) is positive, y is increasing as x increases. Moreover, when the slope is a large positive number, y is increasing rapidly compared to the rate of increase of x and, if the slope is small, y is increasing slowly compared to x.

(2) At any point at which the slope is zero, the tangent line is parallel to the x-axis. If the slope changes from positive to negative at this point, the point is a maximum point; if it changes from negative to positive, the point is a minimum point (Art. 213).

(3) In any region in which the slope is negative, y is decreasing as x increases and the rate of decrease is rapid or slow according as the slope is numerically large or small.

EXAMPLE 2. Find and discuss the slope of the curve $y = x^3 - 3x$ at a point P on the curve.

Since the slope at the point P equals the value of the derivative at P, we have (Art. 209)

$$m = \frac{dy}{dx} = 3(x^2 - 1).$$

This expression is positive if $x < -1$ or if $x > 1$. It equals zero if $x = \pm 1$ and is negative if $-1 < x < 1$. Hence, if $x < -1$, y increases with x, rapidly

if x is numerically large and slowly if x is near -1. It reaches the value 2 when $x = -1$. It then decreases to -2 as x increases to 1. When x increases beyond 1, y increases, slowly at first, then more rapidly as x increases. The curve has a maximum point at $(-1, 2)$ and a minimum point at $(1, -2)$ (Fig. 142).

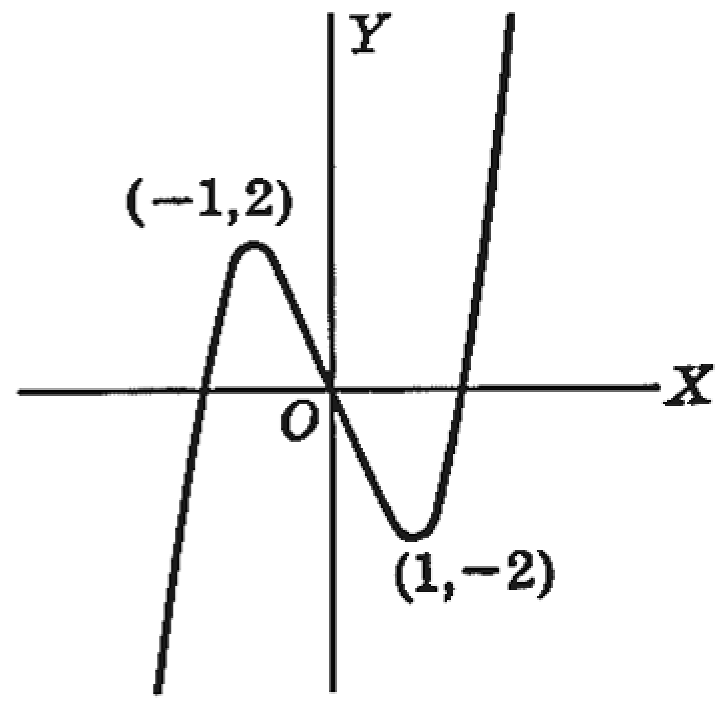


FIG. 142

219. Drawing the Graph. When it is required to draw the graph of a given algebraic equation, we first discuss the equation, as in the preceding article. We then plot points on the curve, taking care to plot some points on each branch of the curve and to plot the points most numerously at the places where the curve changes its direction most rapidly. We then draw a smooth curve through these points in such a way that it satisfies the conditions deduced from the preceding discussion of the equation.

EXAMPLE 1. Discuss the equation $x^2y + a^2y - a^3 = 0$ and draw the curve. (**The Witch**)

This curve is symmetric with respect to the y-axis which it intersects at $(0, a)$. It has the x-axis as an asymptote and does not intersect it. If we solve the equation for x and y, we have

$$x = \pm a\sqrt{\frac{a-y}{y}}, \text{ and } y = \frac{a^3}{x^2 + a^2}.$$

It is seen from the first equation that x is imaginary if $y > a$ or if $y < 0$. It follows from the second equation that y has its largest value, a, when $x = 0$ and that it decreases continually as the numerical value of x increases.

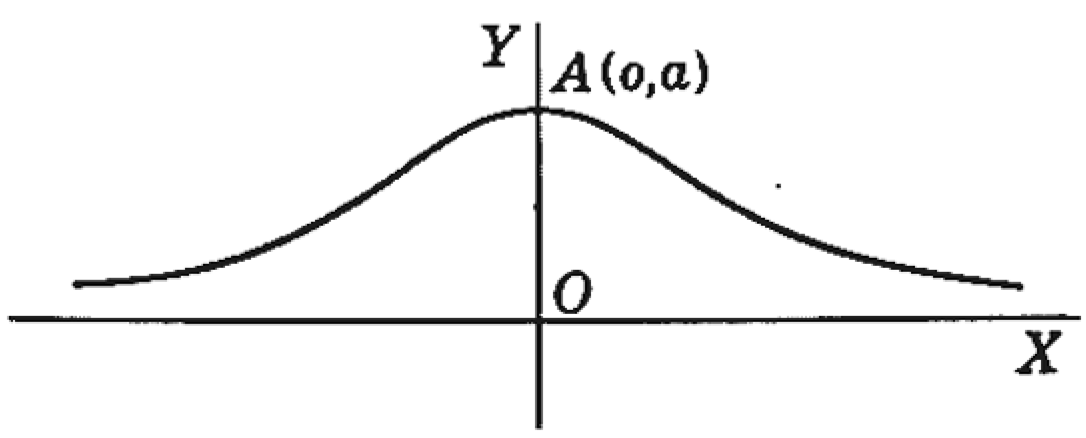


FIG. 143

Figure 143 can now be drawn quite accurately by plotting a number of points on the locus and drawing the curve so that the conditions stated in the discussion are satisfied.

EXAMPLE 2. Discuss the equation $x^3 + xy^2 - 2ay^2 = 0$ and draw the curve. (**The Cissoid**)

The curve is symmetric with respect to the x-axis. It meets the axes only at the origin and its tangents at that point are defined by $y^2 = 0$. The line $x - 2a = 0$ is a vertical asymptote. If $x < 0$ or if $x > 2a$, the values of y are imaginary.

The slope of the tangent line at $P_1(x_1, y_1)$ is

$$m = \frac{x_1^2\,(3a - x_1)}{y_1(2a - x_1)^2} = \pm\frac{(3a - x_1)}{(2a - x_1)}\sqrt{\frac{x_1}{2a - x_1}}.$$

For points on the curve other than the origin, the slope agrees in sign with y_1. If x_1 is small, the slope is numerically small but it becomes numerically large as x_1 approaches $2a$ (Fig. 144).

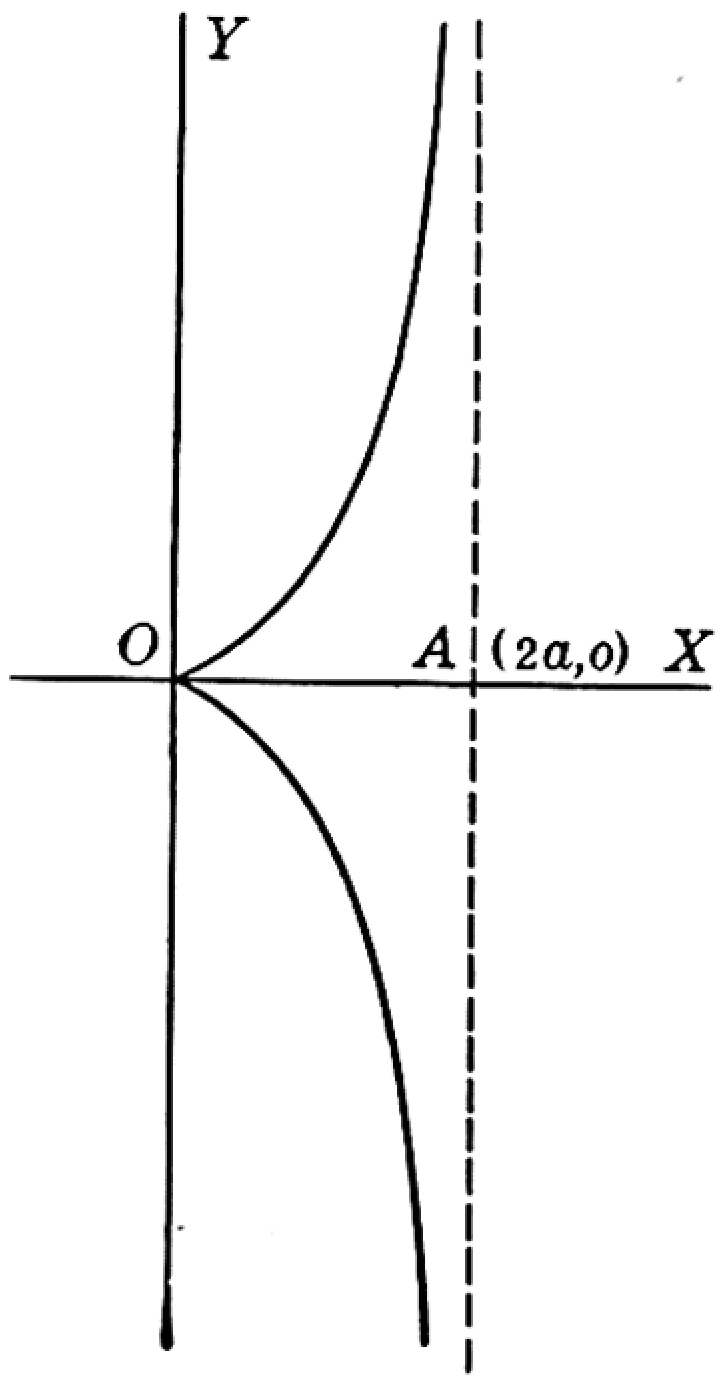


Fig. 144

Example 3. Discuss the equation $y^2 = x(x - a)(x - b)$ and draw the curve.

We shall suppose that $0 < a < b$. The curve is composed of two separate branches and is called a *bipartite cubic curve.*

It is symmetric with respect to the x-axis, touches the y-axis at the origin and intersects the x-axis again at $(a, 0)$ and $(b, 0)$. The value of y is real only if $0 \leq x \leq a$ or if $x \geq b$.

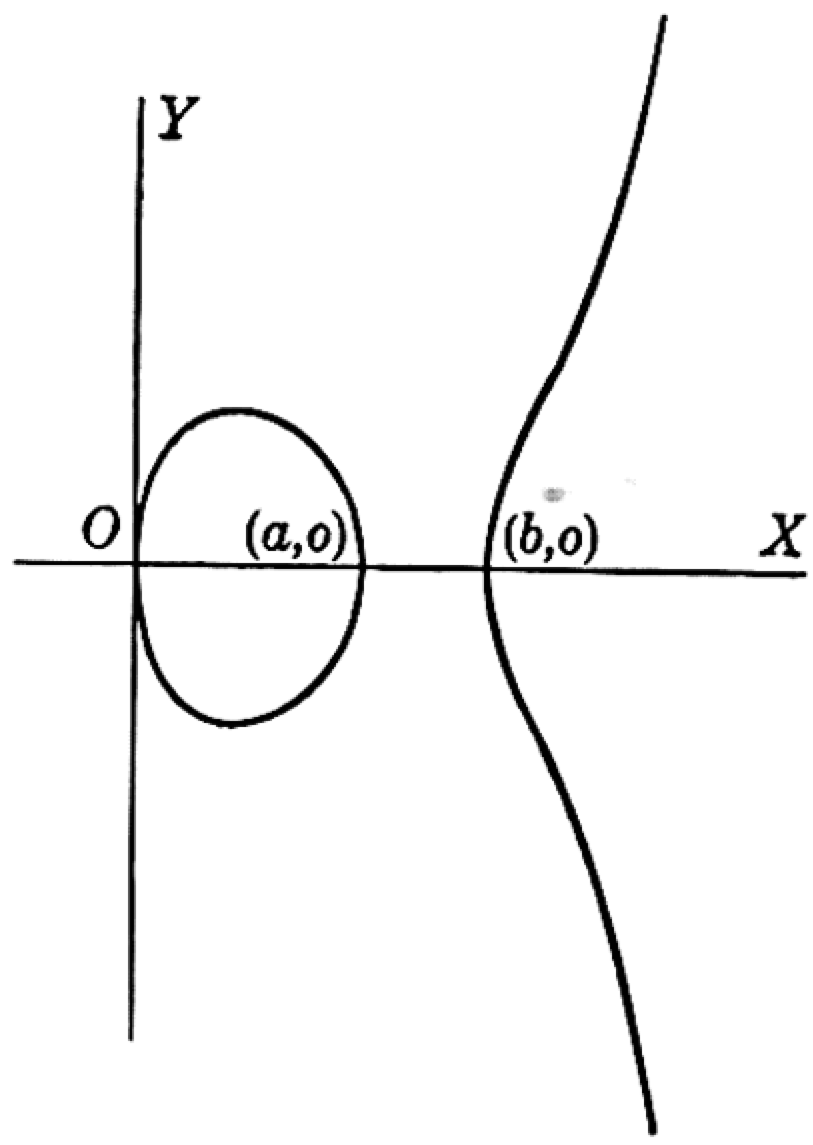


Fig. 145

The slope of the tangent at $P_1(x_1, y_1)$ on the curve is

$$m = \frac{3x_1^2 - 2(a + b)x_1 + ab}{2y_1}.$$

The branch of the curve between $x = 0$ and $x = a$ is an oval, somewhat more pointed to the right than to the left. The branch for $x \geq b$ extends to infinity in the first and fourth quadrants without approaching any asymptote.

Exercises

Discuss the given equation, plot several points on the locus, and draw the curve.

1. $xy + 2x + 2y = 0$.
2. $x^2 - xy - 4y = 0$.
3. $y = x^3 - 2x^2$.
4. $y = x(x + 1)^2$.
5. $x^2y = 1$.
6. $y^4 = x^3$.
7. $y = x(x + 1)(x + 2)$.
8. $y^2 = x(x + 1)(x + 2)$.
9. $y = x^2(9 - x^2)$.
10. $y^2 = x^2(9 - x^2)$.
11. $y = (x^2 - 4)^2$.
12. $(x^2 - 1)y = 1$.
13. $(x^2 - 4)y = x$.
14. $x^2y^2 = x^2 + y^2 + 1$.
15. $xy^2 - x + 1 = 0$.
16. $x^4 + x^2y^2 = y^2$.
17. $y = \dfrac{1}{x^2(x + 1)}$.
18. $y = \dfrac{1}{x + 3} + \dfrac{1}{x - 3}$.
19. $x^2y^2 = x^2 + 1$.
20. $xy^2 = x^2 + 1$.

21. The force F of gravitational attraction between two bodies is expressed as a function of the distance r between them by the equation $F = k/r^2$, where k is a constant. Show this relation graphically.

22. The horsepower H which a shaft can safely transmit is expressed as a function of the diameter d of the shaft by the equation $H = kd^3$, where k is a constant. Show this relation graphically.

In the following exercises, find the equation of the locus of the point $P(x, y)$ and draw the curve.

23. The product of the distances of P from two fixed points $(-a, 0)$ and $(a, 0)$ equals a^2.

24. The inclination of the line joining P to a fixed point $(2a, 0)$ is three times the inclination of the line joining P to the origin.

25. A variable line through the origin intersects the circle $x^2 + y^2 - ax = 0$ at the origin and at a point B and the line $x - a = 0$ in a point A. A vertical line through B intersects a horizontal line through A at P.

26. The line joining P to the origin intersects the circle $x^2 + y^2 - ax = 0$ at the origin and at a variable point B. The undirected distance PB equals a.

27. The line joining P to a fixed point $(-a, 0)$ intersects the y-axis in a variable point B. The undirected distance PB equals the undirected distance from the origin to B.

28. Solve Ex. 27 if the undirected distance PB equals a constant k.

II. Transcendental Curves

220. The Trigonometric Curves. The process of drawing the graphs of the trigonometric functions was discussed in Art. 116 and, of their inverses, in Art. 135. We shall now show how we can obtain from these curves certain modifications of them which have been found useful in the applications.

The curves defined by the equations

$$y = a \sin b(x - c), \quad \text{and} \quad y = a \cos b(x - c) \tag{1}$$

are modified forms of the sine and cosine curves. The number a is the **amplitude,** b is the **periodicity factor,** and c determines the **phase angle.**

To draw these curves, if $c \neq 0$, we first translate the origin to the point $(c, 0)$, giving

$$y' = a \sin bx', \quad \text{and} \quad y' = a \cos bx'. \tag{2}$$

The graphs of equations (2), referred to the new axes, will also be the graphs of (1) referred to the old axes. These graphs can readily be drawn, on the new axes, by first drawing the curves $y' = \sin x'$ and $y' = \cos x'$, as shown in Art. 116, then multiplying all the ordinates by a and then, finally, dividing all the abscissas by b.

et period of one full function

EXAMPLE. Draw the curve $y = \frac{3}{2} \sin 2x$ from $x = 0$ to $x = 2\pi$.

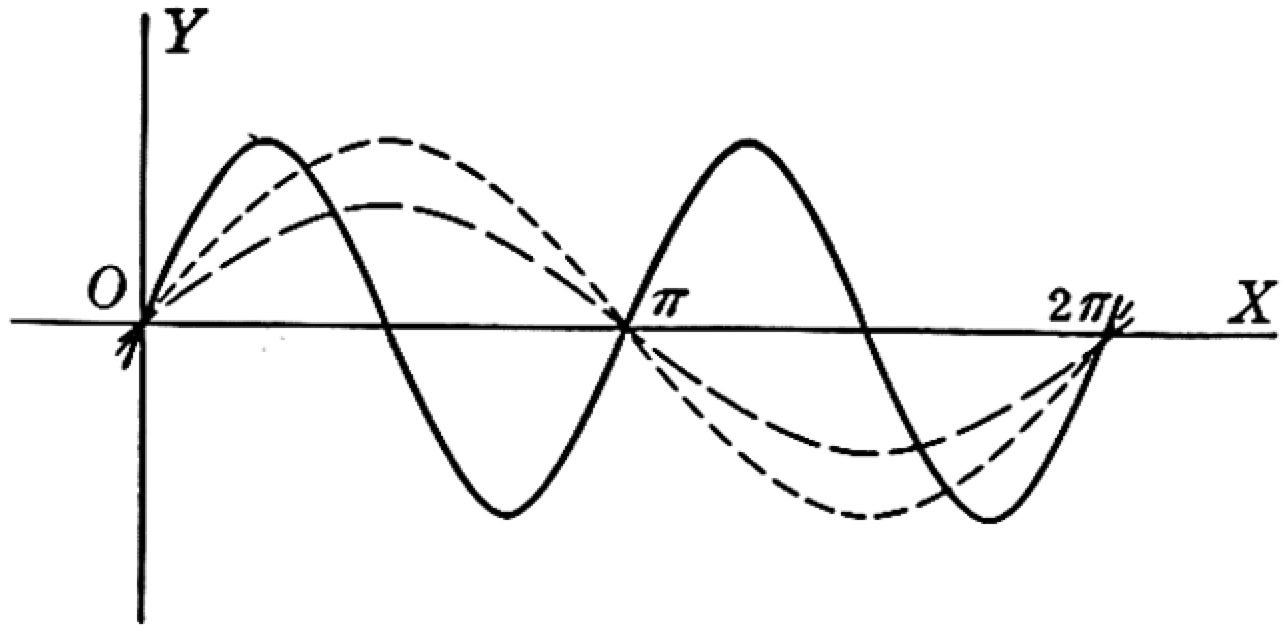


FIG. 146

Since $c = 0$, we first draw the sine curve, $y = \sin x$, for one complete period, 0 to 2π. Next, we construct the curve $y = \frac{3}{2} \sin x$ by multiplying all the ordinates for the sine curve by the amplitude $\frac{3}{2}$. The required curve is then obtained by dividing all the abscissas for this curve by the periodicity factor 2 and repeating the figure so obtained as many times as is necessary (Fig. 146).

Exercises

Draw the graphs of the following functions for two periods.

1. $y = \sin \pi x$. **2.** $y = \cos (x - 2)$.

3. $y = 3 \sin x$. **4.** $y = \sin 3x$.

5. $y = 2 \cos 3(x + 1)$. **6.** $2y = 3 \sin (2x - 1)$.

7. $2y = 5 \tan x$. **8.** $3y = 4 \sec 2x$.

9. $y = - 2 \cot 3(x - 1)$. **10.** $4y = - \csc 2(x + 1)$.

11. $2y = \sin^{-1} 3x$. **12.** $y = 2 \cos^{-1} (x - 2)$.

HINT. Draw the curve $3x = \sin 2y$.

13. $y = 3 \cos^{-1} 2x$. **14.** $2y + 4 = \cos^{-1} 3x$.

221. The Logarithmic and Exponential Curves. The graph of the logarithmic equation,

$$y = \log_a x, \tag{3}$$

was discussed in Art. 85. The graph of the exponential equation

$$y = a^x, \quad \text{or} \quad x = \log_a y, \tag{4}$$

is shown in Figure 147.

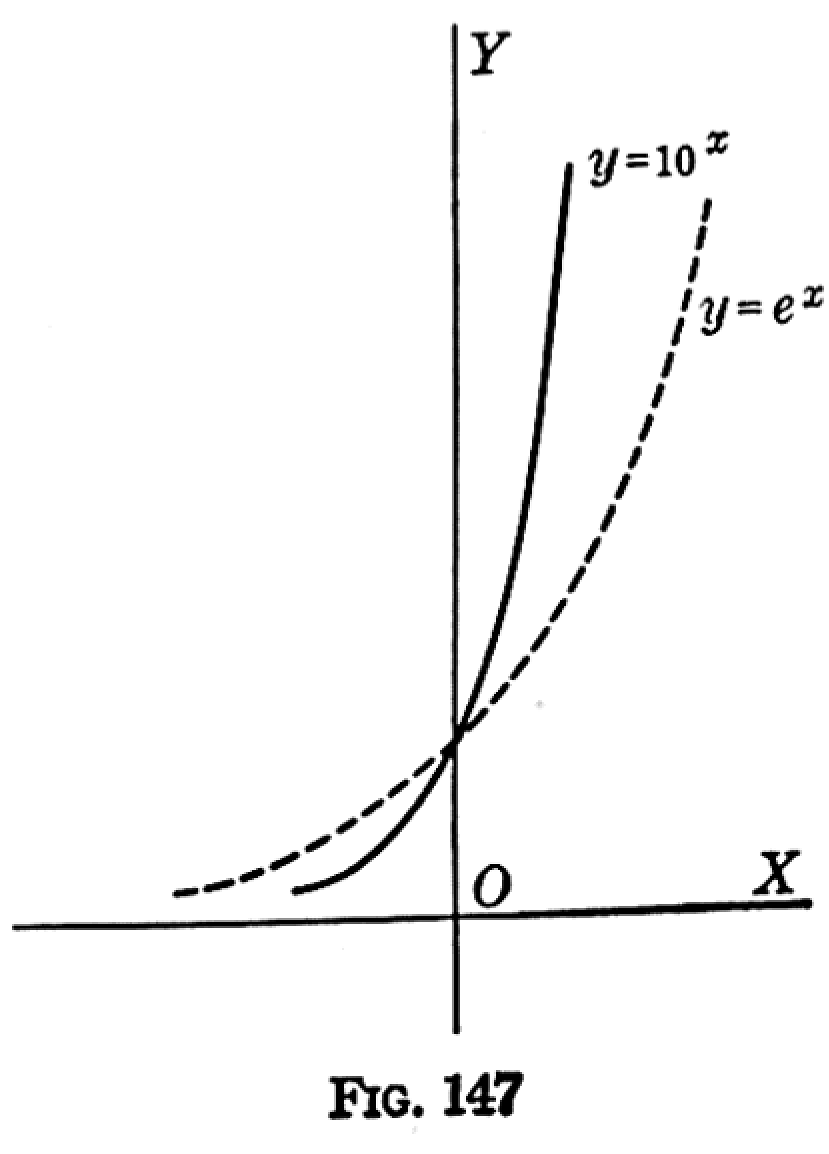


FIG. 147

Since the second of equations (4) differs from (3) only in that x and y are interchanged, it follows that: *the exponential curve differs from the logarithmic only in that it is placed on the figure so that its position with respect to the x- and y-axes is interchanged.* Because of this relation between their graphs, the functions a^x and $\log_a x$ are said to be **inverse functions.**

In the applications, the exponential equation usually appears in the form

$$y = ae^{bx}, \tag{5}$$

where a and b are constants and $e = 2.71828^+$ is the base of the natural system of logarithms (Art. 86). The graph of equation (5) may be found by taking the logarithms to the base 10 of both sides of the equation and writing the result in the form

$$\log_{10} y = bx \log_{10} e + \log_{10} a. \tag{6}$$

If a set of values is now assigned to y, the corresponding values of x may be computed from this equation with the aid of a table of logarithms.

Example 1. Draw the graph of the exponential equation $y = 1.84\, e^{0.57x}$.

Take the logarithms to the base 10 of both sides of this equation,

$$\log_{10} y = 0.57x \log_{10} e + \log_{10} 1.84.$$

We have $\log_{10} e = 0.4343$ and $\log_{10} 1.84 = 0.2648$. On substituting these values in the preceding equation, we get

$$\log_{10} y = 0.2476x + 0.2648.$$

If we solve this equation for x, we have

$$x = 4.04 \log_{10} y - 1.07.$$

By assigning values to y and computing the corresponding values of x, we obtain the following table.

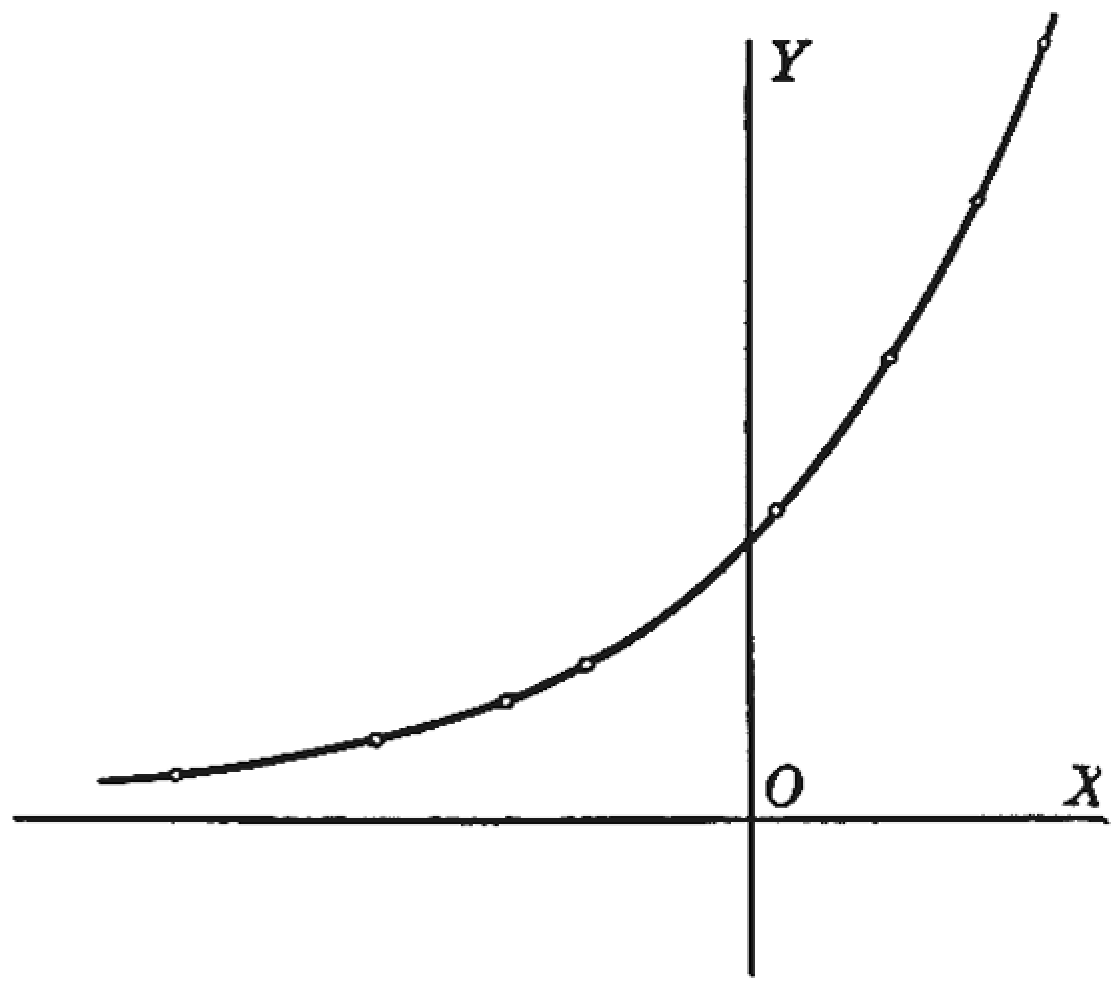


Fig. 148

x	-3.50	-2.29	-1.57	-1.07	0.146	0.858	1.36	1.75
y	0.25	0.5	0.75	1	2	3	4	5

If we plot these points and draw a smooth curve through them, we obtain Figure 148.

Example 2. Draw the graph of the logarithmic equation $y = \log_{10} 2(x + 1)^3$.

Write the equation in the equivalent form

$$y = 3 \log_{10} (x + 1) + \log_{10} 2.$$

By assigning to x values greater than -1 and computing the corresponding values of y, we obtain the following table.

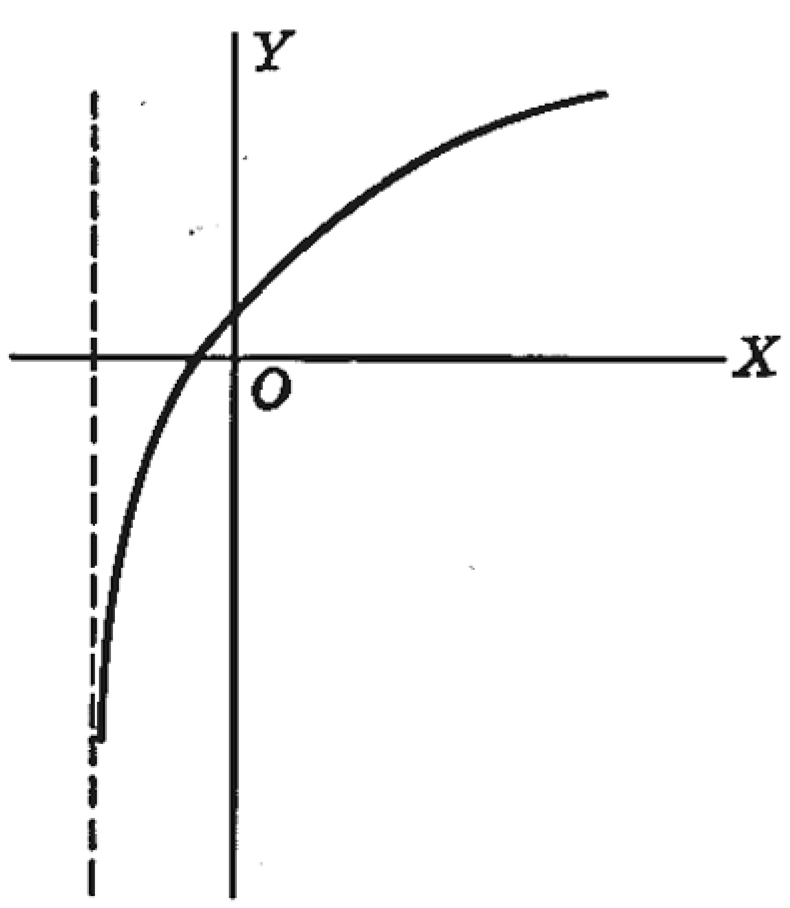


Fig. 149

x	-0.9	-0.75	-0.5	-0.25	0	0.5	1	2
y	-2.70	-1.51	-0.602	-0.074	0.301	0.829	1.20	1.73

Figure 149 is obtained by plotting these points and drawing a smooth curve through them.

Exercises

Draw the graphs of the following equations.

1. $y = \log_{10}(-x)$.
2. $y = \log_{10}(x + 3)$.
3. $y = \log_{10} 3x$.
4. $y = \log_{10} x^5$.
5. $y = \log_2 (x - 5)$.
6. $y = \log_e (1 + x^2)$.
7. $y^2 = \log_{10} x$.
8. $y = \log_{10} 5\sqrt[3]{x + 2}$.
9. $y = e^{-x}$.
10. $y = -e^x$.
11. $3y = e^{\frac{x}{2}}$.
12. $y = 0.23e^{1.41x}$.
13. $y = (1.04)^x$.
14. $y = 1.35(2.51)^x$.
15. $y = xe^x$.
16. $y = e^{-x^2}$. *The probability curve.*

17. The amount, y, due on one dollar at the rate i (expressed as a decimal) at the end of a time x (in years), at compound interest, is $y = (1 + i)^x$ and, at simple interest, is $y = 1 + ix$. Draw, on one set of axes, the graph expressing the amount due on one dollar at interest for x years (a) at 5% compound interest and (b) at 6% simple interest. State the meaning of the points of intersection of these curves and find their coördinates to one decimal place.

18. Find from a figure, to one decimal place, the number of years in which one dollar will double at 6% compound interest.

222. Addition of Ordinates. Let it be required to draw the graph of the equation

$$y = x + \sin x.$$

To find the value of y corresponding to a given value x_1, of x, the equation tells us to add together the numbers x_1 and $\sin x_1$. Instead of finding this sum by an arithmetic computation, the following geometric device for performing the addition will often be found easier and more satisfactory.

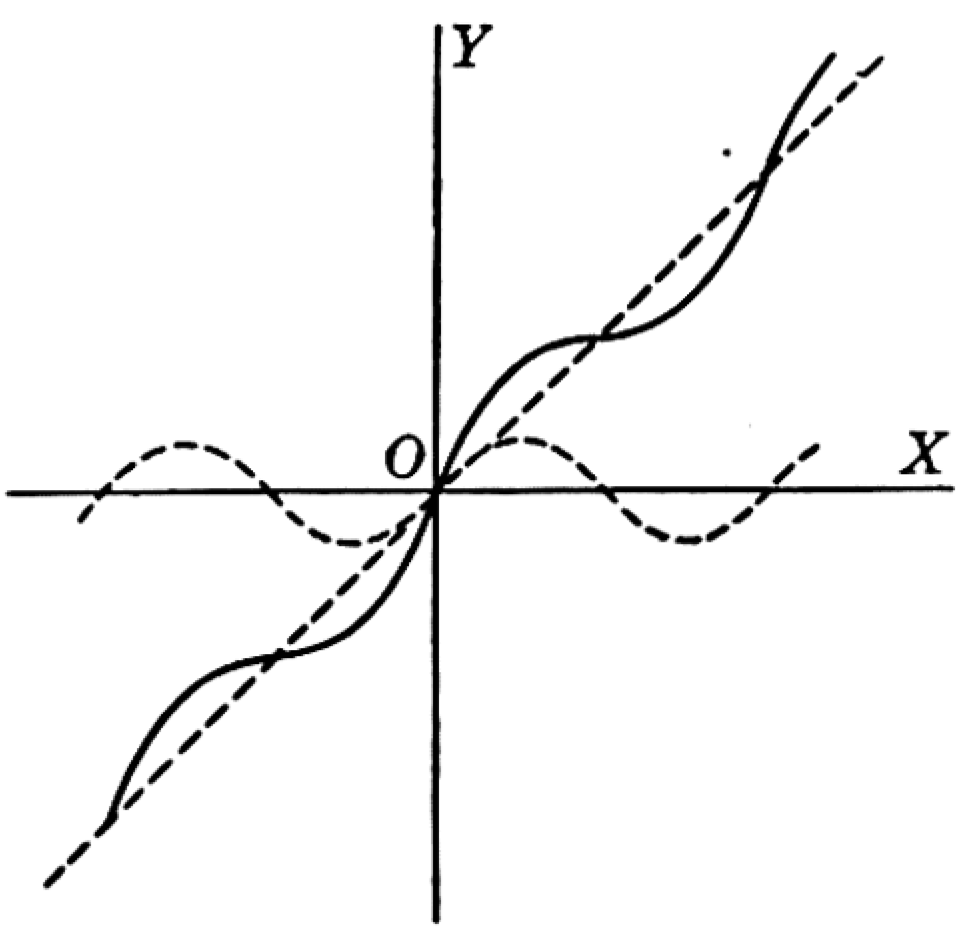


FIG. 150

We first draw, on one set of axis, the curves

$$y = x \quad \text{and} \quad y = \sin x.$$

The values of y for the points on these two curves, when $x = x_1$, are obviously $y = x_1$ and $y = \sin x_1$. Hence, *the algebraic sum of the ordinates of the points on the two curves, for $x = x_1$, is the ordinate of the point on the required curve for $x = x_1$.* We may locate as many points as we please on the required curve, accordingly, by adding graphically the ordinates of the corresponding points on the component curves; that is, by laying off, in a vertical direction from any point on the line $y = x$, a directed distance equal to the ordinate of the point directly above or below it on the curve $y = \sin x$ (Fig. 150).

As a second example of the method of addition of ordinates, consider the ellipse

$$x^2 + 2xy + 2y^2 - 3x - 8y + 8 = 0.$$

If we solve this equation for y, we have

$$y = \tfrac{1}{2}(-x + 4 \pm \sqrt{-x^2 - 2x}).$$

To draw the graph of this equation, we first draw the line

$$y = \frac{-x + 4}{2}.$$

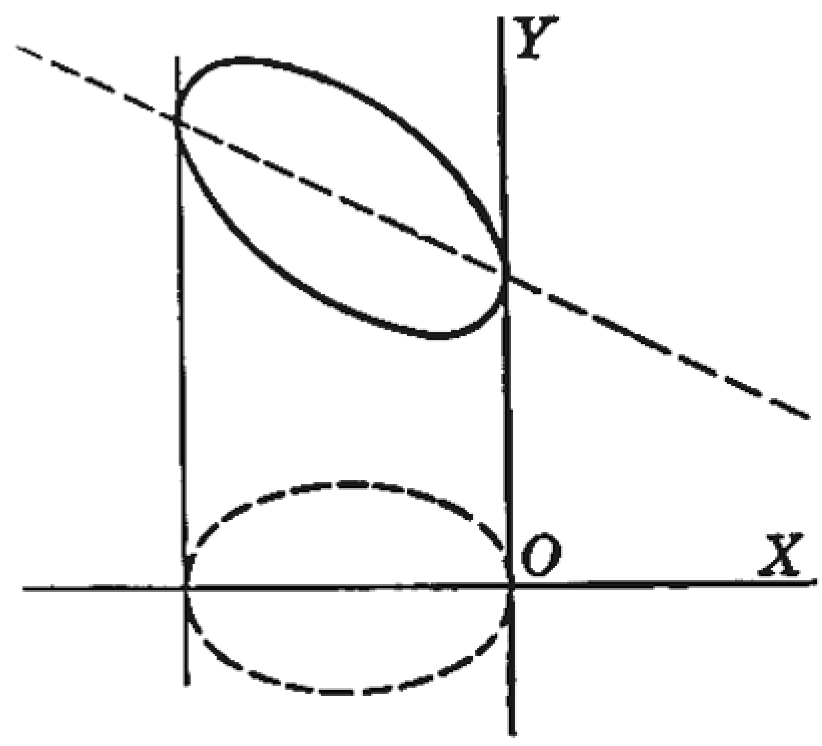


FIG. 151

Next, we draw the ellipse

$$y = \pm \frac{\sqrt{-x^2 - 2x}}{2};$$

that is, the ellipse $(x + 1)^2 + 4y^2 = 1$.

If we now add, graphically, the ordinates of corresponding points on these two curves, we determine the corresponding points on the required curve. An ellipse drawn through these points is the graph of the given equation.

Exercises

Draw the graphs of the following equations, using the method of addition of ordinates.

1. $y = x^3 - x$. **2.** $y = x + \dfrac{1}{x}$.

3. $y = x - \dfrac{1}{x^2}$. **4.** $y = x^2 + \dfrac{2}{x}$.

5. $y = \sin x - \cos x$. **6.** $y = 2 \sin x + \sin 2x$.

7. $y = 3 \sin x - \cos 2x$. **8.** $y = e^{\frac{x}{2}} + x^2$.

9. $y = \sin^2 x$. **10.** $y = \cos^2 x$.

HINT. $\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x$ and $\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x$.

11. $y = x - 1 \pm \sqrt{x}$. **12.** $y = x \pm \sqrt{4 - x^2}$.

13. $4x^2 - 4xy + y^2 - x - 2y + 6 = 0$.

14. $3x^2 - 2xy + y^2 - 6x + 2y - 5 = 0$.

15. $y = \frac{1}{2}(e^x - e^{-x})$. **16.** $y = \frac{1}{2}(e^x + e^{-x})$.

NOTE. The second members of Ex. 15 and 16 are called, respectively, the ***hyperbolic sine of*** x and the ***hyperbolic cosine of*** x. The first is denoted by the symbol $\sinh x$ and the second by $\cosh x$.

17. $y = \dfrac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}}) = a \cosh \dfrac{x}{a}$. *(The catenary)*

NOTE. A perfectly flexible, inextensible cord, suspended between two points, hangs in the form of a catenary.

Review Polar Coordinates

III. Polar Coördinates

223. Introduction. Corresponding to any given point P there are indefinitely many pairs of polar coördinates (Art. 175). If any one of these pairs satisfies the equation of a locus, then P lies on the locus. We shall find, in fact, that sometimes only one pair, and sometimes more than one pair, of coördinates of P will satisfy the equation. The first case is illustrated by the graph of the equation $r = a\theta$ (Art. 227) and the second by the line $r \sin \theta = 1$.

Frequently, also, there are two or more equations for the same curve. For example, the equations $r = 1$ and $r = -1$ define the same circle. Two equations that define the same curve are called **equivalent equations.**

224. Discussion of the Equation. As in the case of rectangular coordinates, the graph can usually be drawn more easily and more accurately from its polar equation if one first observes from the equation certain outstanding properties of the curve.

(*a*) *Symmetries.* If the equation of a curve remains unchanged, or is changed to an equivalent equation, when:

(1) θ is replaced by $-\theta$, or when (r, θ) are replaced by $(-r, \pi - \theta)$, the curve is symmetric with respect to the polar axis.

(2) θ is replaced by $\pi - \theta$, or when (r, θ) are replaced by $(-r, -\theta)$, it is symmetric with respect to the 90°-axis.

(3) θ is replaced by $\pi + \theta$, or when r is replaced by $-r$, the curve is symmetric with respect to the origin.

EXAMPLE 1. The curve $r^2(2 - \cos^2 \theta) = 8$ exhibits all of the above symmetries. This curve is an ellipse, as may be seen by writing its equation, $x^2+2y^2=8$, in rectangular coördinates.

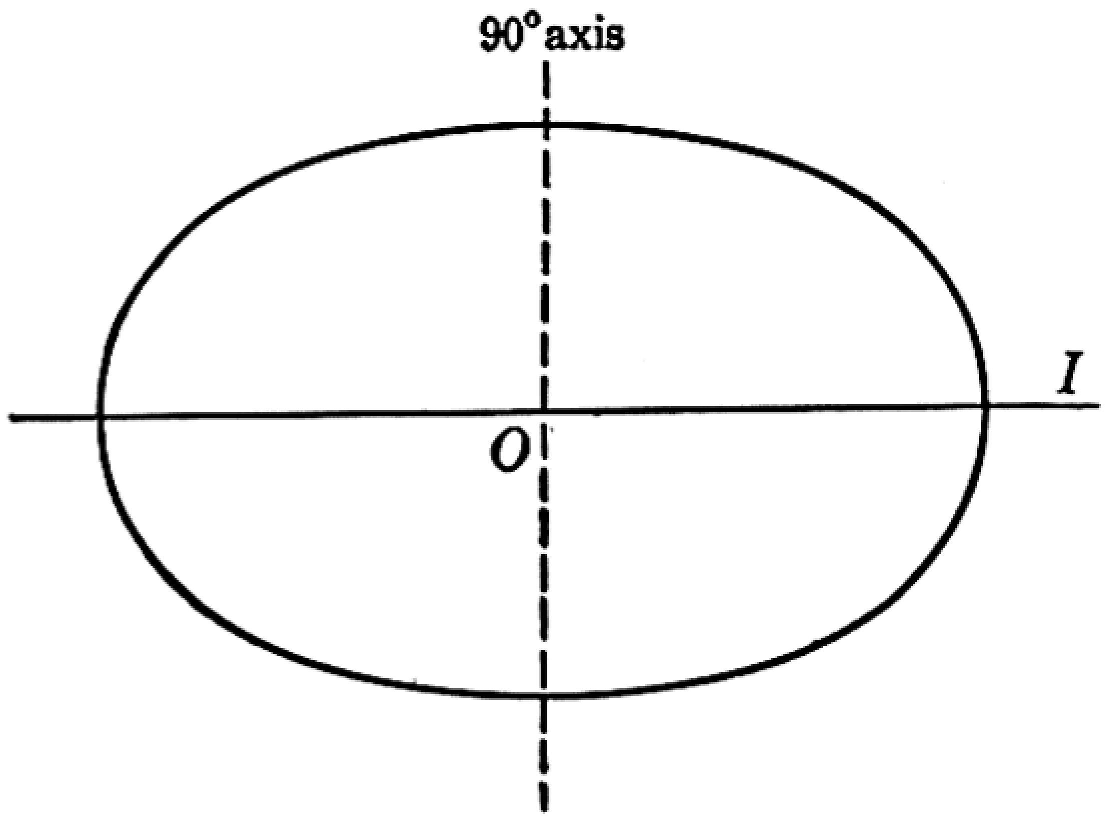


FIG. 152

(*b*) *Intercepts.* To find the intercepts on the polar axis, put $\theta = 0, \pm \pi, \pm 2\pi$, etc., and solve for r. Similarly, to find the intercepts on the 90°-axis, put $\theta = \pm \pi/2$, $\pm 3\pi/2$, etc., and solve for r. This method frequently fails to determine the intersections, if there are any, at the origin, but these intersections will be determined under (*c*).

EXAMPLE 1. The points of intersection of the ellipse $r^2(2 - \cos^2 \theta) = 8$ with the initial line are found in this way to be $(2\sqrt{2}, 0)$ and $(2\sqrt{2}, \pi)$ and its intersections with the 90°-axis are found to be $(2, \pm \pi/2)$.

(*c*) *Tangents at the origin.* If the curve passes through the origin, the angles made by its tangent line, or lines, with the initial line are found by putting $r = 0$ in the equation and solving for θ.

EXAMPLE 2. If, in the equation

$$r = a \sin 3\theta,$$

we put $r = 0$, we have $\sin 3\theta = 0$, so that $3\theta = 0, \pm \pi, \pm 2\pi$, etc. Hence, $\theta = 0, \pm \pi/3, \pm 2\pi/3$, etc. There are thus three tangent lines to this curve at the origin, of inclinations 0, $\pi/3$, and $2\pi/3$, respectively. This curve is called a *three-leaved rose* curve. It belongs to a type that we shall discuss more fully in Art. 226.

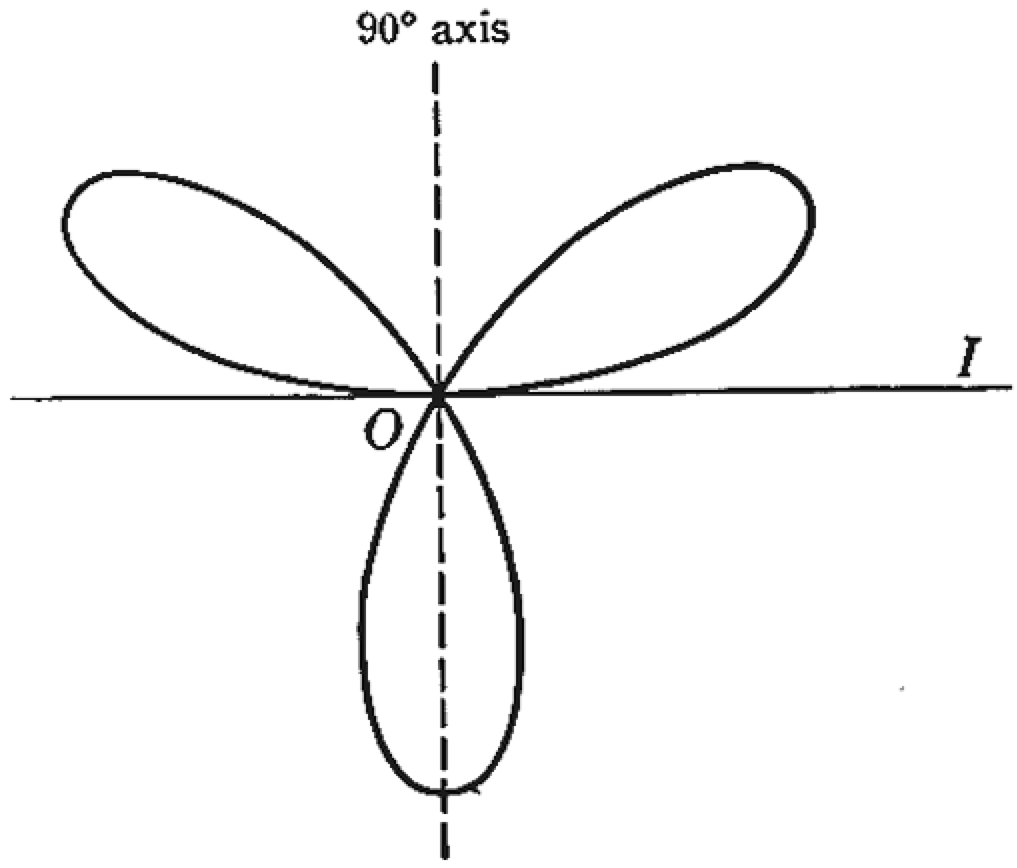


FIG. 153

(*d*) *Directions in which the curve extends to infinity.* To determine the directions in which the curve extends to infinity, we equate to zero the coefficient of the highest power of r in the given equation and solve for θ.

EXAMPLE 3. To find the directions in which the curve

$$r \cos^3 \theta = a \sin^2 \theta$$

extends to infinity, we equate to zero the coefficient of r. This gives $\cos^3 \theta = 0$, so that $\theta = \pm \pi/2, \pm 3\pi/2$, etc. The curve thus extends to infinity in the $\pm 90°$ directions.

This curve is the semi-cubical parabola $ay^2 = x^3$ (Fig. 132) which was studied in Art. 206. It has no rectilinear asymptote.

(*e*) *Excluded intervals.* It is frequently possible, by using the fact that neither $\sin \theta$ nor $\cos \theta$ is ever numerically greater than unity, to assign limits between which the numerical values of r must lie.

Thus, for the three-leaved rose (Fig. 153), the largest numerical value that r can have is a. This numerical value is attained when $\sin 3\theta = \pm 1$, that is, when $\theta = \pm \pi/6, \pm \pi/2, \pm 5\pi/6$, etc.

Similarly, for the line $r \cos \theta = 5$, the smallest numerical value that r can have is 5. This value is reached when $\cos \theta = \pm 1$, that is, when $\theta = 0$, $\pm \pi, \pm 2\pi$, etc.

For the ellipse $r^2(2 - \cos^2 \theta) = 8$ (Fig. 152), we find in a similar way that the largest numerical value that r can have is $2\sqrt{2}$, and the smallest is 2.

In the case of the semi-cubical parabola, $r \cos^3 \theta = a \sin^2 \theta$, $(a > 0)$ (Fig. 132), since $\sin^2 \theta$ cannot be negative, r and $\cos \theta$ must agree in sign. It follows that this curve cannot extend into the second or third quadrants.

(*f*) *Transformation to rectangular coördinates.* It frequently happens that the equation of the curve in rectangular coördinates is one with

which the student is already familiar, or from which it is easier to determine the properties of the equation than it is from its polar equation. We found that the locus of the equation $r^2(2 - \cos^2 \theta) = 8$, for example, was an ellipse by finding its rectangular equation. Similarly, it is usually easier to recognize that the locus of the equation $r \cos \theta = 5$ (or $r = 5 \sec \theta$) is a line from its rectangular equation $x = 5$ than it is from either of the polar forms.

In any event, any information about the curve that is obtained from the discussion of its equation in rectangular coördinates, or by plotting points on it from its rectangular equation, must hold for the required graph.

Equally, if the equation is given to us in rectangular coördinates, it may be possible to simplify the problem of drawing the curve by finding its equation in polar coördinates. It is thus good practice, when the equation of a curve is given to us either in rectangular or in polar coördinates, to discuss its equation in both systems of coördinates before attempting to draw the graph.

(g) *Use of the laws of variation of the trigonometric functions.* If the given equation defines r as equal to a simple expression in terms of the trigonometric functions of θ, a fairly accurate preliminary sketch of the curve can often be obtained quickly by observing how these functions change as θ increases. For this purpose, the graphs of the trigonometric curves in Art. 116 will be found quite helpful. The preliminary sketch obtained in this way may then be corrected by discussing the equation and plotting points on the curve.

EXAMPLE 4. Sketch the curve $r = a \tan \theta$.

From the properties of the tangent function, we find at once that $r = 0$ when $\theta = 0$, that it increases to a when $\theta = \pi/4$, and that it then increases indefinitely as θ increases to $\pi/2$. By following the variation of $\tan \theta$ through the four quadrants, a fairly accurate rough sketch of the curve may thus be quickly obtained. By a more careful discussion of the polar equation, and of the corresponding rectangular equation $x^4 + x^2y^2 = a^2y^2$, and by plotting a number of points on the curve, we obtain Figure 154. The lines $x = \pm a$ are asymptotes. This curve is called the *kappa curve* from its supposed resemblance to the Greek letter kappa.

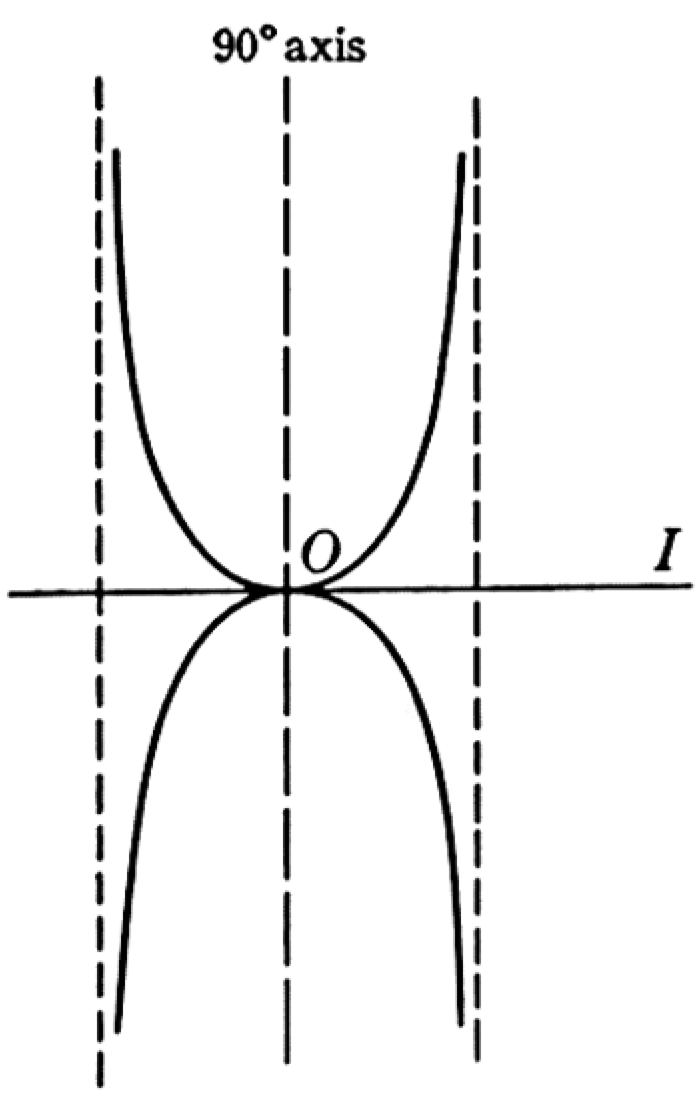


FIG. 154

225. Drawing the Graph. To draw the graph of a polar equation, one should first discuss the equation by the methods outlined in the preceding article, then plot a suitable number of points on the curve, and draw a smooth curve through these points.

EXAMPLE 1. Discuss the equation $r^2 = a^2 \cos 2\theta$ and draw the curve. (**The Lemniscate**)

The curve is symmetric with respect to the polar axis and the 90°-axis. It intersects the polar axis at $(\pm a, 0)$ and these are the points on the graph farthest from the origin. It passes through the origin and touches, at that point, the lines making angles $\pm \pi/4$ and $\pm 3\pi/4$ with the initial line. The radius vector r is imaginary if $\cos 2\theta$ is negative; that is, if $\pi/4 < \theta < 3\pi/4$, or $5\pi/4 < \theta < 7\pi/4$, etc. By plotting several points corresponding to values of θ in the interval 0 to $\pi/4$, and using the properties of symmetry, we obtain Figure 155.

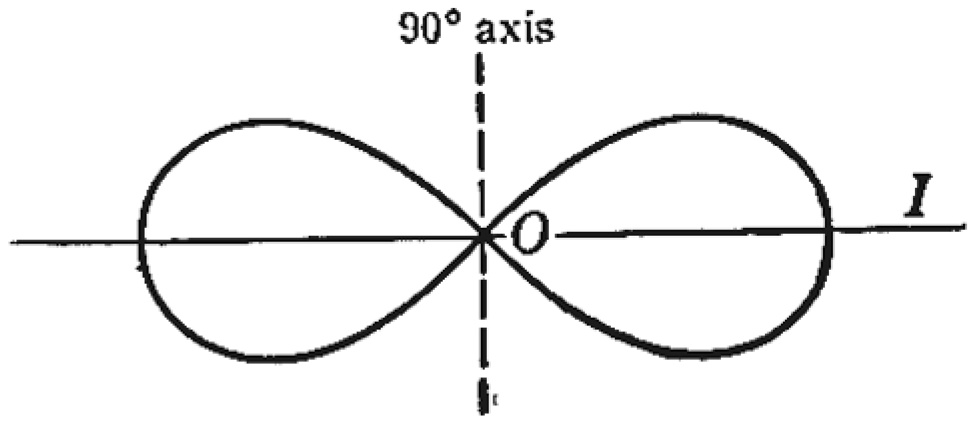


FIG. 155

EXAMPLE 2. Discuss the equation $r = a(1 - \cos \theta)$ and draw the curve. (**The Cardioid**)

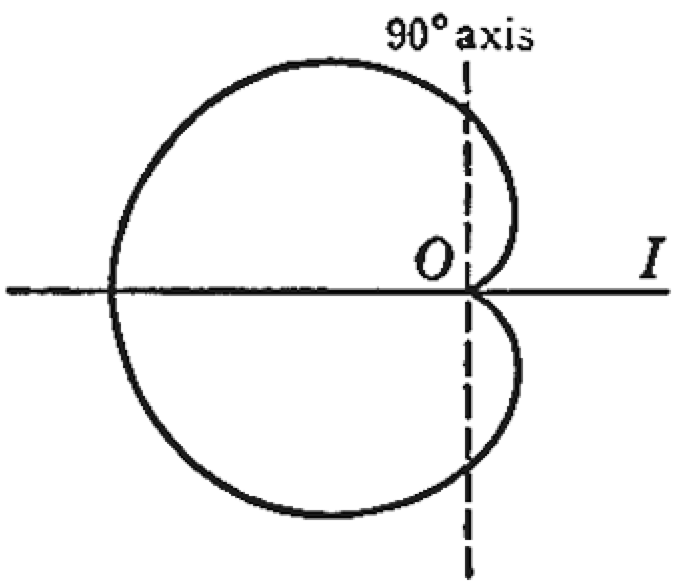


FIG. 156

This curve is symmetric with respect to the polar axis. It is tangent to the initial line at $\theta = 0$. The point on it that lies farthest from the origin is $(2a, \pi)$. It crosses the 90°-axis at $(a, \pm \pi/2)$. By plotting points corresponding to values of θ from 0 to π, and using symmetry, we obtain Figure 156.

EXAMPLE 3. Discuss the equation $r = a \csc \theta \pm b$ and draw the curve. (**The Conchoid**)

There are three cases according as $a \lesseqgtr b$. In the following discussion, and in the figure, we have taken $a < b$. The discussion of the other two cases is left as an exercise for the student.

The curve is symmetric with respect to the 90°-axis and intersects it at the origin and at the points $r = a \pm b$. It intersects the initial line only at the origin, at which point its tangents make angles $\theta = \csc^{-1} (\pm b/a)$ with the initial line. It extends to infinity in such a way that each of its two branches approaches the horizontal line $r = a \csc \theta$ as an asymptote.

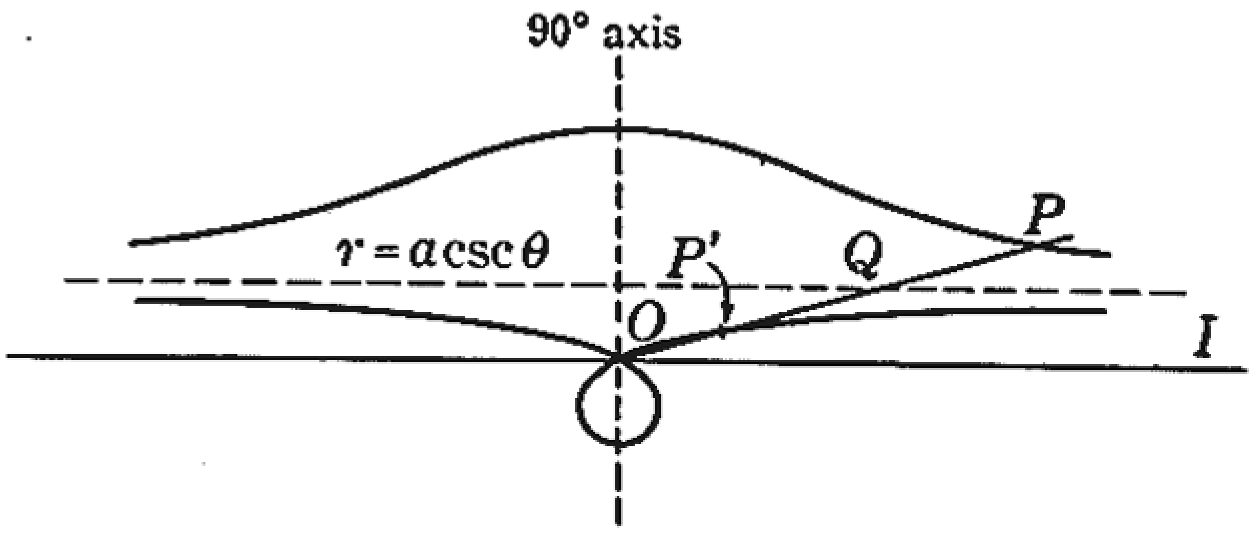


FIG. 157

This curve may be constructed by points in the following way: Draw the line $r = a \csc \theta$ and let Q be any point on it. Draw the line through O and Q and on it lay off, in opposite directions, the segments $QP = QP' = b$. Then the locus of the points P and P' is the conchoid.

226. The Rose Curves. The curves defined by the equations

$$r = a \cos n\theta \quad \text{and} \quad r = a \sin n\theta,$$

where n is a positive integer, are called **rose curves.** Each loop extending out from the origin is a "leaf" of the rose. If n is an odd integer, the number of leaves is n but, if n is an even integer, the number of leaves is $2n$.

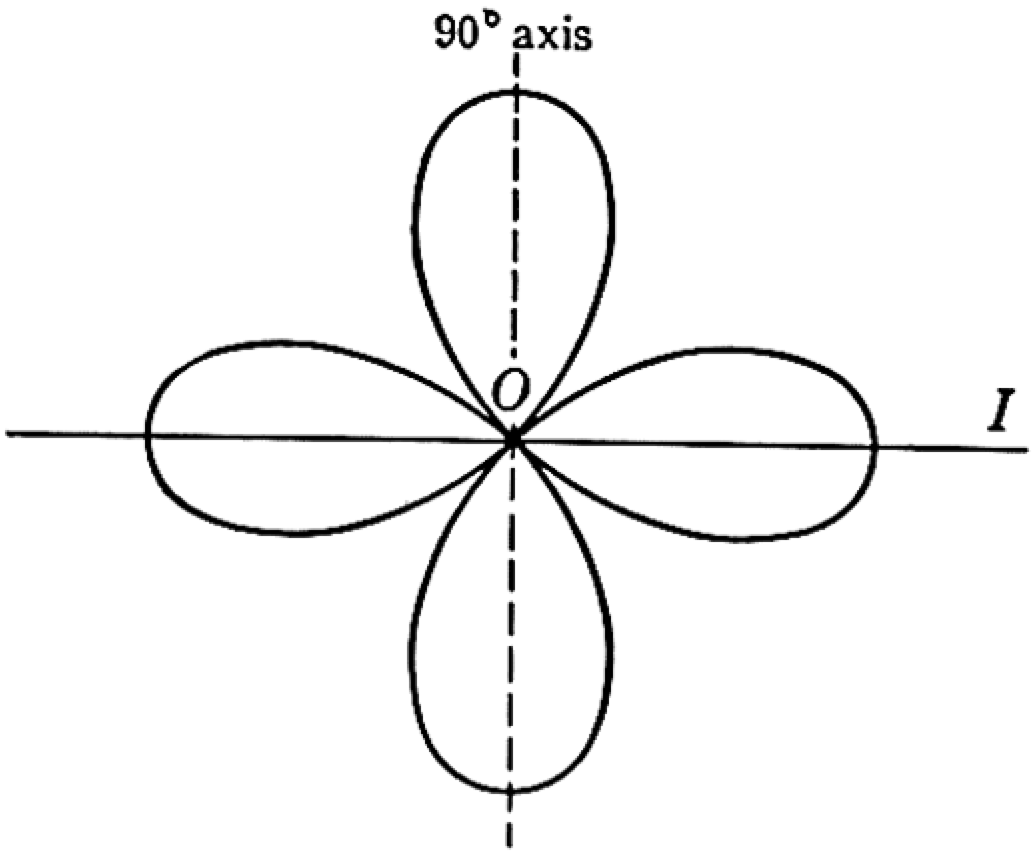


FIG. 158

The three-leaved rose, $r = a \sin 3\theta$, is shown in Figure 153.

EXAMPLE. Discuss the equation $r = a \cos 2\theta$ and draw the curve. **(The Four-Leaved Rose)**

The curve is symmetric with respect to the polar axis and the 90°-axis. It intersects these axes at $(a, 0)$, $(-a, \pi/2)$, (a, π) and $(-a, 3\pi/2)$ and these are the points on the curve farthest from the origin. The tangent lines at the origin are defined by $\theta = \pm \pi/4, \pm 3\pi/4$, etc.

227. The Spirals. If a curve, or one of its branches, winds indefinitely many times about the origin in such a way that r increases (or decreases) continuously as θ increases or decreases continuously, then the curve is called a **spiral.**

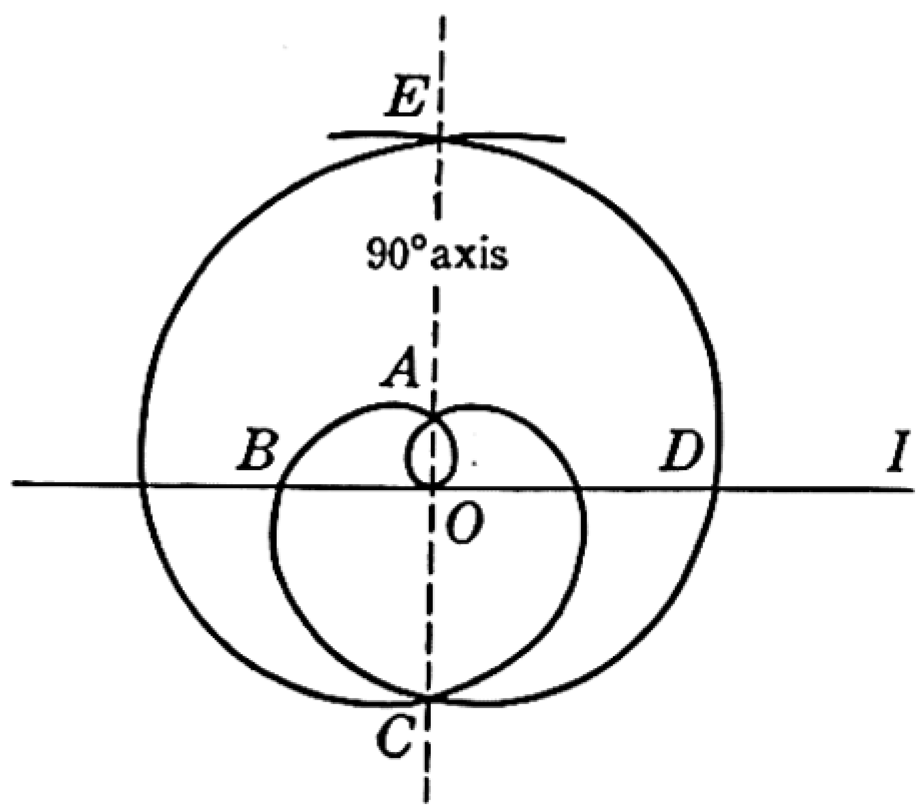


FIG. 159

EXAMPLE 1. Discuss the equation $r = a\theta$ and draw the curve. **(The Spiral of Archimedes)**

The curve is symmetric with respect to the 90°-axis since, if (r, θ) lies on the curve, so, also, does $(-r, -\theta)$. It touches the polar axis at the origin and the rate of increase of r is proportional to that of θ. It meets the axes at $(0, 0)$, $(a\pi/2, \pi/2)$, $(-a\pi/2, \pi/2)$, etc.

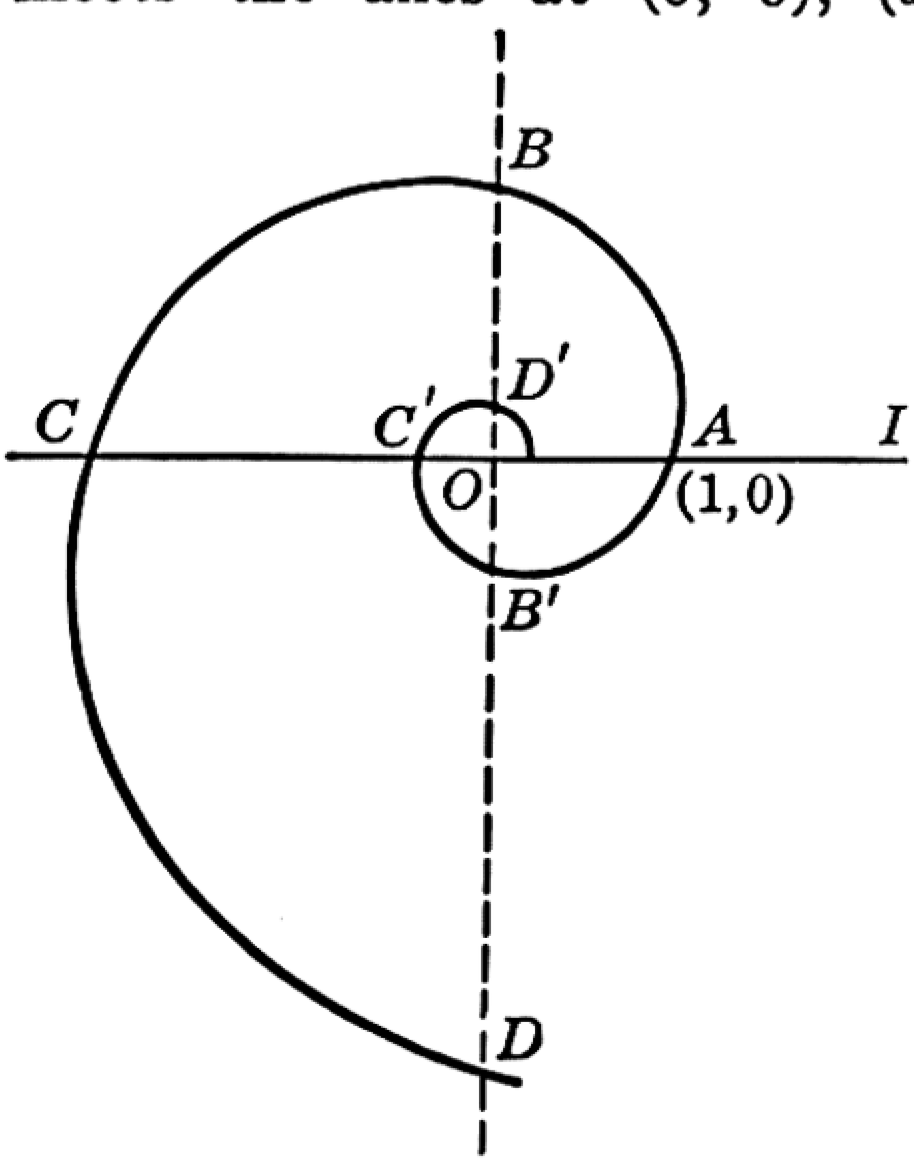


FIG. 160

EXAMPLE 2. Discuss the equation $r = e^{a\theta}$ (or $\log_e r = a\theta$) and draw the curve. **(The Logarithmic Spiral)**

For $\theta = 0$, we have $r = 1$. As θ increases from zero, r increases more and more rapidly and becomes very large as θ increases indefinitely. If θ decreases from zero, r decreases more and more slowly and approaches zero as θ decreases indefinitely.

To plot points on this curve, take the logarithms to the base 10 of both sides of the equation. This gives

$$\log_{10} r = a\theta \log_{10} e.$$

We can now assign values to r and determine the corresponding values of θ with the aid of a table of logarithms.

Exercises

Discuss the following equations and draw the curves. If just one literal constant appears in the equation, assign to it a convenient positive value. If two constants, a and b, appear, consider the three cases $a < b$, $a = b$, and $a > b$. State the name of the curve if you know it.

1. $r \cos \theta = a$.
2. $r = a \cos \theta$.
3. $r^2 \sin 2\theta = a^2$.
4. $r^2 = a^2 \sin 2\theta$.
5. $r^2(4 - \sin^2 \theta) = 12$.
6. $r^2(1 - 4 \sin^2 \theta) = 12$.
7. $r(1 + \sin \theta) = a$.
8. $r = a(1 + \sin \theta)$.
9. $r = a \cos 3\theta$.
10. $r = a \sin 2\theta$.
11. $r = a \cos 4\theta$.
12. $r = a \sin 5\theta$.
13. $r = a - b \sin \theta$. (*Limaçon*)
14. $r = a + b \cos \theta$. (*Limaçon*)
15. $r = a \sec \theta \pm b$. (*Conchoid*)
16. $r^2\theta = a^2$. (*Lituus*)
17. $r^2 \cos^3 \theta = a^2 \sin \theta$. (*Cubical parabola*)
18. $r^2 = a^2\theta$. (*Parabolic spiral*)
19. $r\theta = a$. (*Hyperbolic, or reciprocal, spiral*)
20. $r^2 = a^2 \cos \theta$.
21. $r^2 \cos \theta = a^2$.
22. $r = a \cos (\theta/2)$.
23. $r = a \sin (\theta/3)$.

EVEN INTEGER GIVES A ROSE PEDEL 2 times number

odd integer give a rose pedel the sa
number of pedal

Chapter 28

Parametric Equations

228. Introduction. Instead of representing a curve by a single equation connecting x and y, it is sometimes preferable to use two equations which express the coördinates of the points on the curve in terms of a third variable. This third variable is called a **parameter** and the two equations which express x and y in terms of this parameter are **parametric equations** of the curve.

229. Parametric Equations of the Circle. Let there be given a circle with center at the origin and radius a (Fig. 161). Let $P(x, y)$ be any point on this circle and denote the angle XOP by ϕ. From the figure, we obtain

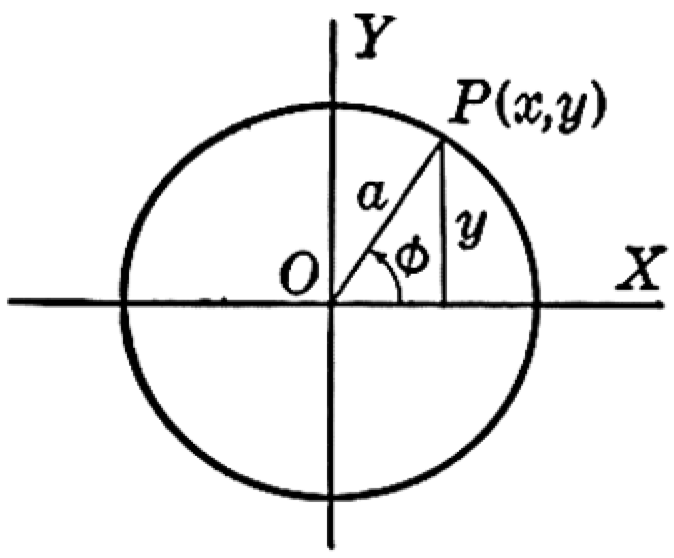


FIG. 161

$$x = a \cos \phi, \qquad y = a \sin \phi. \tag{1}$$

These two equations, which express the coördinates of any point P on the circle in terms of the parameter ϕ, are *parametric equations of the given circle.*

If the parametric equations of a curve are given, the rectangular equation may be found by eliminating the parameter between the two given equations. Thus, from (1), if we square both members of each equation, add, and simplify the result, we have

$$x^2 + y^2 = a^2, \tag{2}$$

which is the rectangular equation of the given circle.

If the rectangular equation of the curve is given, however, various parametric equations can be found for it, depending on the choice of the parameter. For the circle (2), for example, we may choose as the parameter the slope m of the line through any point P on the circle and the fixed point $(-a, 0)$. The equation of this line is $y = m(x + a)$. Since P lies on this line and also on the circle, we can find the coördinates of P in terms of m by solving the equation of the line and the equation (2) of the circle as simultaneous. The result is

$$x = a\frac{1 - m^2}{1 + m^2}, \qquad y = a\frac{2m}{1 + m^2}. \tag{3}$$

These two equations, also, constitute a pair of parametric equations of the circle (2).

230. Parametric Equations of the Ellipse. Any point whose coördinates satisfy the parametric equations,

$$\begin{aligned} x &= a \cos \phi, \\ y &= b \sin \phi, \end{aligned} \tag{4}$$

wherein ϕ is the parameter, lie on an ellipse. For, if we multiply the first equation by b, the second by a, square, add, and simplify, we get

$$b^2x^2 + a^2y^2 = a^2b^2, \qquad (5)$$

which is the equation of an ellipse.

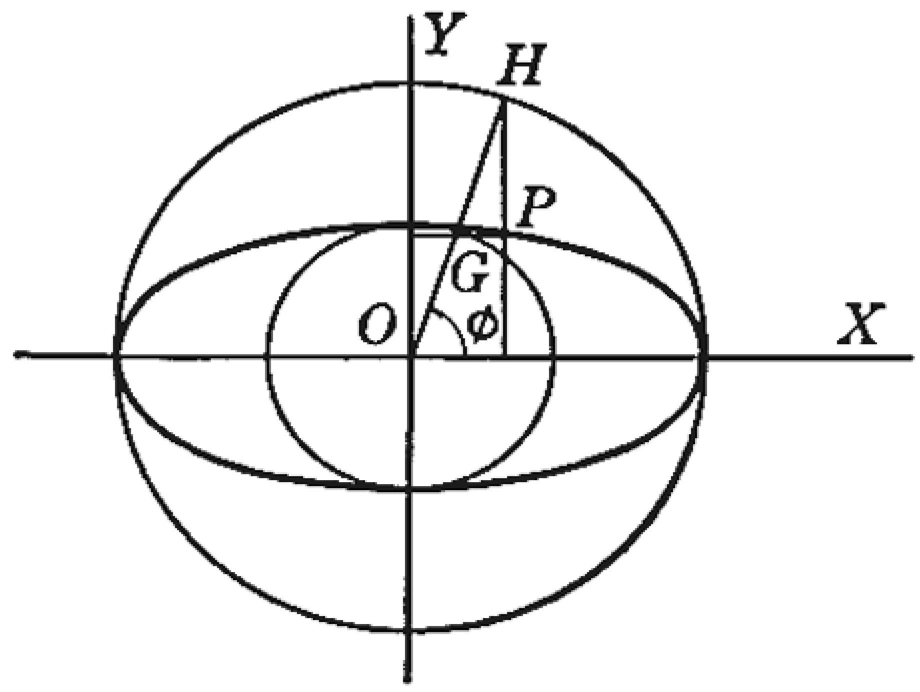


FIG. 162

The circle with center at the origin and radius a is the *major auxiliary circle* of the ellipse (5). By the preceding article, its parametric equations are

$$x = a \cos \phi, \quad y = a \sin \phi. \qquad (6)$$

The circle with center at the origin and radius b is the *minor auxiliary circle* of the ellipse. Its equations are, similarly,

$$x = b \cos \phi, \qquad y = b \sin \phi. \qquad (7)$$

From equations (4), (6), and (7), we deduce the following device for plotting points on the ellipse (5): Draw through O any half-line intersecting the minor auxiliary circle at G and the major at H. Draw through G and H lines parallel to OX and OY, respectively. The point P of intersection of these lines lies on the ellipse. For, denote the angle XOH by ϕ. Since the abscissa of P equals that of H, and the ordinate of P equals that of G, it follows that the coördinates of P are

$$x = a \cos \phi, \qquad y = b \sin \phi.$$

Hence, from (4), P lies on the ellipse.

231. Path of a Projectile. If a projectile is fired from the origin with an initial velocity v_0, in a direction making an angle α with OX, and if it moves subject only to the attraction of gravitation, it is shown in the textbooks on physics that its position at the end of t seconds is given by the equations

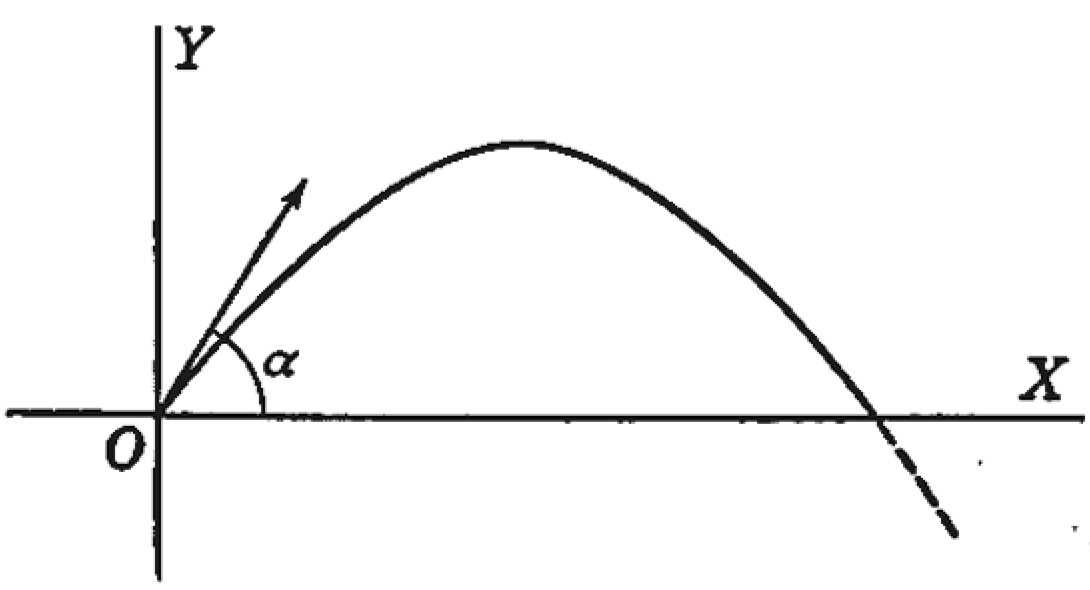


FIG. 163

$$x = tv_0 \cos \alpha, \quad y = tv_0 \sin \alpha - \tfrac{1}{2}gt^2,$$

where g is a constant. These two equations constitute the parametric equations of the path of the projectile in terms of the parameter t.

To find the rectangular equation of the path, we solve the first equation for t, substitute in the second equation, and simplify. The result is

$$y = x \tan \alpha - \frac{gx^2}{2v_0{}^2} \sec^2 \alpha.$$

This equation defines the path of the projectile but it does not tell where the body is in its path at a given time. The parametric equations not only define the path but also state the law according to which the projectile traverses its path.

Exercises

Draw the following curves by assigning values to the parameter, plotting the corresponding points, and drawing a smooth curve through them. Find also the rectangular equation by eliminating the parameter.

1. $x = 2t,\ y = 5 + t.$ **2.** $x = 2pt^2,\ y = 2pt.$

3. $x = a \sec \phi,\ y = b \tan \phi.$ **4.** $x = 2 \cos \phi - 1,\ y = 3 \sin \phi - 2.$

5. $x = t^2 + 3t,\ y = t^2 + t.$ **6.** $x = t + 1/t,\ y = t - 1/t.$

7. $x = \dfrac{t+3}{t},\ y = \dfrac{t-1}{t+3}.$ **8.** $x = \dfrac{t}{t^2+2},\ y = \dfrac{t^2}{t^2+2}.$

9. $x = at,\ y = at^3.$ **10.** $x = at^2,\ y = at^3.$

11. $x = t^2,\ y = (\sqrt{a} - t)^2.$ **12.** $x = a \cos^3 \phi,\ y = a \sin^3 \phi.$

13. $x = \dfrac{3am}{1+m^3},\ y = \dfrac{3am^2}{1+m^3}.$ **14.** $x = \dfrac{(2-m)m^2}{1-m},\ y = \dfrac{(2-m)m^3}{1-m}.$

15. Show that $x = a + bt,\ y = c + dt$ are parametric equations of a line and find its slope.

16. Find parametric equations for the cissoid (Art. 219, Ex. 2) by finding its intersections with the line $y = mx$ through the origin.

17. A projectile is fired from the origin with an initial velocity of 3000 feet per second and at an angle of 35°. Find, to three significant figures, the abscissa of the point where it strikes the x-axis and the time when it arrives at that point. Take $g = 32$ feet per second.

18. A crank OA, 9 inches long, turns around the origin at the rate of 4 radians per second. A rod AB, 27 inches long, has one end attached at A while the other end slides along the x-axis. Using the time as parameter, find the equations of the path of a point on the rod $3k$ units from A.

232. The Cycloid. *The path of a point fixed on the circumference of a circle that rolls along a fixed line is a* **cycloid.**

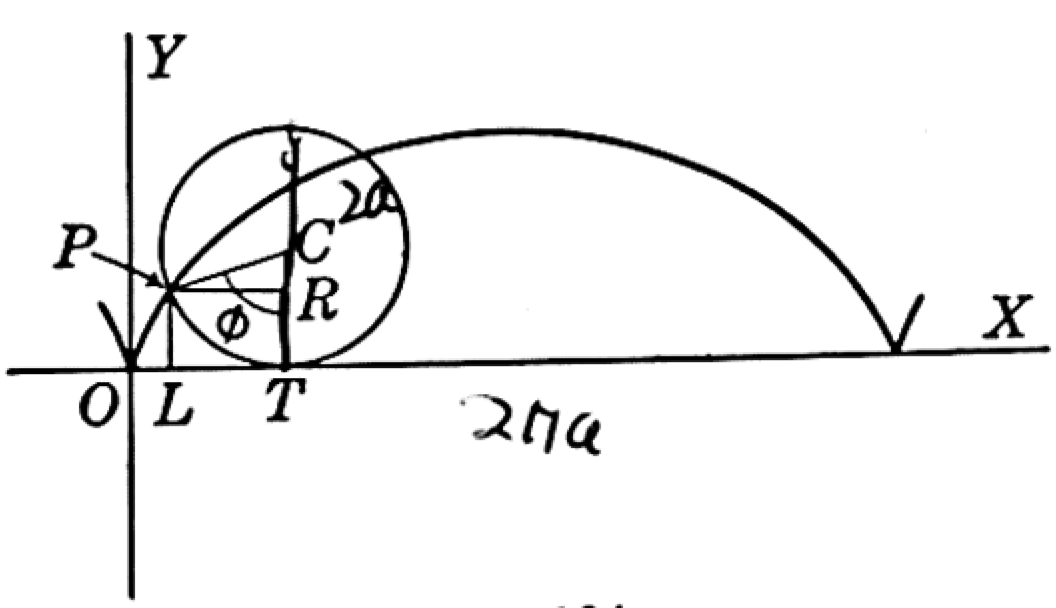


Fig. 164

We shall find the parametric equations of the cycloid when the fixed line on which the circle rolls is taken as the x-axis and any one of the positions at which the tracing point comes in contact with this line is taken as origin.

Let a be the radius of the rolling circle, $P(x, y)$ be any position of the tracing point, and let ϕ be the number of *radians* in the angle through

which the circle has rolled from its position when the tracing point was at the origin.

From the figure,

$$x = OL = OT - LT, \qquad y = LP = TR = TC - RC. \tag{8}$$

Since the circle has rolled from O to T, OT = arc TP and, since ϕ is measured in radians, arc $TP = a\phi$. Hence $OT = a\phi$. Also, $LT = PR = a \sin \phi$, $RC = a \cos \phi$, and $TC = a$.

On making all these substitutions in equations (8), we have, as the required parametric equations of the cycloid in terms of ϕ as parameter,

$$\begin{aligned} x &= a(\phi - \sin \phi), \\ y &= a(1 - \cos \phi). \end{aligned} \tag{9}$$

To find the rectangular equation of the cycloid, first solve the second of equations (9) for $\cos \phi$, giving $\cos \phi = (a - y)/a$. From this equation, we find the values of ϕ and $\sin \phi$ and substitute in the first equation (9). The result is

$$x = a \cos^{-1} \frac{a - y}{a} \pm \sqrt{2ay - y^2}.$$

For most practical purposes, this equation is less convenient than the parametric equations (9).

Exercises

1. Taking values of ϕ at intervals of $\pi/4$ radians, sketch one arch of the cycloid.

2. Find the length of the base and the coördinates of the highest point of one arch of the cycloid. 2π

3. If the abscissas of two points on the cycloid differ by $2\pi a$, show that their ordinates are equal and interpret geometrically.

4. Find the parametric equations of the cycloid when the origin is translated to the top of an arch.

5. If the tracing point P lies on a fixed radius (or radius produced) of the rolling circle, at a distance $b \neq a$ from the center, show that the equations of its path are

$$x = a\phi - b \sin \phi, \qquad y = a - b \cos \phi.$$ Learn this

If $b > a$, this curve is a **prolate cycloid**; if $b < a$, it is a **curtate cycloid**. In either case, it is also called a **trochoid**.

6. Draw the graph of the curtate cycloid $x = 10\phi - 5 \sin \phi$, $y = 10 - 5 \cos \phi$.

7. Draw the graph of the prolate cycloid $x = 10\phi - 15 \sin \phi$, $y = 10 - 15 \cos \phi$.

8. A circle of radius a feet rolls along a line at the rate of b radians per second. At the instant a certain radius extends vertically downward, a particle

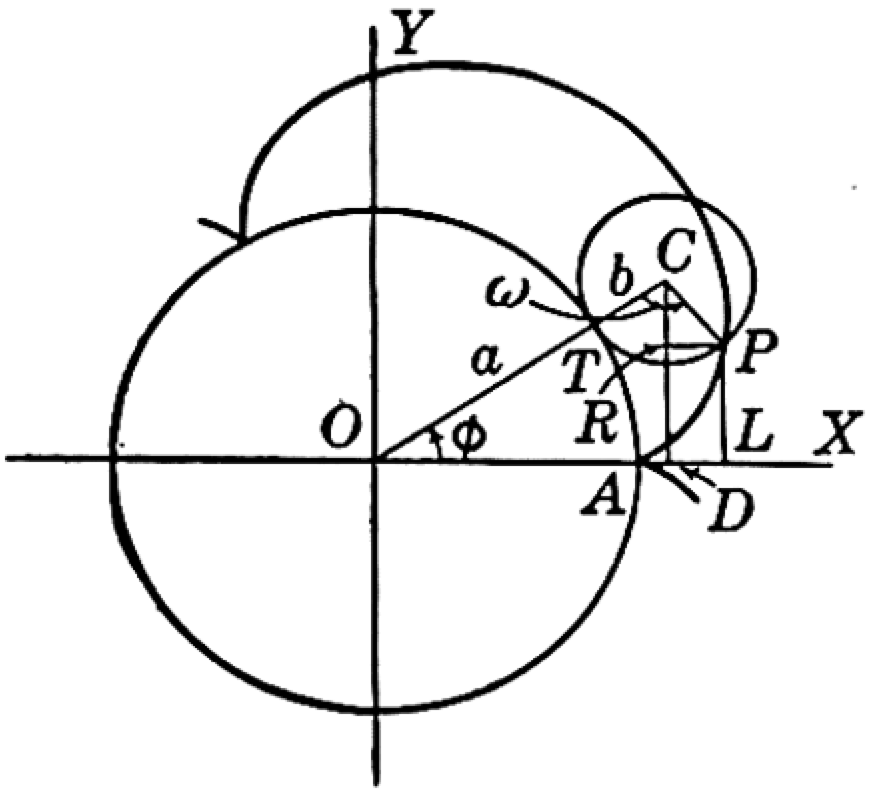


FIG. 165

starts from the center along that radius at the rate of c feet per second. Find the path of the particle.

233. The Epicycloid. *The path of a point fixed on the circumference of a circle that rolls tangent externally to a fixed circle is an* **epicycloid.**

Denote the radius of the fixed circle by a, of the rolling circle by b, and take the coördinate axes as shown in Figure 165. Denote the angle XOC by ϕ (radians) and the angle OCP by ω. Then

angle $DCO = \pi/2 - \phi$, and angle $DCP = \omega -$ angle $DCO = \phi + \omega - \pi/2$.

$$\text{Also,}\quad x = OL = OD + DL = OD + RP,$$
$$\text{and}\quad y = LP = DR = DC - RC. \tag{10}$$

Since $OC = OT + TC = a + b$, we have

$$OD = (a + b)\cos\phi, \qquad RP = b\sin(DCP) = -b\cos(\phi + \omega).$$
$$DC = (a + b)\sin\phi, \qquad RC = b\cos(DCP) = b\sin(\phi + \omega).$$

On making these substitutions in equations (10), we have

$$x = (a + b)\cos\phi - b\cos(\phi + \omega),$$
$$y = (a + b)\sin\phi - b\sin(\phi + \omega). \tag{11}$$

Since the outside circle rolls on the fixed one, arc $AT =$ arc PT; that is, $a\phi = b\omega$, or $\omega = a\phi/b$.

On substituting this value of ω in equations (11), we have

$$\mathbf{x = (a + b)\cos\phi - b\cos\frac{a + b}{b}\phi,}$$
$$\mathbf{y = (a + b)\sin\phi - b\sin\frac{a + b}{b}\phi.} \tag{12}$$

These are the parametric equations of the epicycloid.

234. The Hypocycloid. *The path of a point fixed on the circumference of a circle that rolls tangent internally to a fixed circle is an* **hypocycloid.**

The derivation of the equations of the hypocycloid, which parallels that of the epicycloid, is left as an exercise for the student. The resulting equations are

$$\mathbf{x = (a - b)\cos\phi + b\cos\frac{a - b}{b}\phi,}$$
$$\mathbf{y = (a - b)\sin\phi - b\sin\frac{a - b}{b}\phi.} \tag{13}$$

It should be observed that these equations differ from those of the epicycloid only in that b is replaced by $-b$.

Know rec and parametr equation of this.

In Figure 166, we have taken $b = a/4$. This curve is of special interest and is called the **four-cusped hypocycloid.**

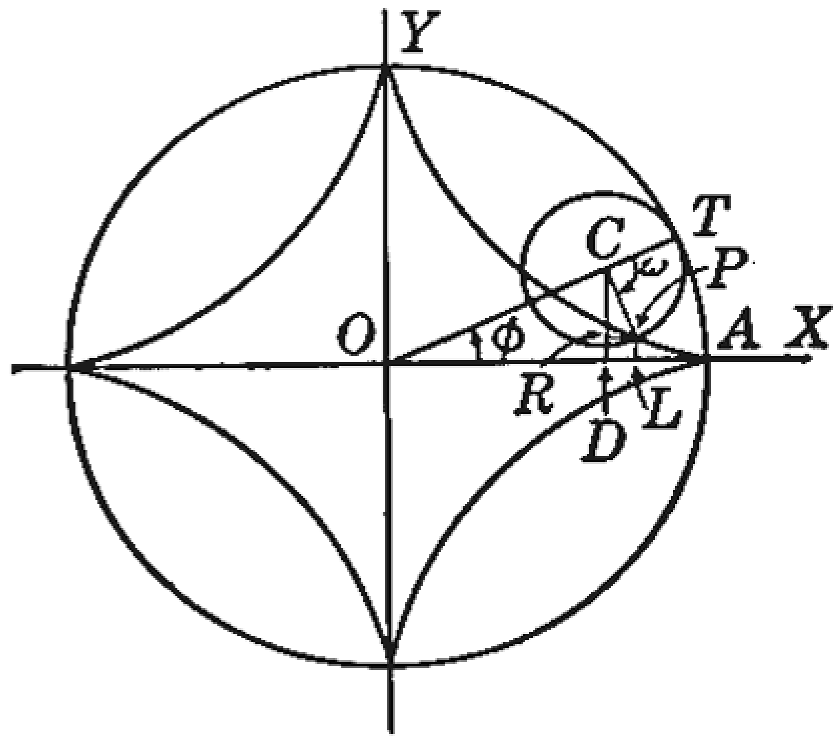


FIG. 166

If we put $b = a/4$ in equations (13) and simplify by means of the trigonometric identities

$$\cos 3\phi = 4\cos^3\phi - 3\cos\phi, \qquad \sin 3\phi = 3\sin\phi - 4\sin^3\phi,$$

we obtain
$$x = a\cos^3\phi, \qquad y = a\sin^3\phi,$$

as the parametric equations of the curve.

By eliminating ϕ between these equations, we get

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}},$$

as the rectangular equations of the four-cusped hypocycloid.

Exercises

1. Sketch the epicycloids, given:

(*a*) $a = 3b$; (*b*) $a = 2b$; (*c*) $a = b$; (*d*) $3a = 2b$.

2. Sketch the hypocycloids, given:

(*a*) $a = 3b$; (*b*) $a = 2b$; (*c*) $2a = 3b$.

3. If a thread is unwound from around a fixed circle, and is held taut in the plane of the circle, any point fixed on the thread will describe a curve called an *involute of the circle*. Take the center of the circle as origin, the radius as a, and let the generating point start from $(a, 0)$. Derive the equations of the curve in the form

$$x = a(\cos\phi + \phi\sin\phi), \qquad y = a(\sin\phi - \phi\cos\phi).$$

Chapter 29

Progressions

235. Sequences. The Continuation Notation. A set of numbers, arranged in a definite order, is called a **sequence** of numbers. The numbers themselves are the **terms** of the sequence and are spoken of as the first term, the second term, and so on according to their position in the sequence.

Thus, $$2, 5, 8, 11, 14, 17, 20, 23$$

is a sequence in which each term after the first is obtained by adding 3 to the preceding term. The first term of this sequence is 2, the second is 5, the third is 8, and so on.

Similarly, $$3, 6, 12, 24, 48, 96$$

is a sequence in which each term after the first is formed by multiplying the preceding term by 2.

The sequence $$1, 4, 9, 16, 25, 36, 49$$

is formed by squaring the number representing the position of the term in the sequence.

It will frequently be impracticable to write out all of the terms of the sequence under consideration. In such cases, we shall write out a few of the terms at the beginning, to show the law of formation of the terms; then insert several dots to indicate that the remaining terms are to be formed according to the indicated law. The last term of the sequence may, or may not, be written after the dots. This notation for a sequence is called the **continuation notation.**

Thus, the illustrative sequences just given would be written, in the continuation notation, respectively as follows

$$2, 5, 8, \cdots, 23,$$
$$3, 6, 12, \cdots, 96,$$

and $$1, 4, 9, \cdots, 49.$$

The dots, in this notation, are read, "and so on." Thus, the first illustration should be read, "two, five, eight, and so on to twenty-three."

Frequently we shall need the sum or the product of the terms of a sequence. To indicate the sum of the terms of the first illustrative sequence in the continuation notation, we write

$$2 + 5 + 8 + \cdots + 23.$$

Their product is written $$2 \cdot 5 \cdot 8 \cdots 23.$$

Exercises

Using the continuation notation, write each of the following sequences, the sum of its terms, and the product of its terms.

1. The odd integers from 1 to 21.
2. The integers divisible by three from 12 to 69.
3. The reciprocals of the integers from 5 to 19.
4. The fractions whose denominators are the integers from 2 to 31 and whose numerators are the squares of the numbers one less than the denominators.
5. The powers of 2 from 2 to 2^n.
6. The positive square roots of the integers from 1 to n.
7. The cubes of the integers from 3 to $n + 2$.
8. Write in full, without using dots, the sum of the terms of the sequence in Ex. 7 for (*a*) $n = 6$, (*b*) $n = 3$, and (*c*) $n = 1$.
9. Write the sum of the sequence in Ex. 5 in full, without using dots, and find its value, given (*a*) $n = 7$, (*b*) $n = 4$, (*c*) $n = 2$.
10. Find the value of the product $1 \cdot 2 \cdot 3 \cdots n$, given (*a*), $n = 6$; (*b*) $n = 4$, (*c*) $n = 1$.

I. Arithmetic Progressions

236. Definitions. A sequence of numbers is an **arithmetic progression** if each term after the first is obtained by adding to the preceding term a fixed number which is called the **common difference.**

We shall denote an arithmetic progression by the abbreviation A.P.

Thus, $$1, 5, 9, 13, 17, \cdots$$

is an A.P. with its first term equal to 1 and its common difference equal to 4.

Similarly, $$11, 6, 1, -4, -9, \cdots$$

is an A.P. in which the first term equals 11 and the common difference equals -5.

Since any desired number of terms may be taken in these sequences, we have not specified the last term.

237. The Elements. We shall denote the first term of an A.P. by a and the common difference by d. We may then write the first n terms of the progression in the form

$$a, a + d, a + 2d, \cdots, a + (n - 1)d. \qquad (1)$$

Associated with these first n terms of the A.P., there are five numbers of special importance which are called its **elements.** They are:

a, *the first term,*
d, *the common difference,*
n, *the number of terms,*
l, *the nth term,*
and s, *the sum of the first n terms.*

238. Equations Connecting the Elements. These five elements are connected by two equations.

From the expressions (1) of the preceding article, we notice that the coefficient of d in the second term is 1, in the third term is 2, and, in general, *in any term the coefficient of d is one less than the number of the term.* For the nth term, which we denote by l, we have, therefore,

$$\boldsymbol{l = a + (n - 1)d.} \tag{2}$$

To find s, the sum of the first n terms, we have, by definition,

$$s = a + (a + d) + (a + 2d) + \cdots + l.$$

Write the numbers in the second member of this equation in the reverse order:

$$s = l + (l - d) + (l - 2d) + \cdots + a.$$

Add corresponding terms in these two equations:

$$2s = (a + l) + (a + l) + (a + l) + \cdots + (a + l),$$

or, since there are n terms in the second member, each equal to $a + l$,

$$2s = n(a + l).$$

Hence
$$\boldsymbol{s = \frac{n(a + l)}{2}.} \tag{3}$$

If, in equation (3), we replace l by its value from (2), we have

$$\boldsymbol{s = \frac{n[2a + (n - 1)d]}{2}.} \tag{4}$$

If we know the values of any three of the five elements of an A.P., we can find the values of the other two by means of equations (2), (3), *and* (4). The number n, however, must always be a positive integer.

EXAMPLE 1. Find the eighth term, and the sum of the first eight terms, of the A.P.: 25, 21, 17, $\cdots$.

We have $a = 25$, $n = 8$, $d = 21 - 25 = -4$.

From equation (2): $l = 25 + (8 - 1)(-4) = -3$.

From equation (3): $s = \dfrac{8(25 - 3)}{2} = 88$.

EXAMPLE 2. Given $a = 2$, $l = 35$, $s = 222$, find n and d and write the first five terms of the progression.

From equation (3): $222 = \dfrac{n(2 + 35)}{2}$, or $444 = 37n$. Hence $n = 12$.

From equation (2): $35 = 2 + (12 - 1)d$, or $33 = 11d$. Hence $d = 3$.

Since $a = 2$ and $d = 3$, the first five terms of the progression are 2, 5, 8, 11, 14.

EXAMPLE 3. The fourth term of an A.P. is -1 and the sixteenth term is 3. Find the thirty-first term.

From equation (2), when $n = 4$, $-1 = a + (4 - 1)d$ and, when $n = 16$, $3 = a + (16 - 1)d$. By solving these two linear equations

$$a + 3d = -1,$$
$$a + 15d = 3,$$

for a and d, we find that $a = -2$, $d = \frac{1}{3}$.

To find the thirty-first term, we put these values a and d in equation (2) and put $n = 31$. We have

$$l = -2 + (31 - 1)\tfrac{1}{3} = 8,$$

which is the required thirty-first term.

239. Arithmetic Means. The terms of an A.P. that lie between two given terms are called **arithmetic means** between those two terms.

If we are given two numbers, a and l, and are required to insert k arithmetic means between these two numbers, we notice that, if we construct an A.P. having a as its first term and l as its $(k + 2)$th term, then there will be precisely k terms of the progression between a and l.

Hence, to insert k arithmetic means between a and l, put $n = k + 2$, and the given values of a and l, in equation (2) and solve for d. As soon as we know a and d, we can write the A.P. from (1). The k terms of this progression that follow a are the required means.

EXAMPLE. Insert five arithmetic means between 4 and 13.

We have $a = 4$, $l = 13$, $n = 5 + 2 = 7$. Hence, from equation (2),

$$13 = 4 + 6d.$$

This gives $d = \frac{3}{2}$ and, since $a = 4$, the required means are

$$5\tfrac{1}{2},\ 7,\ 8\tfrac{1}{2},\ 10,\ 11\tfrac{1}{2},$$

because these five numbers, together with 4 as the first term and 13 as the seventh term, constitute an A.P. of seven terms.

If just *one* arithmetic mean is to be inserted between a and l, this mean is called *the* arithmetic mean between a and l. The student should verify that the value of *the* arithmetic mean between a and l is

$$\frac{a + l}{2}. \tag{5}$$

Exercises

State whether the given sequence is, or is not, an A.P. If it is, state the value of d.

1. 5, 8, 10, 14.
2. 3, 7, 11, 15.
3. 25, 20, 15, 10.
4. 2.8, 3.2, 3.6, 4.0.
5. 9, 5, 1, -2.
6. $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}, 0$.

Find the value of k, given that the sequence is an A.P.

7. $6 + 2k,\ 9 - 2k,\ 5 + k.$
8. $2k + 3,\ 12 - k,\ 3 + 5k.$
9. $3k - 20,\ 4k + 3,\ 8 - k.$
10. $6k - 1,\ 5k - 7,\ 5 - 8k.$

Find the nth term, and the sum of n terms, of the A.P., given:

11. $-5, -1, 3, \cdots, n = 9.$
12. $2, 4.5, 7, \cdots, n = 13.$
13. $2.7, 5.3, 7.9, \cdots, n = 16.$
14. $14, 11\frac{1}{3}, 8\frac{2}{3}, \cdots, n = 22.$

Find the other two elements, given:

15. $a = 17, d = -2, l = -3.$
16. $a = 5, l = 37, n = 9.$
17. $a = -1, d = \frac{1}{3}, n = 37.$
18. $a = 5, l = 37, s = 231.$
19. $d = \frac{7}{5}, l = 72, n = 21.$
20. $l = 17, n = 29, s = 232.$
21. $a = 23, n = 34, s = 527.$
22. $d = -2, l = -19, s = 96.$
23. $a = -7, d = 3, s = 14.$
24. $d = -\frac{2}{5}, n = 61, s = 488.$

25. Insert three arithmetic means between 2 and 16.

26. Insert five arithmetic means between -4 and 12.

27. Insert eleven arithmetic means between 5 and 38.

28. Find the arithmetic mean between 483 and 791.

29. The fifth term of an A.P. is 67 and the thirteenth is 31. Find the thirty-fifth term.

30. The third term of an A.P. is $-\frac{7}{2}$ and the eighth is $-\frac{23}{3}$. Show that -16 is a term of this A.P. and find which term it is.

31. Find the sum of the first n positive odd integers.

32. Find the number of integers between 29 and 86 that are divisible by 4. Find also the sum of these integers.

33. A clock strikes the hours from 1 to 12. How many strokes does it make in 24 hours?

34. A freely falling body falls 16 feet the first second. During each second thereafter, it falls 32 feet farther than it did during the preceding second. How far does it fall (a) during the twelfth second and (b) during the first twelve seconds.

35. A carpenter wishes to make a ladder of 16 rungs which are to diminish uniformly in length from 20 inches at the bottom to 14 inches at the top. Allowing 4 inches for waste, find how long a pole he will need to make these rungs.

36. A steel spring is bent in the form of a spiral lying in a plane. There are twelve complete turns of which the shortest is 0.6 inch and the longest is 4.7 inches. How long is the spring?

37. A man owes \$6300 on which he pays, at the end of each year, \$700 on the principal, and interest at 5% on the amount outstanding during the year. In how many years will he have paid off the debt and how much will he have paid out in principal and interest?

38. A teacher received \$1700 for his first year of teaching with an increase of \$120 each year thereafter until his salary reached \$3500, after which it

remained fixed. He retired at the end of his twenty-third year. Find the total amount of salary he received.

39. There are a trees in line with a pump. The nearest tree is b feet from the pump and the trees are c feet apart. A man carried a bucket of water to each tree and returned the bucket to the pump after watering the last tree. How far did he travel?

40. The lengths of the sides of a series of squares form an A.P. Do (a) the lengths of the diagonals and (b) the areas of the squares also form an A.P.?

II. Geometric Progressions

240. Geometric Progressions. A sequence of numbers is a **geometric progression** if each term after the first is obtained by multiplying the preceding one by a fixed number called the **common ratio.**

We shall denote a geometric progression by the abbreviation G.P.

Thus,
$$7, 14, 28, 56, \cdots$$
is a G.P. with its first term equal to 7 and its common ratio equal to 2.

Similarly,
$$27, -9, 3, -1, \cdots$$
is a G.P. with the first term 27 and the common ratio $-\frac{1}{3}$.

241. The Elements. If the first term of a G.P. is a and the common ratio is r, we may write its first n terms in the form

$$a, ar, ar^2, \cdots, ar^{n-1}. \qquad (6)$$

Associated with these first n terms of the progression are the following five numbers which are its **elements.**

a, *the first term,*
r, *the common ratio,*
n, *the number of terms,*
l, *the nth term,*
and s, *the sum of the first n terms.*

242. Equations Connecting the Elements. The five elements of a G.P. are connected by two equations.

It is seen, from (6), that the exponent of r in any term is *one less than the number of the term.* Hence, for the nth term, which we denote by l, we have

$$l = ar^{n-1}. \qquad (7)$$

For the sum, s, of the first n terms, we have, by definition,

$$s = a + ar + ar^2 + \cdots + ar^{n-1}$$

Multiply by r: $$sr = \quad ar + ar^2 + \cdots + ar^{n-1} + ar^n$$

Subtract: $$s - sr = a \quad - ar^n,$$

or $$(1 - r)s = a(1 - r^n).$$

We shall suppose, throughout this chapter, that $r \neq 1$. With this restriction, we can solve the preceding equation for s, giving

$$s = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}. \tag{8}$$

Since, by equation (7), $l = ar^{n-1}$, we may write equations (8) in the form

$$s = \frac{a - rl}{1 - r} = \frac{rl - a}{r - 1}. \tag{9}$$

If the values of three of the elements of a G.P. are known, the values of the other two can be found by means of equations (7), (8), and (9) provided we can solve the equations resulting from substituting in them the values of the known elements. The values of these known elements must be such that $r \neq 1$ and that n is a positive integer.

Example 1. Find the tenth term, and the sum of the first ten terms, of the G.P.: 5, 10, 20, 40, $\cdots$.

We have $a = 5$, $n = 10$, $r = 10 \div 5 = 2$.
From equation (7), $l = 5 \cdot 512 = 2560$.

From equation (8), $s = \dfrac{5(2^{10} - 1)}{2 - 1} = 5 \cdot 1023 = 5115.$

Example 2. Given $r = \frac{2}{3}$, $n = 6$, $s = \frac{665}{144}$, find a and l.

From equation (8), $\dfrac{665}{144} = a\,\dfrac{1 - (\frac{2}{3})^6}{1 - \frac{2}{3}} = a\,\dfrac{1 - \frac{64}{729}}{\frac{1}{3}} = a\,\dfrac{665}{243}$. Hence, $a = \dfrac{27}{16}$.

From equation (7), $l = \frac{27}{16}(\frac{2}{3})^5 = \frac{27}{16} \cdot \frac{32}{243} = \frac{2}{9}$.

243. Geometric Means. The terms of a G.P. that lie between two given terms are called **geometric means** between those two terms.

If we are given two numbers, a and l, and are required to insert k geometric means between these two numbers, we form a geometric progression having a as its first term and l as its $(k + 2)$th term. The k terms of this progression that lie between a and l are the required k geometric means between a and l.

To construct this progression, we substitute the given values of a and l in equation (7), put $n = k + 2$, and solve for r. Then the required means are

$$ar,\ ar^2,\ ar^3,\ \cdots,\ ar^k.$$

Example. Insert five geometric means between 8 and 5832.

From equation (7), with $a = 8$, $l = 5832$, and $n = 5 + 2 = 7$, we have

$$5832 = 8r^6, \quad \text{or} \quad r^6 = 729.$$

Hence, $r = \sqrt[6]{729} = 3$* and the required means are 24, 72, 216, 648, and 1944.

* When we are solving an equation of the form $r^n = A$ for the common ratio r, we shall consider only the principal nth root of A (Art. 27).

If only one mean is to be inserted between a and l, this mean is called *the* geometric mean between a and l. Let x be this mean. Then

$$r = x/a = l/x, \quad x^2 = al, \quad \text{or} \quad x = \pm \sqrt{al}.$$

In order that x may be real, a and l must agree in sign. Of the two possible values of x, we choose the one that agrees in sign with a and l.

Exercises

State whether the given sequence is, or is not a G.P. If it is, write the next three terms.

1. 3, 6, 12, 24. **2.** 8, 4, 3, 1. **3.** 400, 40, 4, 0.4.
4. $\frac{8}{9}, \frac{4}{3}, 2, 3$. **5.** $50, -10, 2, -0.4$. **6.** $7, \frac{7}{3}, \frac{7}{6}, \frac{7}{12}$.

Find the nth term and the sum of n terms of the G.P., given:

7. $20, 4, 0.8, \cdots, n = 6$. **8.** $3, 6, 12, \cdots, n = 8$.
9. $\frac{1}{6}, \frac{1}{2}, \frac{3}{2}, \cdots, n = 7$. **10.** $\frac{9}{16}, \frac{3}{8}, \frac{1}{4}, \cdots, n = 8$.

Find the remaining two elements, given:

11. $a = 2, r = 3, n = 6$. **12.** $l = 3, r = \frac{1}{5}, s = 468$.
13. $a = 3, n = 7, l = 24$. **14.** $r = \frac{3}{2}, n = 5, s = 1055$.
15. $l = \frac{2}{15}, r = \frac{2}{3}, n = 5$. **16.** $a = 3, r = 4, s = 255$.

17. Find s in terms of l, r, and n.
18. Find r in terms of a, l, and s.
19. Insert two geometric means between 56 and 875.
20. Insert three geometric means between 4 and 324.
21. Insert five geometric means between 21 and 168.

22. Find x, given that $2x - 7$ is the geometric mean between $x - 5$ and $2x + 11$.

23. The fourth term of a G.P. is 6 and the tenth term is 24. Find the nineteenth term.

24. A man has \$2500 invested in a business. At the end of each year he receives 10% on the money invested during the year and immediately reinvests this money in the business. After he has made this reinvestment at the end of the fourth year, how much does he have invested in the business?

25. Bacteria of a certain type increase by the process of each individual splitting into two. Starting from a single individual and assuming no losses, how many bacteria will there be after twelve divisions?

26. At each stroke of an air pump, approximately one-twelfth of the air in a vessel is removed. After six strokes, what fraction of the original amount of air remains in the vessel? Express your answer as a decimal to two significant figures.

27. Given $(1.03)^{12} = 1.42576$, find, to three decimal places, the value of $1 + 1.03 + (1.03)^2 + \cdots + (1.03)^{11}$.

28. Given $(1.045)^{-25} = 0.332731$, find, to three decimal places, the value of $(1.045)^{-1} + (1.045)^{-2} + \cdots + (1.045)^{-25}$.

29. Find an expression for the sum $1 + (1 + i) + (1 + i)^2 + \cdots + (1 + i)^{n-1}$.

30. Find an expression for the sum $(1 + i)^{-1} + (1 + i)^{-2} + \cdots + (1 + i)^{-n}$.

31. Show that the reciprocals of the terms of a G.P. also constitute a G.P. and find its common ratio.

32. Show that the logarithms of the terms of a G.P. form an A.P. and find its common difference.

244. Geometric Progressions with Infinitely Many Terms. Let there be given a G.P. in which the numerical value of r is less than unity. We shall now denote the sum of the first n terms of this G.P. by s_n. From equation (8), we have

$$s_n = \frac{a}{1 - r} - \frac{a}{1 - r} r^n. \tag{10}$$

We wish now to consider what can be said of the sum of the terms of this G.P. if, instead of stopping at some fixed term, the terms are thought of as going on without any end. That is, what is the sum, s, of the G.P. with infinitely many terms

$$s = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots,$$

where the final dots mean that the terms go on unendingly?

We approach this problem by considering what happens to the sum of the first n terms as n increases, so that we keep on adding more and more terms. If, when n increases indefinitely, the sum of the first n terms tends to settle down on some fixed number, we shall call that number the sum of the geometric progression of infinitely many terms.

In the second member of equation (10), n occurs only in the expression $ar^n/(1 - r)$. Since, by hypothesis, r is numerically less than unity, *we can make r^n, and hence $ar^n/(1 - r)$, numerically just as small as we please by taking n sufficiently large.*

For example, if $r = 0.9$, which is quite close to unity, so that r^n becomes small rather slowly, we find, to two significant figures, the following table of pairs of values of n and $(0.9)^n$.

n	10	100	200
$(0.9)^n$	0.35	0.000027	0.00000000071

Since we can make the value of $ar^n/(1 - r)$ numerically just as small as we please by taking n large enough, it follows from equation (10) that we can make the value of s_n be just as near to $a/(1 - r)$ as we please by taking n sufficiently large. We state this fact symbolically in the form

$$\lim_{n \to \infty} s_n = \frac{a}{1 - r}, \tag{11}$$

which is read, "the limit of s_n, as n increases without limit, is $a/(1 - r)$."

We define the sum of the geometric progression of infinitely many terms as this number which s_n approaches as n increases without limit. Hence, from equation (11),

$$s = \frac{a}{1 - r}, \tag{12}$$

is the sum of this geometric progression of infinitely many terms when r is numerically less than unity.

Example 1. Find the sum of the G.P. with infinitely many terms, 98, 70, 50, $\frac{250}{7}, \cdots$.

We have $a = 98$ and $r = \frac{5}{7}$, which is numerically less than unity. Hence,

$$s = \frac{98}{1 - \frac{5}{7}} = 343.$$

Example 2. Express as an ordinary fraction the unending repeating decimal $1.7363636 \cdots$.

We may write this decimal fraction in the form

$$\begin{aligned}1.7 + 0.036 + 0.00036 + 0.0000036 + \cdots \\ = 1.7 + 0.036[1 + (0.01) + (0.01)^2 + (0.01)^3 + \cdots].\end{aligned}$$

The expression in the brackets is a G.P. of infinitely many terms for which $a = 1$ and $r = 0.01$. Hence, by equation (12), its value is $\frac{1}{1 - 0.01} = \frac{1}{0.99}$. The value of the given decimal is, accordingly,

$$1.7 + \frac{0.036}{0.99} = \frac{17}{10} + \frac{36}{990} = \frac{187}{110} + \frac{4}{110} = \frac{191}{110}.$$

This result should be checked by expressing $\frac{191}{110}$ as a decimal fraction by division.

Exercises

Find the sum of the given G.P. of infinitely many terms.

1. $3, \frac{9}{4}, \frac{27}{16}, \cdots$.

2. $5, -\frac{5\sqrt{3}}{2}, \frac{15}{4}, \cdots$.

3. $25, -20, 16, \cdots$.

4. $63, 42, 28, \cdots$.

5. $500, 400, 320, \cdots$.

6. $-\sqrt{12}, -\sqrt{6}, -\sqrt{3}, \cdots$.

Express each of the following repeating decimals as a common fraction. Check by writing the fraction in decimal form.

7. $0.515151 \cdots$.

8. $0.727272 \cdots$.

9. $2.97297297 \cdots$.

10. $25.9259259 \cdots$.

11. $5.242424 \cdots$.

12. $3.1543564356 \cdots$.

13. An automobile, coasting to rest, travels 19 feet the first second and, during each second thereafter, three-fourths as far as during the preceding second. How far will it coast in coming to rest?

14. The first swing of a pendulum is 12 inches and each swing thereafter is nine-tenths as long as the preceding one. How far will it travel in coming to rest?

15. In an unending series of equilateral triangles, the vertices of each triangle after the first are the midpoints of the sides of the preceding triangle. The sides of the first triangle are each one foot long. Find the sum of the perimeters of all the triangles.

16. Find the sum of the areas of all the triangles in Ex. 15, given that the area of an equilateral triangle of side a is $a^2\sqrt{3}/4$.

Chapter 30

The Binomial Theorem

245. The Factorial Notation. The product of all the integers from 1 to n * occurs so frequently in mathematics that a special symbol has been devised to represent it. This symbol is written $n!$ and is read, "n factorial." By definition,

$$n! = 1 \cdot 2 \cdot 3 \cdots n.$$

To this definition, which holds for all positive integral values of n, we add the special definition $0! = 1$.

Thus, $\quad 1! = 1, \quad 2! = 1 \cdot 2 = 2, \quad 3! = 1 \cdot 2 \cdot 3 = 6,$

and so on.

Exercises

Find the value of each of the following expressions.

1. $8!$. **2.** $\dfrac{7!}{4!}$. **3.** $\dfrac{9!}{5!\,4!}$. **4.** $\dfrac{11!}{0!\,8!\,3!}$.

5. $\dfrac{12!}{10!}$. **6.** $\dfrac{(n+1)!}{n!}$. **7.** $\dfrac{(n+1)!}{(n-1)!}$. **8.** $\dfrac{n!}{(n-4)!\,4!}$.

Prove the following identities.

9. $4\dfrac{9!}{5!\,4!} = 9\dfrac{8!}{5!\,3!}$. **10.** $r\dfrac{n!}{(n-r)!\,r!} = n\dfrac{(n-1)!}{(n-r)!\,(r-1)!}$.

11. $\dfrac{9!}{6!\,3!} + \dfrac{9!}{7!\,2!} = \dfrac{10!}{7!\,3!}$. **12.** $\dfrac{13!}{8!\,5!} + \dfrac{13!}{9!\,4!} = \dfrac{14!}{9!\,5!}$.

13. $\dfrac{n!}{(n-r)!\,r!} + \dfrac{n!}{(n-r+1)!\,(r-1)!} = \dfrac{(n+1)!}{(n-r+1)!\,r!}$.

246. The Binomial Formula. By actual multiplication, we find that

$$\begin{aligned}
(a+x)^1 &= a + x,\\
(a+x)^2 &= a^2 + 2ax + x^2,\\
(a+x)^3 &= a^3 + 3a^2x + 3ax^2 + x^3,\\
(a+x)^4 &= a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4.
\end{aligned}$$

In these expressions for $(a + x)^n$, with $n = 1, 2, 3$, and 4, respectively, we observe the following common properties:

1. The first term is always a^n.
2. The second term is always $na^{n-1}x$.
3. If we know any term, we can get the next following one from it by (a) multiplying its coefficient by the exponent of a and dividing the

* Throughout this chapter, n and r are assumed to be positive integers.

result by the exponent of x increased by unity, (*b*) decreasing the exponent of a by unity, and (*c*) increasing the exponent of x by unity.

4. The last term is always x^n.

With the aid of these observed facts, we can write down immediately any one of the given equations.

For example, let us write down the expansion of $(a + x)^4$.

From statements 1 and 2, we can write at once the first two terms, $a^4 + 4a^3x$.

To find the third term, we start from the second term, $4a^3x$. According to statement 3, we must multiply this second term by 3, divide it by 2, decrease the exponent of a by unity and increase the exponent of x by unity. The result is $\dfrac{4 \cdot 3}{2}\, a^{3-1}x^{1+1} = 6a^2x^2$.

To find the fourth term, we start from the third term, $6a^2x^2$, and find, by means of statement 3, $\dfrac{6 \cdot 2}{3}\, a^{2-1}x^{2+1} = 4ax^3$. For the fifth term, we find, in a similar way, $\dfrac{4 \cdot 1}{4}\, a^{1-1}x^{3+1} = x^4$.

Since, by statement 4, the last term is x^4, the result is

$$(a + x)^4 = a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4,$$

which agrees with the result already found by multiplication.

We shall prove, in Art. 249, that *the properties stated in Nos.* 1 *to* 4 *hold for the expansion of* $(a + x)^n$ *for all positive integral values of* n. This statement is called the **binomial theorem.**

From the binomial theorem, we obtain at once the **binomial formula** for the expansion of $(a + x)^n$, where n is any positive integer. This binomial formula is

$$\begin{aligned}(a + x)^n &= a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \cdots \\ &\quad + \frac{n(n-1)\cdots(n-r+2)}{(r-1)!}a^{n-r+1}x^{r-1} + \cdots + x^n. \qquad (1)\end{aligned}$$

In the second member of this equation, we have written explicitly the first three terms. The fourth term may be obtained by putting $r = 4$ in the first term on the second line, the fifth by putting $r = 5$, and so on.

The expansion of $(a + x)^n$ may be obtained, either by using properties Nos. 1 to 4, as stated near the beginning of this article, or by using the binomial formula. The student should familiarize himself with both methods and should use one of them as a check on the other.

Example 1. Expand $(a^2 + 2b)^6$ by the binomial formula, and simplify.

From equation (1), we have

$$\begin{aligned}(a^2 + 2b)^6 &= (a^2)^6 + 6(a^2)^5(2b) + \frac{6 \cdot 5}{1 \cdot 2}(a^2)^4(2b)^2 + \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3}(a^2)^3(2b)^3 \\ &\quad + \frac{6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4}(a^2)^2(2b)^4 + \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}a^2(2b)^5 \\ &\quad + \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}(2b)^6 \\ &= a^{12} + 12a^{10}b + 60a^8b^2 + 160a^6b^3 + 240a^4b^4 + 192a^2b^5 + 64b^6.\end{aligned}$$

EXAMPLE 2. Expand $(\sqrt[3]{x} - \sqrt{y})^4$ by the binomial formula, and simplify.

It is usually best to replace the radicals by fractional exponents before writing out the expansion.

$$\begin{aligned}(x^{\frac{1}{3}} - y^{\frac{1}{2}})^4 &= (x^{\frac{1}{3}})^4 + 4(x^{\frac{1}{3}})^3(-y^{\frac{1}{2}}) + \frac{4\cdot 3}{1\cdot 2}(x^{\frac{1}{3}})^2(-y^{\frac{1}{2}})^2 \\ &\quad + \frac{4\cdot 3\cdot 2}{1\cdot 2\cdot 3}(x^{\frac{1}{3}})(-y^{\frac{1}{2}})^3 + \frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4}(-y^{\frac{1}{2}})^4 \\ &= x^{\frac{4}{3}} - 4xy^{\frac{1}{2}} + 6x^{\frac{2}{3}}y - 4x^{\frac{1}{3}}y^{\frac{3}{2}} + y^2 \\ &= \sqrt[3]{x^4} - 4x\sqrt{y} + 6\sqrt[3]{x^2}y - 4\sqrt[3]{x}\sqrt{y^3} + y^2.\end{aligned}$$

EXAMPLE 3. Expand $(1.04)^5$ by the binomial formula and find its value to five decimal places.

$$\begin{aligned}(1.04)^5 = (1 + 0.04)^5 &= 1 + 5(0.04) + 10(0.04)^2 + 10(0.04)^3 + 5(0.04)^4 + (0.04)^5 \\ &= 1 + 0.2 + 0.016 + 0.00064 + 0.0000128 + 0.0000001024 \\ &= 1.21665, \text{ to five decimal places.}\end{aligned}$$

Exercises

Expand by the binomial formula and simplify.

1. $(a + 2)^4$. **2.** $\left(2x + \frac{y}{2}\right)^5$. **3.** $\left(x^3 - \frac{3}{x}\right)^4$.

4. $(s + t)^7$. **5.** $(x^2 - y^2)^5$. **6.** $(2\sqrt{a} - b)^6$.

7. $(\sqrt[4]{x} + \sqrt[3]{y^2})^4$. **8.** $(\sqrt[5]{y} + 2x^2)^5$. **9.** $\left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right)^6$.

10. $(x^{-1} + y^{-2})^7$. **11.** $(3a^{\frac{1}{3}} - 2b^{-\frac{2}{3}})^3$. **12.** $(\sqrt{2x} + \sqrt[3]{2y})^5$.

13. $\left(\frac{a^{-2}}{\sqrt{3}} + \frac{b}{\sqrt{2}}\right)^4$. **14.** $(e^{\frac{x}{a}} - e^{-\frac{x}{a}})^5$. **15.** $(a^{\frac{1}{2}}b^{-2} + c^{-\frac{1}{2}}d)^4$.

16. $29^3 = (30 - 1)^3$. **17.** $(1.05)^4$. **18.** $(0.96)^4$.

19. By first putting $b + c = x$, expand $(a + b + c)^2$ by the binomial formula.

20. Expand $(a + b + c)^3$ by the binomial formula.

Find the first four terms, only, of the expansion of:

21. $(x^2 - 3t)^9$. **22.** $(r^3 + 4\sqrt[3]{t})^{10}$. **23.** $(\sqrt{3u} - v^2)^8$.

24. $(a^2 - 2b^3)^{12}$. **25.** $\left(\frac{1}{x} - \frac{2}{y}\right)^{14}$. **26.** $(a^{-2} + b^2)^{15}$.

27. $(x^3 + \sqrt{5y^3})^{14}$. **28.** $\left(t^5 + \frac{2}{t^2}\right)^{16}$. **29.** $\left(a^{\frac{1}{4}} - \frac{1}{2}b^{\frac{1}{3}}\right)^{16}$.

The amount due on P dollars, at the end of n years, at the rate i (expressed as a decimal), is $P(1 + i)^n$. Find, to the nearest cent, the amount due, given:

30. $P = \$100$, $i = 0.03$, $n = 4$. **31.** $P = \$150$, $i = 0.05$, $n = 6$.

32. $P = \$400$, $i = 0.06$, $n = 12$. **33.** $P = \$700$, $i = 0.04$, $n = 16$.

247. The General Term of $(a + x)^n$. It is sometimes required to write a specified term of the expansion of $(a + x)^n$ without finding the preceding terms. We can do this with the aid of the expression given in the binomial formula (equation 1) for the rth term of this expansion, namely,

$$\textbf{the } r\textbf{th term} = \frac{n(n-1)\cdots(n-r+2)}{(r-1)!}a^{n-r+1}x^{r-1}. \qquad (2)$$

EXAMPLE 1. Find the ninth term of the expansion of $\left(2z - \frac{1}{y}\right)^{13}$.

We have, in formula (2), $a = 2z$, $x = -\frac{1}{y}$, $n = 13$, and $r = 9$. If we substitute these values in equation (2), we find, as the required ninth term,

$$\frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}(2z)^{13-9+1}\left(-\frac{1}{y}\right)^8 = 41184\frac{z^5}{y^8}.$$

We shall sometimes need, not the rth term, but the term of the expansion that involves x^r. It will be seen from equation (2) that this term is the $(r + 1)$th term of the expansion. The expression for it is found, by replacing r by $r + 1$ in equation (2), to be

$$\textbf{the term involving } x^r = \frac{n(n-1)\cdots(n-r+1)}{r!}a^{n-r}x^r. \qquad (3)$$

Whether we should use equation (2) or equation (3), in determining a required term of the expansion, depends on whether we wish to find the rth term or the term involving x^r.

EXAMPLE 2. Find the term of the expansion of $(u^2 - v^3/2)^{17}$ that contains v^{15}.

We have $a = u^2$, $x = -v^3/2$, and $n = 17$. To find, in formula (3), the value of r that gives the required term, we neglect, for the moment, numerical coefficients and put $x = v^3$, giving $x^r = (v^3)^r = v^{3r}$. Since v^{3r} must equal v^{15}, we must have $3r = 15$, or $r = 5$. Putting $a = u^2$, $x = -v^3/2$, $n = 17$, and $r = 5$ in formula (3), we now have, as the required term,

$$\frac{17 \cdot 16 \cdot 15 \cdot 14 \cdot 13}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}(u^2)^{12}\left(-\frac{v^3}{2}\right)^5 = -\frac{1547}{8}u^{24}v^{15}.$$

EXAMPLE 3. Find the term of the expansion of $\left(\frac{1}{t} - 3t^4\right)^{13}$ that contains t^2.

We have $a = 1/t$, $x = -3t^4$, and $n = 13$. To find the required value of r in formula (3), we ignore, momentarily, numerical coefficients and put $a = 1/t$. $x = t^4$ Then

$$a^{13-r}x^r = \left(\frac{1}{t}\right)^{13-r}(t^4)^r = t^{-13+r+4r} = t^{5r-13}.$$

Since this must equal t^2, we have $5r - 13 = 2$, or $r = 3$. Formula (3) now gives

$$\frac{13 \cdot 12 \cdot 11}{1 \cdot 2 \cdot 3}\left(\frac{1}{t}\right)^{10}(-3t^4)^3 = -7722t^2.$$

Exercises

Find the required term without finding the preceding ones.

1. Seventh term of $(a+b)^{10}$.
2. Sixth term of $\left(u-\dfrac{1}{v}\right)^9$.
3. Twelfth term of $(x-y)^{15}$.
4. Fifth term of $(a^2+2b)^{11}$.
5. Sixth term of $\left(t^2-\dfrac{1}{t}\right)^{18}$.
6. Tenth term of $\left(2h^5-\dfrac{1}{2h^3}\right)^{16}$.
7. Fifth term of $\left(\sqrt[4]{2x^2}+\sqrt{\dfrac{y}{3}}\right)^{10}$.
8. Fourth term of $(x^r+y^s)^7$.
9. Middle term of $(2x^2-y^3)^6$.
   HINT. $(a+x)^n$ contains $n+1$ terms.
10. Middle term of $\left(5x^4+\dfrac{1}{5x^3}\right)^{12}$.
11. Two middle terms of $\left(\dfrac{r}{2}+s^2\right)^5$.
12. Two middle terms of $(x^3-y^2)^7$.
13. Term involving w^{14} in $(u-w^2)^{10}$.
14. Term involving k^{12} in $(k^3+t)^6$.
15. Term involving a^{10} in $(a^2+\sqrt{2b})^9$.
16. Term involving w^9 in $(v^2+2w^3)^{10}$.
17. Term involving x^9 in $\left(x^3+\dfrac{1}{x}\right)^7$.
18. Term involving u^7 in $(u^{\frac{1}{2}}+2u^{\frac{3}{2}})^8$.

248. Mathematical Induction. Mathematical induction is a form of reasoning that enables us to prove that certain theorems are true for all integral values of n that are greater than some definitely fixed integer.

The method of reasoning used in the proof of a theorem by mathematical induction is illustrated by the following example.

EXAMPLE 1. Prove by mathematical induction that the sum of the first n odd integers equals n^2, that is,

$$1+3+5+\cdots+(2n-1)=n^2.$$

We observe, first, that the theorem is true for $n=1$, since the equation then reduces to $1=1^2$.

We next prove that, if k is any positive integral value of n for which the theorem is true, then it must also be true for $n=k+1$.

By hypothesis,

$$1+3+5+\cdots+(2k-1)=k^2$$

is a true equation since we have limited k to values for which the theorem is true.

Add $2k+1$ to both sides of this equation. Then

$$1+3+5+\cdots+(2k-1)+(2k+1)=k^2+2k+1=(k+1)^2$$

is also a true equation since it was obtained by adding $2k+1$ to both sides of the preceding one. Since $2k+1=2(k+1)-1$, the last equation may be written in the form

$$1+3+5+\cdots+[2(k+1)-1]=(k+1)^2,$$

which is precisely the formula we are to prove for $n = k + 1$. Hence, if the formula is true when $n = k$, it must also be true for $n = k + 1$.

Suppose, now, that we wish to know whether the theorem is true for some given positive integral value of n, say $n = 17$. We reason as follows: we know, by actual computation, that the theorem is true for $n = 1$. It follows from the proof just given that it is true for $n = 2$. If it is true for $n = 2$, it must be true for $n = 3$. Continuing in this way, we find that it is true for $n = 17$. Hence,

$$1 + 3 + 5 + \cdots + 33 = 17^2 = 289$$

is a true equation. In the same way, we can show that the theorem is true for any other positive, integral value of n.

The proof of a theorem by mathematical induction always consists of the following two parts:

1. *A verification that the theorem is true for some one integral value of n.*
2. *A proof that, if k is any integral value of n for which the theorem is true, then $k + 1$ is also a value for which it is true.*

A proof by mathematical induction has not been completed until both parts of the proof have been given. If both parts have been proved, then *the theorem is true for all integral values of n greater than, or equal to, the value used in the proof of Part* 1.

Exercises

Prove the following formulas for all positive, integral values of n by mathematical induction.

1. $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$.

2. $2 + 2^2 + 2^3 + \cdots + 2^n = 2(2^n - 1)$.

3. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$.

4. $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \dfrac{1}{5 \cdot 7} + \cdots + \dfrac{1}{(2n-1)(2n+1)} = \dfrac{n}{2n+1}$.

5. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$.

6. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

7. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$.

8. $1^4 + 2^4 + 3^4 + \cdots + n^4 = \dfrac{n(n+1)(6n^3 + 9n^2 + n - 1)}{30}$.

9. Show that $1 + 3 + 5 + \cdots + (2n - 1) = n^2 + 1$ satisfies Part 2 of the proof by mathematical induction but not Part 1.

10. Show that $1 + 3 + 5 + \cdots + (2n - 1) = n^2 + (n - 1)(n - 2)(n - 3)$ satisfies Part 1 of the proof by mathematical induction (for $n = 1$, 2, or 3) but not Part 2.

249. The Binomial Theorem. We shall prove by mathematical induction that the binomial formula (Equation 1, Art. 246) is true for all positive, integral values of n. This statement is the **binomial theorem.**

Part 1. For $n = 1$, the binomial formula gives the result

$$(a + x)^1 = a + x.$$

Since this equation is true, the formula is true for $n = 1$.

Part 2. Let k be any positive, integral value of n for which the formula is true. Then, by hypothesis, the equation

$$(a + x)^k = a^k + ka^{k-1}x + \cdots + \frac{k(k-1)\cdots(k-r+2)}{(r-1)!}a^{k-r+1}x^{r-1}$$
$$+ \frac{k(k-1)\cdots(k-r+1)}{r!}a^{k-r}x^r + \cdots + x^k \qquad (4)$$

is a true equation. In this equation we have written both the rth and the $(r + 1)$th terms explicitly.

We shall multiply both members of this equation by $a + x$.

The product of the first member by $a + x$ is

$$(a + x)^k(a + x) = (a + x)^{k+1}.$$

To multiply the second member by $a + x$, we must multiply each of its terms by a and by x and add the results.

Multiply the second member of equation (4) by a. We obtain

$$a^{k+1} + ka^kx + \cdots + \frac{k(k-1)\cdots(k-r+1)}{r!}a^{k-r+1}x^r + \cdots + ax^k. \qquad (5)$$

Multiply the same expression by x. We have

$$a^kx + \cdots + \frac{k(k-1)\cdots(k-r+2)}{(r-1)!}a^{k-r+1}x^r + \cdots + kax^k + x^{k+1}. \qquad (6)$$

Add expressions (5) and (6) and combine the terms involving a and x to the same powers. In combining the coefficients of these corresponding terms, notice that

$$\frac{k(k-1)\cdots(k-r+1)}{r!} + \frac{k(k-1)\cdots(k-r+2)}{(r-1)!}$$
$$= \frac{k(k-1)\cdots(k-r+2)}{(r-1)!}\left(\frac{k-r+1}{r} + 1\right)$$
$$= \frac{(k+1)k\cdots(k-r+2)}{r!}.$$

By the aid of this expression for the sum of the coefficients, we obtain, as the sum of expressions (5) and (6),

$$a^{k+1} + (k+1)a^k x + \cdots + \frac{(k+1)k \cdots (k-r+2)}{r!} a^{k-r+1} x^r + \cdots + x^{k+1}. \qquad (7)$$

The expression (7) is precisely the expansion of $(a + x)^{k+1}$ by the binomial formula. Hence, if the formula is true for $n = k$, it is also true if $n = k + 1$.

We have proved (Part 1) that the binomial formula is true for $n = 1$ and also (Part 2) that, if it is true for $n = k$, it must be true for $n = k + 1$. It follows that the formula is true for all positive, integral values of n.

Chapter 31

Permutations and Combinations

250. Fundamental Principle. *If one act can be done in any one of m different ways, and if, after it has been done in any one of these ways, a second act can be done in any one of n different ways, then both acts can be performed, in the order stated, in mn different ways.*

The reasoning on which this theorem is based is illustrated by the following example.

EXAMPLE 1. In an election, there are three candidates for senator and four for governor. In how many ways can a ballot be marked for both of these offices?

The ballot can be marked for senator in any one of three ways. With any one of these three ways, we can associate any one of the four ways in which it can be marked for governor. The total number of ways in which it can be marked for both offices is thus $3 \times 4 = 12$ different ways.

We have indicated these twelve possible ways of marking the ballot in the adjoining diagram, in which Aa means a vote for A for senator and a for governor, and so on.

Aa	Ab	Ac	Ad
Ba	Bb	Bc	Bd
Ca	Cb	Cc	Cd

In the problems discussed in this chapter, we shall break up the entire action to be performed into a succession of component acts, as was done in Example 1, and determine the number of ways in which each of these component acts can be performed. It will then follow from the fundamental principle that the number of ways in which the entire action can be carried out is the product of the number of ways of doing, in succession, its component acts.

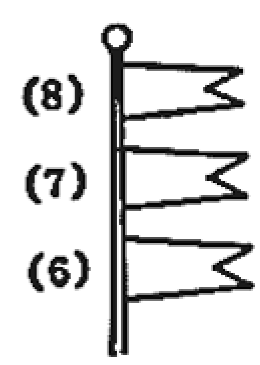


FIG. 167

It will frequently be found helpful to draw a diagram to indicate the successive steps in the problem and to mark on the figure the number of ways in which the successive steps can be carried out, as in Figure 167. Such a diagram should always be drawn whenever the problem is difficult or whenever the method of solving it is found to be uncertain.

EXAMPLE 2. A signal officer has eight flags of different colors. How many different signals can he form by placing three flags, one above the other, on a flagpole?

He can place any one of the 8 flags at the top, then any one of the remaining 7 flags just below it, then any one of the 6 flags still remaining at the bottom (Fig. 167).

The total number of ways in which he can perform these three acts in succession is, by the fundamental principle, $8 \times 7 \times 6 = 336$. It follows that the total number of such signals is 336.

Exercises

1. A man can leave one building by any one of four doors and enter another by any one of seven doors. In how many ways can he leave the first building and enter the second?

2. A club consists of 12 seniors and 8 juniors. In how many ways can a senior be chosen as president and a junior as vice-president?

3. In how many ways can 4 people arrange themselves in the 4 positions at a bridge table? What is the number if A and C must be partners?

4. A railroad has 24 stations. How many tickets must be printed if there is to be a ticket from each station to each of the other stations? How many if each ticket may be used in either direction between the stations named on it?

5. There are 4 roads from A to B, 6 from B to C, and 2 from C to D. In how many ways can one go from A to D by going first to B, then to C. and then to D?

6. In how many ways can 3 men choose a hotel in a town that has 6 hotels? What is the number if A and B refuse to stay at the same hotel?

7. A real estate company, in building houses in a subdivision, decided on 4 types of houses, 3 styles of fronts, and 6 colors of paint. How many houses, differing in appearance, can they build?

8. A man has 12 books of fiction, 7 biographies, and 4 books of essays. In how many ways can he choose one book of each type to take on a trip?

9. From among 7 boys and 4 girls, in how many ways can a game of tennis doubles be arranged if each side consists of a boy and a girl?

10. A signal officer has 2 flags of different colors, each of which can occupy any one of 6 positions. How many signals can he form, using one or both flags, if both flags cannot occupy the same position at the same time?

11. In how many ways can 4 people seat themselves in a seven-passenger car if the driver's seat must always be occupied?

12. There are 7 chairs in a row. In how many ways can A and B seat themselves in consecutive chairs?

251. Permutations. Suppose we have n different things. We are to choose r from among these n things and to arrange the r things so chosen in a specified order. Each arrangement that can be formed in this way is called a **permutation** *of the n things taken r at a time.*

Thus, the permutations of the three letters, a, b, and c, taken two at a time, are ab, ba, ac, ca, bc, and cb.

It is required to determine the *number* of permutations that can be formed from n things, taken r at a time. We shall denote this number by

What is a permutation?

the symbol $P(n, r)$, which is read, "the number of permutations of n things taken r at a time."

EXAMPLE 1. From a society of 10 members, in how many ways can a president, a vice-president, and a secretary be chosen?

We can choose for president any one of the 10 members; for vice-president, any one of the remaining 9 members; and, for secretary, any one of the other 8 members. By the fundamental principle, the required number of choices is $10 \cdot 9 \cdot 8 = 720$.

This number is equal to $P(10, 3)$, since it is the number of ways in which 3 can be selected from among the given 10 people and arranged in the stated offices. Hence, $P(10, 3) = 10 \cdot 9 \cdot 8 = 720$.

The value of $P(n, r)$ can be found by the reasoning used in the preceding example. The first of the r places can be filled in any one of n ways, then the second can be filled in any one of $n - 1$ ways, and so on. To fill the rth place, we have $n - (r - 1) = n - r + 1$ things to choose from. Hence,

$$P(n, r) = n(n - 1)(n - 2) \cdots (n - r + 1). \tag{1}$$

In particular, if $r = n$, we have $P(n, n)$, which is the number of ways in which all of n things can be arranged among themselves.

Since, in this case, the last factor in equation (1) is $n - n + 1 = 1$, we have

$$P(n, n) = n(n - 1)(n - 2) \cdots 1 = n!, \tag{2}$$

that is, *the number of ways in which all of n things can be arranged among themselves is n factorial.*

EXAMPLE 2. In how many ways can 4 from among 8 books be arranged on a shelf?

This number is $P(8, 4) = 8 \cdot 7 \cdot 6 \cdot 5 = 1680$.

Exercises

1. Find the value of $P(6, 2)$, $P(4, 3)$, $P(8, 5)$, $P(16, 3)$, and $P(32, 2)$.
2. How many radio broadcasting stations can be named with three different letters? How many if the first letter must be K? If one of the letters must be K?
3. Find n, given that $P(n, 4) = 56P(n, 2)$.
4. How many numbers, each consisting of 5 different digits, can be formed from the ten digits? (The first digit must not be zero).
5. Solve Ex. 4 if repetition of the digits is allowed.
6. In Ex. 4, how many of these numbers are (*a*) even numbers, (*b*) how many are less than 60,000, and (*c*) how many are less than 60,000 and divisible by 5?

7. In how many batting orders can a baseball team be arranged if the fielders must be the first three to bat and the pitcher must bat last?

8. In how many ways can 8 men stand in a row if A and B cannot stand side by side and C must stand beside B?

9. In how many ways can 11 men stand in a row if A must stand in the middle and B and C must stand at the ends?

10. In how many relative positions can 6 people be seated at a circular table?

HINT. Choose one person and let him sit in a specified seat. Permutations such as this, in which only the relative positions are considered, are called **circular permutations.**

11. In how many relative positions can 5 men and 5 women sit at a circular table if men and women alternate?

12. In how many ways can 5 men and 5 women be seated in a row if men and women alternate?

13. In how many ways can 5 women and 3 men be seated in a row if no two men sit together and each woman sits beside one, and only one, man?

14. A man and his wife invite four couples to dinner. After the host and hostess have been placed at the ends of the table, in how many ways can the guests be arranged if men and women sit alternately and no man sits beside his wife?

15. Show that $P(n, r) = n!/(n-r)!$.

252. Permutations of n Things Not All Different. Let it be required to find the number P of distinct permutations, seven at a time, of the seven letters of the word *receive*. Among these letters, e occurs three times and any two arrangements which differ only by an interchange of the letters e among themselves would be indistinguishable and should count as only a single one among the required arrangements.

To find the value of P, take any one of these permutations and, to distinguish the letters e in it, assign subscripts to them, e_1, e_2, e_3. We can now permute these three distinct letters among themselves, leaving the other four letters fixed, in $3!$ ways. If we do this for each of the P permutations of the letters r, e, c, e, i, v, e, we obtain $P \cdot 3!$ permutations of the seven distinct letters $r, e_1, c, e_2, i, v, e_3$. But these seven distinct letters, taken seven at a time, have $7!$ permutations. Hence,

$$P \cdot 3! = 7!, \quad \text{or} \quad P = \frac{7!}{3!}.$$

Using precisely the same reasoning, we find, if we have n things, of which n_1 are alike, n_2 others are alike, n_3 others are alike, and so on, that the number P of distinct permutations of these n things, taken n at a time, is

$$P = \frac{n!}{n_1!\, n_2!\, n_3! \cdots}. \tag{3}$$

Exercises

1. Find the number of distinct permutations of the letters of the word *sees*, taken all at a time, and write these permutations out in full.

2. Find the number of distinct permutations of the letters of the word *addresses*, taken all at a time.

3. Find the number of distinct permutations, taken all at a time, of the letters of the word *carelessness* if the vowels must occupy the second, fourth, sixth, and tenth places.

4. How many integers of ten digits each can be formed of the digits 1, 2, 3, and 4 if 2 occurs twice, 3, three times, and 4, four times.

5. Solve Ex. 4 if the first four digits are 1, 2, 3, and 4, in some order.

6. A signal man has 4 flags of each of 3 colors. How many signals can he form by displaying all of them at once, one above another, on a flagpole?

253. Combinations. Suppose that, from among n things, we select r things *without regard to the order of arrangement.* Any such selection is a **combination** *of the n things taken r at a time.*

Thus, the combinations of the letters a, b, c, taken two at a time, are ab, ac, bc.

The essential difference between a permutation and a combination of n things taken r at a time lies in the fact that, in the permutation, the r things chosen are arranged in a definite order among themselves whereas, in the combination, the order of arrangement of the things chosen is disregarded.

Thus, there are just 10 combinations of the 5 letters a, b, c, d, and e, 3 at a time, namely, abc, abd, abe, acd, ace, ade, bcd, bce, bde, and cde. There are, however, $P(5, 3) = 60$ permutations of these 5 letters taken 3 at a time. These 60 permutations can be formed by taking each of the combinations already listed and arranging the three letters in it in the $P(3, 3) = 3! = 6$ possible orders. For example, the first combination, abc, yields the 6 permutations abc, acb, bac, bca, cab, and cba, and similarly for each of the others.

It is required to find the number of combinations of n things taken r at a time. We shall denote this number by $C(n, r)$, which is read, "the number of combinations of n things r at a time."

To find $C(n, r)$, we notice that, as in the illustration just given, we can form the $P(n, r)$ permutations by taking each of the $C(n, r)$ combinations and arranging the r things in this combination in all of the $P(r, r) = r!$ possible ways. It follows that

$$C(n, r) \cdot r! = P(n, r),$$

or

$$C(n, r) = \frac{P(n, r)}{r!}. \tag{4}$$

EXAMPLE 1. How many triangles are determined by nine points, no three of which lie on a line?

Any three of the points, taken without regard to their order of arrangement, determine a triangle. The required number of triangles is, accordingly,

$$C(9, 3) = \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} = 84.$$

If, in equation (4), we replace $P(n, r)$ by its value from equation (1), we have

$$C(n, r) = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}. \tag{5}$$

Further, if we multiply the numerator and the denominator of the second member of equation (5) by $(n-r)!$, the new numerator is

$$\begin{aligned} n(n-1)(n-2)&\cdots(n-r+1)\cdot(n-r)! \\ &= n(n-1)(n-2)\cdots(n-r+1)\cdot(n-r)(n-r-1)\cdots 1 \\ &= n!. \end{aligned}$$

Equation (5) now takes the easily remembered form

$$C(n, r) = \frac{n!}{r!(n-r)!}. \tag{6}$$

In equation (6), if we everywhere replace r by $n - r$, we have, since $n - (n - r) = r$,

$$C(n, n-r) = \frac{n!}{(n-r)!\,r!}.$$

By comparing this result with equation (6), we find that

$$C(n, r) = C(n, n-r). \tag{7}$$

Equation (7) merely expresses the fact that the number of ways in which we can choose r from among n things equals the number of ways in which we can leave the remaining $n - r$ things unchosen.

Equation (7) is useful in computing $C(n, r)$ when $n - r$ is less than r, as in the following example.

EXAMPLE 2. In how many ways can 50 cards be chosen from among 52 playing cards?

We have, from equation (7),

$$C(52, 50) = C(52, 2) = \frac{52 \cdot 51}{2 \cdot 1} = 1326.$$

254. The Binomial Coefficients. By comparing formula (5) of the preceding article with equation (3) of Art. 247, we find that the numerical coefficient of $a^{n-r}x^r$ in the expansion of $(a + x)^n$ is precisely $C(n, r)$. It follows that the binomial formula [Art. 246, equation (1)] can be written in the form

$$\begin{aligned}(a + x)^n = a^n &+ C(n, 1)a^{n-1}x + C(n, 2)a^{n-2}x^2 + \cdots \\ &+ C(n, r-1)a^{n-r+1}x^{r-1} + \cdots + C(n, n)x^n. \end{aligned}\tag{8}$$

As a particular consequence of this theorem, we find, by putting $a = x = 1$, that

$$(1 + 1)^n = 1 + C(n, 1) + C(n, 2) + \cdots + C(n, r - 1) + \cdots + C(n, n),$$

or

$$\mathbf{C(n, 1) + C(n, 2) + \cdots + C(n, r - 1) + \cdots + C(n, n) = 2^n - 1.} \quad (9)$$

Equation (9) states that: *the total number of combinations of n things taken* 1 *at a time,* 2 *at a time, and so on to n at a time is* $2^n - 1$.

Exercises

1. Find the value of $C(8, 3)$, $C(12, 6)$, $C(16, 4)$, $C(19, 15)$, and $C(38, 35)$.

2. Express by a symbol and compute: the number of combinations of 9 things 5 at a time.

3. In a league of 12 teams, each team played each of the other teams. How many games were played? How many games did each team play?

4. A contractor employs 15 men. In how many ways can he choose 5 of them to do a certain job?

5. Find n, given $C(n, 2) = 78$.

6. Find n, given $P(n, 5) = 120C(n, 3)$.

7. Given 18 points in a plane no 3 of which lie on a line. How many lines can be drawn each of which passes through 2 of the points?

8. Given 10 points in space no 4 of which lie in a plane. How many planes are there each of which passes through 3 of the points? How many of these planes contain the line through 2 specified points of the set?

9. In how many ways can a committee of 5 boys and 4 girls be chosen from a club of 10 boys and 7 girls?

10. In how many ways can 52 playing cards be dealt equally to 4 players if the order of the hands but not of the cards in the hands is considered? Write the result using the factorial notation.

11. How many triangles are formed by 12 lines if each line meets each of the other lines but no three of the lines meet in a point? How many of these triangles have a given line as a side?

12. How many rectangles are formed when 8 vertical lines are intersected by 5 horizontal lines?

13. In Ex. 12, if the intervals between successive parallel lines is one inch, how many of the rectangles are squares?

14. Two men and their wives, 3 single men, and 3 single women form a club. How many committees of 5 can be formed if no man and his wife are on the same committee?

15. Find the total number of combinations of 8 things.

16. A merchant has a 1-, 2-, 4-, 8-, and 16- ounce weight. How many different weights can he form?

17. How many different sums of money can be formed from a cent, a nickel, a dime, a quarter, a half-dollar, and a dollar?

Chapter 32

Probability

255. Definitions. A **trial** of an event is an occasion on which the event may occur or fail to occur. If, on a trial, the event occurs, it is said to **succeed** on that trial; if it does not occur, it is said to **fail.**

Suppose, for example, we are considering the probability that a coin, if tossed, will come up heads. The act of tossing the coin is a trial. On any given trial, the event of coming up heads succeeds if the coin comes up heads; otherwise, it fails.

We desire to set up a *numerical measure* of the probability that a certain event will succeed on a future trial. There are two ways in which, under certain assumptions, we can sometimes set up such a measure. One of these ways leads to a measure which is called **mathematical probability**; the other to **empirical probability.** In both cases, it must always be remembered that the measures are valid only if the assumptions on which they are based are correct assumptions.

256. Mathematical Probability. *If an event can succeed in any one of s ways and can fail in any one of f ways, if any one of these $s + f$ ways is equally likely to occur and if one and only one of them must occur, then the probability, p, that the event will succeed, and the probability, q, that it will fail, on a given trial are, respectively,*

$$p = \frac{s}{s+f}, \quad \text{and} \quad q = \frac{f}{s+f}. \tag{1}$$

From the foregoing definitions, it follows that, if $f = 0$ and $s > 0$, then $p = 1$. Hence, *certainty of success is expressed by a probability $p = 1$.* Similarly, if $s = 0$ and $f > 0$, then $p = 0$; that is, *certainty of failure is expressed by a probability $p = 0$.* In any case, if all of the possibilities have been considered,

$$p + q = 1.$$

This equation constitutes a useful check on the accuracy with which p and q have been computed.

Example 1. From a bag containing 12 white balls and 18 red ones, one ball is drawn at random. What is the probability that the ball so drawn is white?

Here $s = 12, f = 18$, and $s + f = 30$. Hence, $p = \frac{12}{30} = \frac{2}{5}$.

As a (partial) check, we compute $q = \frac{18}{30} = \frac{3}{5}$ and observe that $p + q = \frac{2}{5} + \frac{3}{5} = 1$.

EXAMPLE 2. A committee of 5 is to be chosen by lot from 7 men and 5 women. What is the probability that the committee will consist of 3 men and 2 women?

The committee can be chosen in any one of $C(12, 5) = 792$ ways. Three men can be chosen in $C(7, 3) = 35$ ways and two women in $C(5, 2) = 10$ ways. Hence, $s = 35 \cdot 10 = 350$, $s + f = 792$, and $p = \frac{350}{792} = \frac{175}{396}$.

257. Empirical Probability. In many cases of considerable practical importance, such as life or fire insurance or business forecasting, it is not possible to determine the mathematical probability of the success of an event. It is sometimes possible, in such cases, to determine p approximately by observing a large number of cases and recording the relative frequency of successes. The probability, determined experimentally in this way, is called **empirical, or a posteriori, probability.**

If n (a large number) is the total number of observed trials made under a certain set of conditions, and if s is the observed number of successes, *we define, provisionally,*

$$p = \frac{s}{n},$$

as the empirical probability of success in one trial under the given conditions. This result is, of course, an approximate one and is subject to revision as the experimental data accumulate.

In practice, it is always necessary to remember that the required probability may be greatly modified by special conditions pertaining to the particular event under consideration. It would be obviously inaccurate, for example, to apply the life insurance tables of life expectation to a man in an advanced stage of tuberculosis or to compute the probability that it will rain here tomorrow from data compiled at another time or place. With questions of this sort, we shall not concern ourselves. We shall assume, throughout, that the observed probability is valid for the individuals, or the groups, to which we intend to apply it.

258. Mathematical Expectation. *If a person is to receive M dollars in case a certain event occurs, and if the probability that the event will occur is p, then the value of his expectation is Mp dollars.*

EXAMPLE. A man may take any one of four envelopes of which one contains \$10 and the other three are empty. What is the value of his expectation?

The probability that he will receive \$10 is $p = \frac{1}{4}$. Hence, the value of his expectation is $\$10 \times \frac{1}{4} = \$2.50$.

Exercises

1. A ball is drawn at random from a bag containing 7 red and 9 white balls. What is the probability that the ball drawn is red?

2. One card is drawn from a pack of 52 playing cards. What is the probability that it is (*a*) a club, (*b*) an ace?

3. Four coins are tossed. Find the probability that they will turn up (*a*) four heads, (*b*) two heads and two tails.

4. In a certain registration district, 24 boys and 30 girls were born in a certain month. Find the probability that the first child born that month was a boy.

5. A committee of 5 was chosen by lot from among 9 men. Find the probability that A was and B was not a member of this committee.

6. Nine students are seated at random at a round table. What is the probability that A and B sit side by side?

7. Solve Ex. 6 if they are seated in a row.

8. Six boys and 2 girls are seated at random at two bridge tables. Find the probability that the two girls are (*a*) partners, (*b*) seated at different tables.

9. Two dice were thrown. Find the probability that the sum of the numbers that turned up was (*a*) exactly eight, (*b*) greater than eight.

10. In Ex. 9, find the probability that, of the two numbers that turned up, (*a*) precisely one was a six, (*b*) neither was a six.

11. Five red and 4 green books are placed at random on a shelf. Find the probability that the middle and end positions are occupied by (*a*) red books, (*b*) green books.

12. From a pack of playing cards, 3 cards are drawn. Find the probability that they are an ace, a king, and a queen, all of different suits.

13. A man wrote 3 letters and addressed the corresponding envelopes. A servant put the letters at random in the envelopes. What is the probability (*a*) that every letter was put in its right envelope, (*b*) that no letter was put in its right envelope?

14. In a certain year, there were 5861 automobiles registered in a certain district and 238 accidents were reported. Ten years later, there were 9528 automobiles and 357 accidents. Was the probability of an accident to a given automobile greater or less during the latter year, and how much?

15. Each of 4 boys called at random on one of 3 girls. Find the probability (*a*) that at least one of the girls was not called on, (*b*) that a certain girl was not called on.

16. The prize in a lottery is $150. If there are 500 tickets, what is the value of the expectation of one of them?

17. A man is to receive $12 if, when 4 coins are tossed, 3 heads and 1 tail turn up. What is the value of his expectation?

18. A man is to receive $270 if, when 3 dice are thrown, the sum of the numbers that turn up is exactly 15. Find the value of his expectation.

19. A man insured his car against theft for one year for $650. Of 24,731 cars registered in his district, 94 were stolen during the year. What was the value of his expectation?

20. A man, 40 years old, takes out a $1000 endowment insurance policy which is to be paid to him if he is alive at the end of 20 years and to his estate

if he dies before that time. Using the data of Ex. 22, Art. 41, find to two significant figures his expectation and that of his estate.

21. In Ex. 20, find the probability that the man will be alive at age 55 and dead at age 60.

259. Mutually Exclusive Events. A set of events, any one of which may occur on a given trial, are **mutually exclusive** if the happening of any one of them on a trial excludes the possibility that any other one will happen on that trial.

Thus, if one ball is drawn from a bag containing red, white, and blue balls, the events that the ball drawn is red, or white, or blue are mutually exclusive since, if the ball drawn is of one of these colors, it cannot be either of the other colors.

Let $E_1, E_2, \cdots, E_k$ be a set of mutually exclusive events, let $p_1, p_2, \cdots, p_k$ be their respective probabilities, and let p be the probability that some one of these events will happen on a given trial. Then

$$p = p_1 + p_2 + \cdots + p_k, \tag{2}$$

that is, *the probability that some one of a set of mutually exclusive events will happen on a single trial is the sum of the probabilities for the separate events.*

For, suppose that a trial can result in any one of n equally probable ways and suppose that, of these ways, E_1 can succeed in s_1 ways, E_2 in s_2 ways, and so on. Then

$$p_1 = \frac{s_1}{n}, \quad p_2 = \frac{s_2}{n}, \cdots, \quad p_k = \frac{s_k}{n}.$$

Also, s, the number of ways in which some one event of the given set can succeed is

$$s = s_1 + s_2 + \cdots + s_k,$$

and

$$p = \frac{s}{n} = \frac{s_1}{n} + \frac{s_2}{n} + \cdots + \frac{s_k}{n} = p_1 + p_2 + \cdots + p_k.$$

EXAMPLE. From a pack of playing cards, two hearts, nine clubs, and six diamonds have already been drawn. What is the probability that the next card drawn will belong to one of these suits?

The probability of drawing a heart is $p_1 = \frac{11}{35}$; a club, is $p_2 = \frac{4}{35}$; and a diamond, is $p_3 = \frac{7}{35}$. The probability of drawing a card of one of these suits is

$$p = \tfrac{22}{35} = \tfrac{11}{35} + \tfrac{4}{35} + \tfrac{7}{35} = p_1 + p_2 + p_3.$$

260. Independent and Dependent Events. If $E_1, E_2, \cdots, E_k$ is a set of events such that the occurrence of any one of them does not affect the probability that any one of the others will also occur, then the events

of this set are said to be **independent**; if the occurrence of one of them does affect the probability that the others will also occur, they are **dependent.**

Thus, the event of drawing a white ball out of a bag containing white and black balls, and of throwing a six with a single die, are independent events. The event of passing an examination in a course in mathematics, and of passing the course, are dependent events.

If the events $E_1, E_2, \cdots, E_k$ are independent, and if their respective probabilities are $p_1, p_2, \cdots, p_k$, then the probability that all of them will succeed on a single trial is

$$p = p_1 \cdot p_2 \cdots p_k. \tag{3}$$

For simplicity, let $k = 2$. Suppose that E_1 can result in any one of n_1 ways, all equally likely to occur, and let s_1 of these ways be successes. Suppose further, that E_2 can result in n_2 ways, all equally likely, and let s_2 of these be successes. Then

$$p_1 = \frac{s_1}{n_1}, \quad \text{and} \quad p_2 = \frac{s_2}{n_2}.$$

By the fundamental principle of Art. 250, both events can occur in n_1n_2 ways of which s_1s_2 are successes. Hence,

$$p = \frac{s_1s_2}{n_1n_2} = \frac{s_1}{n_1} \cdot \frac{s_2}{n_2} = p_1 \cdot p_2,$$

and similarly for $k = 3$ or more.

If $E_1, E_2, \cdots, E_k$ are dependent events, if the probability that E_1 will succeed before any of the others have been tried is p_1, if the probability that E_2 will succeed after E_1 has succeeded but before any one of the remaining events has been tried is p_2, that E_3 will succeed after E_1 and E_2 has succeeded and none of the others have been tried is p_3, and so on, then it follows, by the reasoning given in the preceding case, that *the probability that all of the events will succeed in the order $E_1, E_2, \cdots, E_k$ is*

$$p = p_1 \cdot p_2 \cdots p_k, \tag{4}$$

that is, the probability is the same as though the events were independent *provided the order of trial is the same as the order for which the successive probabilities have been determined.*

Example 1. A and B are running in different races. The probability that A will win his race is $\frac{1}{2}$ and that B will win his is $\frac{1}{5}$. What is the probability that both will win?

The events are independent and

$$p = \tfrac{1}{2} \cdot \tfrac{1}{5} = \tfrac{1}{10}.$$

EXAMPLE 2. The probability that, at the time of a race, the track will be muddy is $\frac{1}{30}$. The probability that, in case the track is muddy, A will win the race is $\frac{2}{3}$. What is the probability that the track will be muddy and that A will win the race?

The probability that both events will occur in the order named is

$$p = \tfrac{1}{30} \cdot \tfrac{2}{3} = \tfrac{1}{45}.$$

261. Repeated Trials. Let p be the probability of success, and $q = 1 - p$ the probability of failure, of an event on a single trial. Further, we shall suppose that this probability remains unchanged throughout the entire series of trials under consideration. Then, *the probability P of precisely r successes among n trials is*

$$P = C(n, r)p^r q^{n-r}. \tag{5}$$

For, by Art. 253, we can choose r specified ones among these n trials to succeed, and the rest to fail, in any one of $C(n, r)$ ways. By equation (4), the probability that the r specified trials will succeed, and that the rest will fail, is $p^r q^{n-r}$. Further, if any one of these $C(n, r)$ choices of r successes does occur, then no other one of them can occur; that is, the choices of orders of successes and failures are mutually exclusive events. It now follows from equation (2) that the probability that some one of these choices will actually occur is $C(n, r)p^r q^{n-r}$, as stated in equation (5).

By the aid of equation (5), together with the identity $C(n, r) = C(n, n - r)$, we can show that *the probability of* **at least** *r successes in n trials is*

$$p^n + C(n, 1)p^{n-1}q + C(n, 2)p^{n-2}q^2 + \cdots + C(n, n - r)p^r q^{n-r}. \tag{6}$$

For, from (5), the probability of success in all the trials is p^n, in all but one is $C(n, n - 1)p^{n-1}q = C(n, 1)p^{n-1}q$, and so on. Since these events are mutually exclusive, it follows from equation (2) that the probability that some one of them will succeed is given by (6).

EXAMPLE 1. A die is cast five times. What is the probability that it will turn up a six precisely three times?

The probability that it will turn up a six on any one trial is $\frac{1}{6}$. Hence, from equation (4), the probability that it will turn up a six on precisely three out of five trials is

$$p = C(5, 3) \cdot \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2 = 10 \cdot \frac{5^2}{6^5} = \frac{125}{3888}.$$

EXAMPLE 2. A man bets five times on a gambling device on which his chances of winning on a single trial are $\frac{1}{3}$. Find the probability that he will win at least two of the five times.

$$p = (\tfrac{1}{3})^5 + 5(\tfrac{1}{3})^4(\tfrac{2}{3}) + 10(\tfrac{1}{3})^3(\tfrac{2}{3})^2 + 10(\tfrac{1}{3})^2(\tfrac{2}{3})^3 = \tfrac{131}{243}.$$

EXAMPLE 3. An automobile driver habitually engages in a practice in which the probability of an accident is 0.01. What is the probability that he can take this chance 100 times without an accident? In how many trials will the probability of an accident be 0.5?

The probability of avoiding an accident in every one of 100 trials is $p = (0.99)^{100} = 0.37$, approximately.

In n trials, the probability of avoiding an accident is $(0.99)^n$. If we equate this number to 0.5 and solve for n by logarithms, we find $n = 69$, approximately.

Exercises

1. A bag contains 9 red, 5 white, and 4 blue balls. Three balls are drawn at random and each is replaced before the next one is drawn. Find the probability that the first ball drawn is white, the second red, and the third blue.

2. Solve Ex. 1 if the balls drawn are not replaced.

3. In Ex. 1, find the probability that one of the balls drawn is white, another red, and another blue.

4. The probability that A will win a certain prize is $\frac{2}{7}$ and that B will win it is $\frac{1}{3}$. Find the probability that one of them will win it.

5. In Ex. 4, if A and B are competing for different prizes, what is the probability that at least one of them will win?

6. One bag contains 3 white and 5 black balls; another 4 white and 7 black balls. A man chooses one bag at random and draws one ball. Find the probability that the ball drawn is white.

7. A and B take turns in tossing a coin. The first to toss a head is to receive \$15. If A tosses first, find the value of his expectation.

8. The probability that A will win a prize of \$120 if B does not compete is $\frac{3}{4}$ and, if B does compete, is $\frac{1}{6}$. The probability that B will compete is $\frac{5}{8}$. Find the value of A's expectation.

9. A, B, and C, having tied for first place in a tournament, agree that each shall play one game against each of the others. If anyone wins two games, he wins the tournament. If no one wins two games, the winner is not decided. If each man's probability of winning a game is $\frac{1}{2}$, find the probability that A will win the tournament.

10. In a lottery, there are three prizes, one of \$300, one of \$200, and one of \$100. If there are 2000 tickets, what is the expectation of one of them?

11. A plays a set of 9 games against B. If A's probability of winning a game is $\frac{3}{5}$, and if B has already won the first two games, what is the probability that A will win at least five games?

12. Six students were assigned a problem. If that probability that any one student can solve this problem is $\frac{2}{3}$, what is the probability that at least three of them will solve it?

13. A coin is tossed 8 times. What is the most probable number of times that heads will come up and what is this probability?

14. Two dice are thrown until either a 7 or an 11 comes up. What is the probability that a 7 will come up first?

15. A man pays \$1 for the right to play a game in which his chances of winning are $\frac{1}{5}$. If he wins, he receives \$3; if he loses he receives nothing. If he starts playing with \$3, what is the probability that he can play at least 9 games?

16. In a football game, each contestant made 3 touchdowns. The probability that either contestant would kick goal after a touchdown is $\frac{2}{3}$. Find the probability that the score was a tie.

17. A club of 60 members meets 52 times a year. At each meeting, one member, chosen by lot, receives a prize. Find, to two significant figures, the probability that Mr. A will receive the prize (*a*) at least once and (*b*) at least twice, during the year.

$i^2 = -1 \qquad i^4 = +1$

Chapter 33

Complex Numbers

262. Definitions. A **complex number** is one that can be written in the form

$$a + bi,$$

where a and b are real numbers and $i = \sqrt{-1}$.*

The number a is the **real part** and bi is the **imaginary part** of the complex number. If $b = 0$, the complex number is a **real number**; if $a = 0$ and $b \neq 0$, it is a **pure imaginary number.**

Thus, $5 - 4i$ is a complex number whose real part is 5 and whose imaginary part is $-4i$; $4 = 4 + 0i$ is a real complex number; and $8i = 0 + 8i$ is a pure imaginary number.

Two complex numbers, $a + bi$ and $a - bi$, which differ only in the sign of the imaginary part, are **conjugate complex numbers** and either of them is said to be the **conjugate** of the other.

Thus, $-7 + 3i$ and $-7 - 3i$ are conjugate complex numbers. The number conjugate to $5 - 2i$ is $5 + 2i$.

Two complex numbers, $a + bi$ and $c + di$, are equal if, and only if, $a = c$ and $b = d$. In particular, *if $a + bi = 0$, then $a = 0$ and $b = 0$.*

263. Operations with Complex Numbers. The operations of addition, subtraction, multiplication, and division of complex numbers are performed according to the ordinary rules of algebra. The results should be simplified, whenever possible, by putting $i^2 = -1$.

Thus,

$$3 + i + (-2 + 6i) = 3 - 2 + (1 + 6)i = 1 + 7i.$$
$$5 - 2i - (-3 + 7i) = 5 + 3 + (-2 - 7)i = 8 - 9i.$$
$$7(5 + 2i) = 7 \cdot 5 + 7 \cdot 2i = 35 + 14i.$$
$$(-3 + 2i)(4 + 7i) = -12 - 21i + 8i + 14i^2 = -26 - 13i.$$
$$(a + bi)(c + di) = ac - bd + (ad + bc)i.$$

To express the quotient of two complex numbers, we write the required quotient as a fraction, then multiply both the numerator and the denominator by the complex number conjugate to the denominator; we have

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{ac + bd + (bc - ad)i}{c^2 + d^2}$$
$$= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} i.$$

* Throughout this chapter, the letter i denotes $\sqrt{-1}$. All other literal numbers are assumed to be real.

Thus, $\dfrac{2+7i}{4-3i} = \dfrac{2+7i}{4-3i}\cdot\dfrac{4+3i}{4+3i} = \dfrac{8+28i+6i+21i^2}{16+9} = \dfrac{-13}{25}+\dfrac{34}{25}i.$

If a number is given in a form involving radicals, or fractional or negative exponents, it should be written in the form $a+bi$ before any operations are performed with it. In particular, if b is positive, the expression $a+\sqrt{-b}$, or $a+(-b)^{\frac{1}{2}}$, should be replaced by $a+\sqrt{b}i$. *Failure to do this may lead to incorrect results.*

Thus, to multiply $\sqrt{-2}$ by $\sqrt{-3}$ correctly, we proceed as follows:

$$\sqrt{-2}\,\sqrt{-3} = \sqrt{2}i\cdot\sqrt{3}i = \sqrt{6}i^2 = -\sqrt{6}.$$

The following computation is *erroneous:*

$$\sqrt{-2}\,\sqrt{-3} = \sqrt{(-2)(-3)} = \sqrt{6}.$$

Exercises

Simplify the following numbers and write them in a form involving i.

1. $\sqrt{-4}$. **2.** $-\sqrt{-12}$. **3.** $7\sqrt{-36}$. **4.** $-15\sqrt{-0.16}$.

5. $\sqrt{-9x^2}$. **6.** $3+\sqrt{-5}$. **7.** $8-2\sqrt{-18}$. **8.** $\dfrac{\sqrt{45}-\sqrt{-27}}{3}$.

9. Simplify i^3, i^4, i^9, i^{67}, i^{-1}.

Write the conjugate of each given complex number.

10. $2+7i$. **11.** $-5-3i$. **12.** $9-\sqrt{-5}$. **13.** $3x+4yi$.

Perform the indicated operations and write each expression in the form $a+bi$.

14. $(8+6i)+(-3+2i)$.

15. $(-7+3i)+(4-5i)$.

16. $(11-5i)-(6+3i)$.

17. $(-1+8i)-(-6+9i)$.

18. $(5-2i)+(3+4i)-(1-3i)$.

19. $(-5+7i)-(3+4i)-(-11+9i)$.

20. $(3+\sqrt{-4})+(5-\sqrt{-36})$.

21. $-\sqrt{2}+\sqrt{-20}-(\sqrt{18}-\sqrt{-80})$.

22. $\sqrt{-4}\,\sqrt{-25}$.

23. $\sqrt{-12}\,(\sqrt{3}-\sqrt{-45})$.

24. $(3-2i)(2+5i)$.

25. $(2+3i)(-5+7i)$.

26. $(2+7i)^2$.

27. $(x+yi)^2$.

28. $(3+\sqrt{-8})\,(5-\sqrt{-2})$.

29. $(5-\sqrt{-3})\,(2+\sqrt{-7})$.

30. $(4-5i)^3$.

31. $(2+i)^4$.

32. $\dfrac{4-5i}{3+4i}$.

33. $\dfrac{-6-5i}{2-i}$.

34. $(-8+3i)\div(1+3i)$.

35. $(11+5i)\div(4-i)$.

36. $\dfrac{5-\sqrt{-6}}{3+\sqrt{-2}}$.

37. $\dfrac{\sqrt{2}+\sqrt{-5}}{\sqrt{3}-\sqrt{-7}}$.

38. $(1-4i)^{-1}$.

39. $(2-\sqrt{-3})^{-2}$.

Find the real numbers x and y, given:

40. $x + yi = 5 - \sqrt{-9}$.
41. $(x + 1) - (3y - 6)i = 0$.
42. $(2x - 1) + (3y + 1)i = 3 - 8i$.
43. $(7x + 4y) + (2x + 5y)i = 10 - i$.
44. $(xy - 1) + (2x + y - 9)i = 5 - 2i$.
45. $(2x^2 + y^2 - 1) + (x^2 - 2y^2)i = 8 + 2i$.

Perform the indicated multiplications.

46. $[x - (1 + 2i)][x - (1 - 2i)]$.
47. $[x - (3 + 2i)][x - (3 - 2i)]$.
48. $\left(x - \dfrac{1 + \sqrt{3}i}{2}\right)\left(x - \dfrac{1 - \sqrt{3}i}{2}\right)$.
49. $[x - (a + bi)][x - (a - bi)]$.

Form a quadratic equation whose roots are the given numbers.

50. $3 + i,\ 3 - i$.
51. $-5 + 2i,\ -5 - 2i$.
52. $-2 + \sqrt{-7},\ -2 - \sqrt{-7}$.

Write each expression as the product of two factors, each linear in x.

53. $x^2 - 4x + 13$.
54. $x^2 + 8x + 17$.

HINT. Equate each expression to zero and solve for x. Then x minus each root is a factor of the given expression.

264. Graphical Representation of Complex Numbers. In dealing with complex numbers, it has been found helpful to represent them graphically by points in a plane. We first set up a pair of coördinate axes in the usual way (Fig. 168). Then, *the point having the coördinates* (a, b) *is the graphical representation of the complex number* $a + bi$.

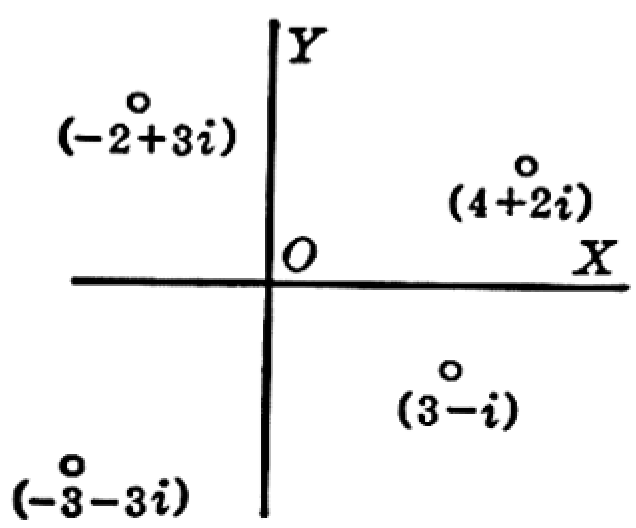


FIG. 168

In Figure 168, we have plotted the points representing the complex numbers $4 + 2i$, $-2 + 3i$, $-3 - 3i$, and $3 - i$.

The points representing the real numbers, $a + 0i$, have their y-coördinates equal to zero and lie on the x-axis. This line is, accordingly, called the **axis of reals.** Similarly, the points representing the pure imaginary numbers, $0 + bi$, lie on the y-axis, which is now called the **axis of imaginaries.** The plane on which the complex numbers are plotted is called the **complex plane.**

Exercises

Represent the following complex numbers graphically.

1. $2 + 3i$.
2. $-1 + 3i$.
3. $-4 - 9i$.
4. $5 - 2i$.
5. $8 + 0i$.
6. $0 - 2i$.
7. $4 - \sqrt{-3}$.
8. $5 + \sqrt{-7}$.

Plot the following points and write the complex number that each represents. Write, also, the conjugate complex number in each case and plot the point that represents it.

9. $(4, 3)$.
10. $(-8, -5)$.
11. $(0, 3)$.
12. $(3, -7)$.
13. $(9, 0)$.
14. $(-4, 9)$.
15. $(8, 3)$.
16. $(1 + \sqrt{5}, -4)$.

17. Let the points A and B represent the complex numbers $a + bi$ and $c + di$, respectively. Draw OA and OB and complete the parallelogram $OACB$ having OA and OB as adjacent sides. Show that C represents the sum of the given complex numbers $a + bi$ and $c + di$.

265. The Trigonometric Form of a Complex Number. Let P (Fig. 169) represent the complex number $a + bi$. In a system of polar coördinates having O as origin and OX as initial side, let r and θ be the polar coördinates of P, these coördinates being chosen so that r is *positive*, or zero. From the formulas for changing from rectangular to polar coördinates, and conversely (Art. 176), we have,

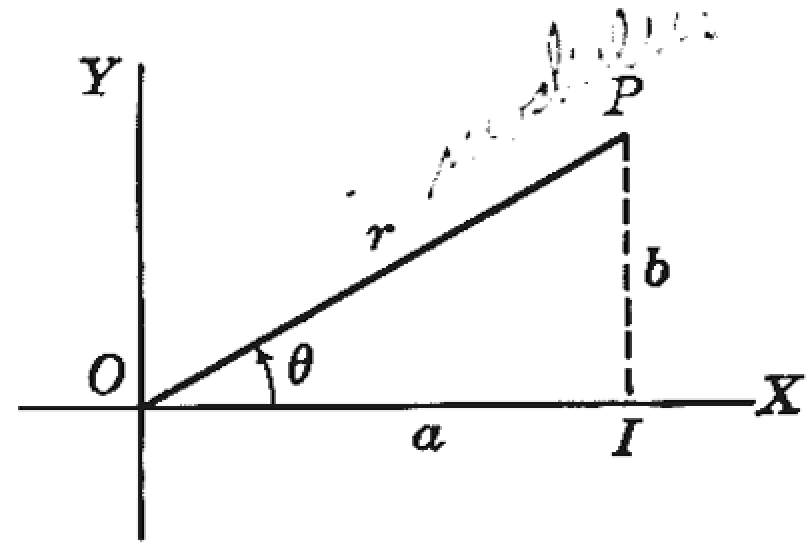


FIG. 169

$$r = \sqrt{a^2 + b^2}, \quad \text{and} \quad \tan \theta = \frac{b}{a}; \tag{1}$$

and

$$a = r \cos \theta, \quad \text{and} \quad b = r \sin \theta. \tag{2}$$

By substituting the values of a and b from equations (2) in the expression $a + bi$ we obtain

$$a + bi = r(\cos \theta + i \sin \theta). \tag{3}$$

The expression $r(\cos \theta + i \sin \theta)$ is called the **trigonometric, or polar, form** of the complex number and $a + bi$ is its **rectangular form.** The angle θ is the **angle, amplitude,** or **argument** of the complex number and the *positive* (or zero) number r is its **modulus, or absolute value.**

For the angle θ, of the complex number, we may take any angle having OX as its initial side and OP as its terminal side. Since these angles differ by integral multiples of 360°, we may write, in place of equation (3),

$$a + bi = r[\cos (\theta + k360°) + i \sin (\theta + k360°)], \tag{4}$$

where k is any positive or negative integer, or zero.

There are thus an unlimited number of values for the angle of a complex number. These values differ by integral multiples of 360°. Moreover, *two complex numbers are equal if their absolute values are equal and their angles differ by an integral multiple of* 360°.

To find the trigonometric form of a complex number given in the rectangular form, we first plot the point representing the complex number and determine the quadrant in which it lies. We then find r and θ with the aid of a table of square roots and a table of natural tangents, choosing θ so that its terminal side lies in the required quadrant. These values of r and θ, when substituted in the expression $r(\cos \theta + i \sin \theta)$, define the given number in the trigonometric form.

EXAMPLE 1. Write the number $-1+\sqrt{3}i$ in the trigonometric form

Plot the point $-1+\sqrt{3}i$ (Fig. 170).

From equation (1), $r=\sqrt{(-1)^2+(\sqrt{3})^2}=2$ and $\tan\theta=-\sqrt{3}$. Since the point representing $-1+\sqrt{3}i$ lies in the second quadrant, we take $\theta=120°$. Then $-1+\sqrt{3}i=2(\cos 120°+i\sin 120°)$. The second member of this equation is the required trigonometric form of the given number.

EXAMPLE 2. Write the number $12-5i$ in the trigonometric form.

The point $12-5i$ lies in the fourth quadrant (Fig. 171). We have $r=\sqrt{(12)^2+(-5)^2}=13$ and $\tan\theta=-\frac{5}{12}=-0.4167$. With the aid of Table III, we find $\theta=337°\,23'$, approximately. Hence,

$$12-5i=13(\cos 337°\,23'+i\sin 337°\,23').$$

To represent the number $r(\cos\theta+i\sin\theta)$ graphically, construct (geometrically or with the aid of a protractor) the angle θ having O

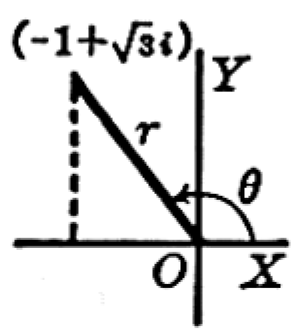


FIG. 170

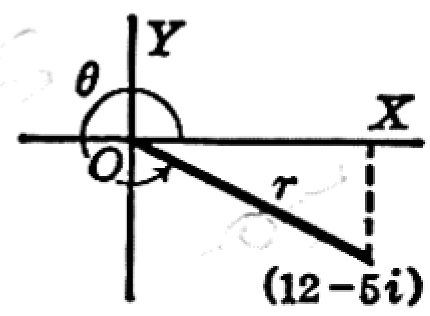


FIG. 171

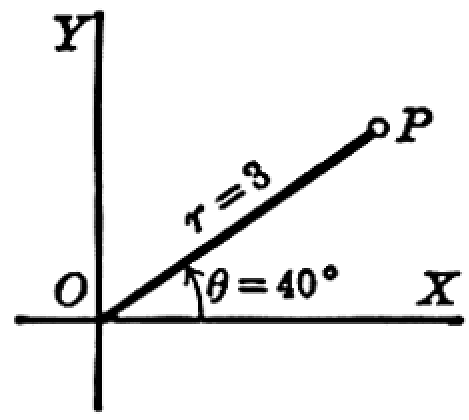


FIG. 172

as its vertex and OX as its initial side. On the terminal side of the angle, lay off $OP=r$. Then P is the point representing the given complex number.

EXAMPLE 3. Represent the complex number $3(\cos 40°+i\sin 40°)$ graphically and write it in the rectangular form.

Plot the point whose polar coördinates are $(3, 40°)$ (Fig. 172).

To write this number in the rectangular form, we replace $\cos 40°$ and $\sin 40°$ by their values from Table III. We have

$$3(\cos 40°+i\sin 40°)=3(0.7660+i\,0.6428)=2.298+i\,1.928.$$

Exercises

Represent each of the following numbers graphically and write it in the trigonometric form.

1. $5+5i$. **2.** $-3+3i$. **3.** $1-\sqrt{3}i$. **4.** $-\sqrt{2}-\sqrt{2}i$.

5. $7i$. **6.** $-\sqrt{12}+2i$. **7.** $-\sqrt{-9}$. **8.** $\sqrt{6}-\sqrt{-2}$.

9. -5. **10.** $\sqrt{8}-\sqrt{-8}$. **11.** $5-2i$. **12.** $-\sqrt{5}-\sqrt{-11}$.

Represent each of the following numbers graphically and write it in the rectangular form.

13. $4(\cos 30^\circ + i \sin 30^\circ)$.
14. $2(\cos 120^\circ + i \sin 120^\circ)$.
15. $6(\cos 225^\circ + i \sin 255^\circ)$.
16. $8(\cos 300^\circ + i \sin 300^\circ)$.
17. $7(\cos 180^\circ + i \sin 180^\circ)$.
18. $5(\cos 0^\circ + i \sin 0^\circ)$.
19. $3(\cos 90^\circ + i \sin 90^\circ)$.
20. $9(\cos 270^\circ + i \sin 270^\circ)$.
21. $10(\cos 34^\circ + i \sin 34^\circ)$.
22. $5(\cos 141^\circ + i \sin 141^\circ)$.
23. $6(\cos 217^\circ 12' + i \sin 217^\circ 12')$.
24. $7(\cos 348^\circ 34' + i \sin 348^\circ 34')$.

266. Product of Two Complex Numbers. *The absolute value of the product of two complex numbers is the product of their absolute values and its angle is the sum of their angles.*

For, by actual multiplication by the method shown in Art. 263, we find

$$r_1(\cos \theta_1 + i \sin \theta_1) \cdot r_2(\cos \theta_2 + i \sin \theta_2) \tag{5}$$
$$= r_1r_2[(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)].$$

On replacing the quantities in parentheses by their values from formulas I and II of Art. 123, we have as the formula for the product of two complex numbers:

$$\boldsymbol{r_1(\cos \theta_1 + i \sin \theta_1) \cdot r_2(\cos \theta_1 + i \sin \theta_2)}$$
$$\boldsymbol{= r_1r_2[\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)].} \tag{6}$$

This process of multiplication may be extended to any number of factors. If there are three factors, for example, we have

$$r_1(\cos \theta_1 + i \sin \theta_1) \cdot r_2(\cos \theta_2 + i \sin \theta_2) \cdot r_3(\cos \theta_3 + i \sin \theta_3)$$
$$= r_1r_2[\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)] \cdot r_3(\cos \theta_3 + i \sin \theta_3)$$
$$= r_1r_2r_3[\cos (\theta_1 + \theta_2 + \theta_3) + i \sin (\theta_1 + \theta_2 + \theta_3)].$$

EXAMPLE. Multiply $2(\cos 60^\circ + i \sin 60^\circ)$ by $10(\cos 150^\circ + i \sin 150^\circ)$.

By equation (6), we have

$$2(\cos 60^\circ + i \sin 60^\circ) \cdot 10(\cos 150^\circ + i \sin 150^\circ) = 20(\cos 210^\circ + i \sin 210^\circ).$$

This multiplication is equivalent to the following one in the rectangular form

$$(1 + \sqrt{3}i)(-5\sqrt{3} + 5i) = -10\sqrt{3} - 10i,$$

which is the product obtained by the method of Art. 263.

267. Quotient of Two Complex Numbers. *The quotient of two complex numbers is given by the equation*

$$\boldsymbol{\frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)} = \frac{r_1}{r_2}[\cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2)].} \tag{7}$$

For, if we multiply both the numerator and the denominator of the first member of equation (7) by $\cos(-\theta_2) + i \sin(-\theta_2)$, and apply to each the rule for multiplication given in the preceding article, we obtain, since $\cos 0^\circ = 1$ and $\sin 0^\circ = 0$,

$$\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)\cdot[\cos(-\theta_2) + i\sin(-\theta_2)]}{r_2(\cos\theta_2 + i\sin\theta_2)\cdot[\cos(-\theta_2) + i\sin(-\theta_2)]}$$

$$= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)],$$

which is the result stated in equation (7).

Exercises

Perform the indicated operations. In Ex. 1, 2, 3, 4, 9, and 10, check your results by performing the operations in the rectangular form.

1. $4(\cos 30° + i\sin 30°)3(\cos 120° + i\sin 120°)$.
2. $6(\cos 60° + i\sin 60°)5(\cos 150° + i\sin 150°)$.
3. $11(\cos 45° + i\sin 45°)4(\cos 90° + i\sin 90°)$.
4. $9(\cos 225° + i\sin 225°)7(\cos 315° + i\sin 315°)$.
5. $6(\cos 45° + i\sin 45°)8(\cos 75° + i\sin 75°)$.
6. $10(\cos 60° + i\sin 60°)6(\cos 50° + i\sin 50°)$.
7. $13(\cos 38° + i\sin 38°)5(\cos 73° + i\sin 73°)$.
8. $15(\cos 84° + i\sin 84°)11(\cos 147° + i\sin 147°)$.
9. $\dfrac{18(\cos 150° + i\sin 150°)}{6(\cos 30° + i\sin 30°)}$.
10. $\dfrac{35(\cos 300° + i\sin 300°)}{7(\cos 240° + i\sin 240°)}$.
11. $\dfrac{28(\cos 123° + i\sin 123°)}{4(\cos 47° + i\sin 47°)}$.
12. $\dfrac{6(\cos 74° + i\sin 74°)}{15(\cos 41° + i\sin 41°)}$.
13. Show that $\dfrac{1}{r(\cos\theta + i\sin\theta)} = \dfrac{1}{r}[\cos(-\theta) + i\sin(-\theta)]$.

268. Integral Powers of a Complex Number. De Moivre's Theorem. If, in equation (6), the two given complex numbers to be multiplied are equal, we have

$$[r(\cos\theta + i\sin\theta)]^2 = r^2(\cos 2\theta + i\sin 2\theta).$$

If we multiply both sides of this equation by $r(\cos\theta + i\sin\theta)$ and simplify the second member by equation (6), we have

$$[r(\cos\theta + i\sin\theta)]^3 = r^3(\cos 3\theta + i\sin 3\theta).$$

If we continue this process of multiplying by $r(\cos\theta + i\sin\theta)$ and simplifying the second member by means of equation (6), we obtain, *for all positive, integral values of n,*

$$\boldsymbol{[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta).} \tag{8}$$

In particular, if $r = 1$, this equation becomes

$$\boldsymbol{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.} \tag{9}$$

The identity (9) is called **De Moivre's Theorem.**

EXAMPLE 1. Find the cube of $2(\cos 45° + i \sin 45°)$.

By equation (8),

$$[2(\cos 45° + i \sin 45°)]^3 = 8(\cos 135° + i \sin 135°).$$

This equation is equivalent to the following one in the rectangular form

$$(\sqrt{2} + \sqrt{2}i)^3 = -4\sqrt{2} + 4\sqrt{2}i,$$

which may be verified by direct multiplication.

EXAMPLE 2. Find the value of $\left(\frac{3\sqrt{3}}{2} + \frac{3i}{2}\right)^5$.

$$\left(\frac{3\sqrt{3}}{2} + \frac{3i}{2}\right)^5 = [3(\cos 30° + i \sin 30°)]^5$$
$$= 3^5(\cos 150° + i \sin 150°) = \frac{-243\sqrt{3}}{2} + \frac{243}{2}i.$$

Exercises

Write each of the following powers in the trigonometric form. Express the result also in the rectangular form.

1. $(1 + i)^3$.
2. $(\sqrt{3} + i)^4$.
3. $(0 + 2i)^5$.
4. $\left(\frac{-3}{2} - \frac{\sqrt{3}i}{2}\right)^5$.
5. $(\sqrt{2} + \sqrt{-6})^4$.
6. $(3 + 0i)^6$.
7. $[2(\cos 60° + i \sin 60°)]^3$.
8. $[\sqrt{3}(\cos 135° + i \sin 135°)]^4$.
9. $[\sqrt[5]{5}(\cos 15° + i \sin 15°)]^{10}$.
10. $(\cos 1° + i \sin 1°)^{45}$.

269. Roots of Complex Numbers. To find the nth roots of a complex number $a + bi$, we first write the number in the form given in equation (4)

$$a + bi = r[\cos (\theta + k360°) + i \sin (\theta + k360°)], \qquad (10)$$

where k is any positive or negative integer, or zero. Let

$$R(\cos \phi + i \sin \phi) \qquad (11)$$

be any nth root of the given number. From the definition of an nth root of a number, we have

$$[R(\cos \phi + i \sin \phi)]^n = r[\cos (\theta + k360°) + i \sin (\theta + k360°)].$$

From equation (8), we have

$$[R(\cos \phi + i \sin \phi)]^n = R^n(\cos n\phi + i \sin n\phi).$$

It follows that

$$R^n(\cos n\phi + i \sin n\phi) = r[\cos (\theta + k360°) + i \sin (\theta + k360°)].$$

If we equate the absolute values and the angles of the two members of this equation, we have

$$R^n = r, \qquad \text{or} \quad R = \sqrt[n]{r},$$

and

$$n\phi = \theta + k360°, \quad \text{or} \quad \phi = \frac{\theta + k360°}{n}.$$

Substitute these values of R and ϕ in the expression (11). We obtain, as *the expression for the nth roots of the complex number* (10),

$$\sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{k360°}{n}\right) + i \sin\left(\frac{\theta}{n} + \frac{k360°}{n}\right)\right], \tag{12}$$

where k is any positive or negative integer, or zero.

If $r \neq 0$, the expression (12) assumes n distinct values when we assign to k the n values $0, 1, 2, \cdots, n-1$. Hence, *any complex number, except zero, has n distinct nth roots. These n roots may be found by assigning to k, in the expression* (12), *the n values* $0, 1, 2, \cdots, n-1$.

EXAMPLE 1. Find the square roots of $9i = 9(\cos 90° + i \sin 90°)$.

From (12), by putting $k = 0$ and $k = 1$, we obtain, as the required roots,

$$3(\cos 45° + i \sin 45°), \quad \text{and} \quad 3(\cos 225° + i \sin 225°),$$

or

$$\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i, \qquad \text{and} \qquad \frac{-3\sqrt{2}}{2} + \frac{-3\sqrt{2}}{2}i.$$

EXAMPLE 2. Find the three cube roots of $4\sqrt{2} - 4\sqrt{2}i$.

Write the given number in the trigonometric form

$$4\sqrt{2} - 4\sqrt{2}i = 8(\cos 315° + i \sin 315°).$$

By putting $k = 0$, 1, and 2, in (12), we obtain the required cube roots:

$2(\cos 105° + i \sin 105°)$, $2(\cos 225° + i \sin 225°)$, and $2(\cos 345° + i \sin 345°)$

or

$$\frac{\sqrt{2} - \sqrt{6}}{2} + \frac{\sqrt{6} + \sqrt{2}}{2}i, \; -\sqrt{2} - \sqrt{2}i, \; \text{and} \; \frac{\sqrt{6} + \sqrt{2}}{2} + \frac{\sqrt{2} - \sqrt{6}}{2}i.$$

Exercises

Find all the required roots in the trigonometric form. When you can without using the tables, write them also in the rectangular form.

1. The square roots of $2 + 2\sqrt{3}i$.

2. The square roots of $-16i$.

3. The cube roots of unity.

4. The cube roots of $-4 + 4\sqrt{3}i$.

5. The fourth roots of 625.

6. The fourth roots of $-8 - 8\sqrt{3}i$.

7. The cube roots of $\dfrac{-27\sqrt{2}}{2} + \dfrac{27\sqrt{2}}{2}i$.

8. The fourth roots of $-16i$.

9. The fifth roots of $16 + 16\sqrt{3}i$.

10. The sixth roots of $-4\sqrt{3} + 4i$.

Solve the following equations by writing the second members in the trigonometric form and finding the required roots.

11. $x^3 = -27$.

12. $x^3 = i$.

13. $x^4 = -50 + 50\sqrt{3}i$.

14. $x^5 = 32$.

15. $x^4 = -81$.

16. $x^9 = 1$.

Chapter 34

Theory of Equations

270. Integral Rational Functions. An expression of the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n, \qquad a_0 \neq 0$$

in which n is a positive integer and the coefficients $a_0, a_1, \cdots a_n$ are constants, is a **polynomial,** or an **integral rational function,** of degree n, in x. The equation formed by equating such a polynomial to zero is a **polynomial equation,** or an **integral rational equation** in x.

An integral rational function is in the **standard form** if it is arranged in decreasing powers of x from a_0x^n to a_n, inclusive. Any missing terms must be supplied with zero coefficients.

Thus, $$8x^3 - 7x^5 + 2x^2 + 3x^6 - 8x,$$

is a polynomial, or integral rational function. If we write this function in the standard form, we have

$$3x^6 - 7x^5 + 0x^4 + 8x^3 + 2x^2 - 8x + 0.$$

From this standard form, we see that the degree of this polynomial is $n = 6$ and that $a_0 = 3$, $a_1 = -7$, $a_2 = 0$, $a_3 = 8$, $a_4 = 2$, $a_5 = -8$, and $a_6 = 0$.

We shall assume throughout this chapter, unless the contrary is stated, that the coefficients a_0, a_1, and so on, are real numbers and that the polynomial is written in the standard form.

Since the purpose of this chapter is to study the properties of integral rational functions and equations, we shall, throughout this chapter, use the symbols $f(x)$, $F(x)$, $q(x)$, etc., only to indicate functions of this particular type.

Exercises

Write each of the following integral rational functions in the standard form.

1. $3x^2 - 2x + 5x^3 - 8.$

2. $5x^3 - 3x^2 - 9 + 7x^4 - x.$

3. $4x - 9x^4 + 2x^2 + 3.$

4. $6x^4 - 8x^5 + 2x - 1.$

5. $(2x - 1)(x^2 - 3) + 5x^2 - 1.$

6. $(x - 2)^3 + (x + 1)^3 + 6x^2.$

7. $(x - 1)(x - 2)(x - 3).$

8. $(x^2 - 5x + 3)(x^2 + 8x + 1).$

9. Given $f(x) = x^3 - 5x^2 + 3x - 1$, find $f(3)$, $f(-2)$, $f(-x)$, $f\left(\frac{x}{2}\right)$, $f(3y)$.

10. Divide $f(x) = x^3 - 4x^2 + 2x + 1$ by $x - 3$ and state the quotient and the remainder. Find, also, $f(3)$.

271. Synthetic Division. In the work that follows, it will be necessary, so many times, to carry through the computation of dividing a polynomial by the special binomial $x - r$ that it will be worth while to shorten the division process as much as possible. The shortened process which we shall arrive at is called **synthetic division.**

In the following example 1, we shall first carry out the division in full, then we shall show, in successive steps, how the computation can be shortened.

EXAMPLE 1. Divide $2x^3 - 9x^2 + 4x + 8$ by $x - 3$.

The computation is written out in full in diagram I. Notice that the divisor has been written to the right of the dividend and the quotient above it.

I

$$\begin{array}{rrrrl}
2x^2 & -\,3x & -\,5 & & \text{(Quotient)}\\ \hline
2x^3 & -\,9x^2 & +\,4x & +\;\;8 & \big|\,\underline{x-3}\\
2x^3 & -\,6x^2 & & & \\ \cline{1-2}
 & -\,3x^2 & +\,4x & & \\
 & -\,3x^2 & +\,9x & & \\ \cline{2-3}
 & & -\,5x & +\;\;8 & \\
 & & -\,5x & +\,15 & \\ \cline{3-4}
\text{(Remainder)} & & & -\;\;7 &
\end{array}$$

II

$$\begin{array}{rrrrl}
2x^2 & -\,3x & -\,5 & & \\ \hline
2x^3 & -\,9x^2 & +\,4x & +\;\;8 & \big|\,\underline{x-3}\\
 & -\,6x^2 & & & \\ \cline{1-3}
 & -\,3x^2 & & & \\
 & & +\,9x & & \\ \cline{2-4}
 & & -\,5x & & \\
 & & & +\,15 & \\ \cline{3-4}
 & & & -\;\;7 &
\end{array}$$

To obtain II, we have rewritten I, leaving out the terms that are always the same as those directly above them.

To obtain III, we have made the work more compact by bringing up the terms that were scattered downward and writing everything on four lines.

III

$$\begin{array}{rrrrl}
2x^2 & -\,3x & -\,5 & & \\ \hline
2x^3 & -\,9x^2 & +\,4x & +\;\;8 & \big|\,\underline{x-3}\\
 & -\,6x^2 & +\,9x & +\,15 & \\ \hline
 & -\,3x^2 & -\,5x & -\;\;7 &
\end{array}$$

IV

$$\begin{array}{rrrrl}
2 & -\,3 & -\,5 & & \\
2 & -\,9 & +\,4 & +\;\;8 & \big|\,\underline{-3}\\
 & -\,6 & +\,9 & +\,15 & \\ \hline
2 & -\,3 & -\,5 & -\;\;7 &
\end{array}$$

To obtain IV, we make the following changes:

(*a*) In the divisor, omit the first term, which is always x.

(*b*) In the rest of the computation, omit x and all its powers, since each of these is indicated by the position of its coefficient.

(*c*) Copy the first coefficient from the second line on the fourth line

To obtain V, which is the final form, we:

(*a*) Omit the coefficients of the quotient, since these appear, in their proper order, on the last line.

(*b*) Change the sign of the term that appears in the divisor.

(*c*) Change all the signs in the second line.

V

$$\begin{array}{rrrrl}
2 & -\,9 & +\,4 & +\;\;8 & \big|\,\underline{3}\\
 & 6 & -\,9 & -\,15 & \\ \hline
2 & -\,3 & -\,5 & (-\,7) &
\end{array}$$

It will be observed, in the final form V, that:

(1) Each number in the second line is the product of the preceding number in the third line by the number that appears in the divisor.

(2) Each number in the third line is the *sum* of the two numbers directly above it.

(3) The last number in the third line is the remainder on the division; that is, the remainder on this division is -7.

(4) The remaining numbers in the third line, in order, are the coefficients of the terms of the quotient; that is, the quotient is $2x^2 - 3x - 5$.

In examples 2 and 3, we shall show how one actually uses this shortened method to perform a division.

EXAMPLE 2. Divide $31x + 2x^4 + 74 - 12x^3$ by $x - 5$.

Write the dividend in the standard form: $2x^4 - 12x^3 + 0x^2 + 31x + 74$.

On the first line, write the coefficients of the dividend, in order, and the constant term of the divisor with its sign changed.

$$\begin{array}{l|l} 2 - 12 + \ \ 0 + 31 \ + 74 & \underline{5} \\ \quad + 10 - 10 - 50 \ - 95 & \\ \hline 2 - \ \ 2 - 10 - 19\ (-21) & \end{array}$$

Copy the first coefficient of the dividend on the third line. Multiply it by 5 and write the result under -12. Write the sum of -12 and 10 on the third line. Continue in this way until the computation is completed.

The quotient is $2x^3 - 2x^2 - 10x - 19$ and the remainder is -21.

EXAMPLE 3. Divide $5x^4 + 11x + 2x^5 - 12$ by $x + 2$.

The dividend, in the standard form, is $2x^5 + 5x^4 + 0x^3 + 0x^2 + 11x - 12$. Further, $x + 2 = x - (-2)$. The synthetic division is performed as follows:

$$\begin{array}{l|l} 2 + 5 + 0 + 0 + 11 \ \ - 12 & \underline{-2} \\ \quad - 4 - 2 + 4 - \ \ 8 \ \ - \ \ 6 & \\ \hline 2 + 1 - 2 + 4 + \ \ 3\ (-18) & \end{array}$$

The quotient is $2x^4 + x^3 - 2x^2 + 4x + 3$ and the remainder is -18.

Exercises

Divide (*a*) by long division, (*b*) by synthetic division.

1. $x^3 - 3x^2 + 5$ by $x - 4$.
2. $2x^3 - 2 + x^4 - x^2$ by $x + 3$.

Find the quotient and remainder by synthetic division.

3. $(2x^3 - 7x^2 - x + 5) \div (x - 2)$.
4. $(5x^3 + 16x^2 + 17x + 35) \div (x + 3)$.
5. $(2x^3 + 5x^2 + x + 8) \div (x + 3)$.
6. $(3x^3 + 17x^2 + 23x - 4) \div (x + 4)$.
7. $(19 - 9x^3 + 5x^4 + 13x) \div (x + 1)$.
8. $(11x^2 + x^4 - 9x^3 + 16x) \div (x - 7)$.
9. $(x^5 - 16x^2 + 9 - 22x^3) \div (x - 5)$.
10. $(x^6 - 6x^4 + 5x^2 + 13) \div (x + 2)$.
11. $(x^3 + 7x^2 - 3x + 5) \div (x - a)$.
12. $(x^3 + 3yx^2 + 4y^2x + 2y^3) \div (x + y)$.

272. The Remainder Theorem. *If a polynomial $f(x)$ is divided by $x - r$ until a remainder independent of x is obtained, then this remainder equals $f(r)$.*

As an illustration of the meaning of this theorem, and of its proof, consider the following example.

EXAMPLE 1. Show that, if $f(x) = x^3 + 4x^2 - 7x - 3$ is divided by $x - 2$, the remainder equals $f(2)$.

In this case, we have $f(2) = 2^3 + 4 \cdot 2^2 - 7 \cdot 2 - 3 = 7$.

If we divide $f(x)$ by $x - 2$, we obtain the quotient $x^2 + 6x + 5$ and the remainder 7.

Hence, in this case at least, the remainder on dividing by $x - 2$ equals $f(2)$ since each of these quantities equals 7.

To show why this is so, suppose we had checked the above division by the complete check formula for division (Art. 10). We would have, identically,

$$f(x) = x^3 + 4x^2 - 7x - 3 = (x - 2)(x^2 + 6x + 5) + 7.$$

This equation, being true for all values of x, is true for $x = 2$. On putting $x = 2$, we have

$$f(2) = 2^3 + 4 \cdot 2^2 - 7 \cdot 2 - 3 = (2 - 2)(2^2 + 6 \cdot 2 + 5) + 7 = 0 \cdot 21 + 7 = 7;$$

that is, $f(2) = 7 =$ the remainder when $f(x)$ is divided by $x - 2$.

The proof in the general case parallels that given in the example. Let

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n.$$

Then
$$f(r) = a_0r^n + a_1r^{n-1} + a_2r^{n-2} + \cdots + a_n.$$

Divide $f(x)$ by $x - r$. Denote the quotient by $q(x)$ and the remainder by R. From the complete check formula for division (Art. 10),

$$f(x) = (x - r)q(x) + R.$$

In this identity, put $x = r$. We have

$$f(r) = (r - r)q(r) + R = 0 \cdot q(r) + R = R;$$

that is,
$$f(r) = R.$$

EXAMPLE 2. Without dividing, find the remainder when $f(x) = x^7 + 5x^4 - 3x^2 + 9$ is divided by $x + 2$.

We have $x + 2 = x - (-2)$. Hence $r = -2$.

$$f(-2) = (-2)^7 + 5(-2)^4 - 3(-2)^2 + 9 = -128 + 80 - 12 + 9 = -51.$$

It follows by the remainder theorem that, if $f(x)$ is divided by $x + 2$, the remainder will be -51.

The student should check this result by performing the synthetic division.

EXAMPLE 3. Given $f(x) = x^3 - 4x^2 + x - 9$. Find, by synthetic division, the value of $f(5)$.

When we divide $f(x)$ by $x - 5$, we obtain the remainder $R = 21$. Hence, by the remainder theorem, $f(5) = 21$.

$$\begin{array}{rrrr|l} 1 & -4 & +1 & -9 & 5 \\ & +5 & +5 & +30 & \\ \hline 1 & +1 & +6 & +(21) & \end{array}$$

The student should check this result by substitution.

273. The Factor Theorem. *If r is a root of the equation $f(x) = 0$, then $x - r$ is a factor of the polynomial $f(x)$, and conversely.*

The statement: r is a root of $f(x) = 0$, means that $f(r) = 0$. Since, by the remainder theorem, $f(r) = R$, it follows that R also is equal to zero. By the complete check formula for division, we now have

$$f(x) = (x - r)q(x),$$

that is, $x - r$ is a factor of $f(x)$.

Conversely, the statement that $x - r$ is a factor of $f(x)$ means that the remainder R, obtained by dividing $f(x)$ by $x - r$, is zero. Hence, by the remainder theorem, $f(r) = 0$ and r is a root of $f(x) = 0$.

Thus, 5 is a root of $f(x) = x^3 - 7x^2 + 8x + 10 = 0$, because $5^3 - 7 \cdot 5^2 + 8 \cdot 5 + 10 = 0$. By the factor theorem, it follows that $x - 5$ must be a factor of $f(x)$. By division, we find that

$$f(x) = x^3 - 7x^2 + 8x + 10 = (x - 5)(x^2 - 2x - 2).$$

Similarly, we find by division that $x + 3$ is a factor of $f(x) = 2x^3 + x^2 - 13x + 6$, since

$$f(x) = 2x^3 + x^2 - 13x + 6 = (x + 3)(2x^2 - 5x + 2).$$

Hence, $x = -3$ must be a root of $f(x) = 0$. By substitution, we find

$$f(-3) = 2(-3)^3 + (-3)^2 - 13(-3) + 6 = -54 + 9 + 39 + 6 = 0.$$

Exercises

Find $f(r)$, using synthetic division and the remainder theorem, given

1. $f(x) = 2x^3 - 9x^2 + 3x + 9$, $r = 4$.
2. $f(x) = 3x^3 + 19x^2 + 23x - 5$, $r = -5$.
3. $f(x) = x^3 - 8x^2 + 9x + 5$, $r = 6$.
4. $f(x) = x^4 - x^3 - 9x + 6$, $r = 2$.
5. $f(x) = 5x^4 + 7x^3 + 8x + 6$, $r = -2$.
6. $f(x) = x^5 - 19x^2 - 17x + 7$, $r = 3$.

Without dividing, find the remainder on the following indicated divisions.

7. $(x^3 - 4x^2 + 2x + 9) \div (x - 3)$. **8.** $(2x^3 + 5x^2 + 7x + 1) \div (x + 4)$.
9. $(x^6 - 7x^4 + 14x^2 + 11x) \div (x - 2)$.
10. $(8x^{14} - 9x^8 + 3x^5 - 7) \div (x + 1)$.

11. $(x^3 + 2ax^2 - 5a^2x - a^3) \div (x - a)$.
12. $[(x - 3)^5 - 2(x + 1)^2 + 4] \div (x - 4)$.
13. $[(x^2 + 3x - 8)^6 - (2x^2 - 5x - 2)^2 + 6x^2 - 4x - 9] \div (x - 2)$.

Answer the following questions, using the factor theorem.

14. Is $x + 2$ a factor of $x^4 + x^3 - 5x^2 - 2x + 8$?
15. Is $x - 3$ a factor of $x^5 - 11x^3 + 11x^2 - 24$?
16. Is $x + y$ a factor of $x^9 + y^9$?
17. Is 4 a root of $2x^3 - 5x^2 - 8x - 20 = 0$?
18. Is -3 a root of $x^3 - 2x^2 - 13x + 6 = 0$?
19. Is -2 a root of $x^4 - x^3 - 8x - 4 = 0$?

In the following exercises, first find $f(r)$, then use the remainder theorem to set up the required equation in k.

Find the value of k, given that:

20. If $x^3 + kx^2 + 3x - 2$ is divided by $x - 3$, the remainder is 7.
21. If $kx^5 + 9x^2 - 3x - 7$ is divided by $x + 1$, the remainder is -4.
22. If $(x^2 + 4x + 7)^2 - k(x^2 - 5)$ is divided by $x + 3$, the remainder is 8.

274. The Factored Form of a Polynomial. Let

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n, \qquad a_0 \neq 0 \qquad (1)$$

be a polynomial of degree n. We shall show that $f(x)$ *can be written in the form*

$$f(x) = a_0(x - r_1)(x - r_2)(x - r_3) \cdots (x - r_n), \qquad a_0 \neq 0 \quad (2)$$

where $r_1, r_2, r_3, \cdots, r_n$ are constants, real or imaginary. The second member of this equation is called the **factored form** of $f(x)$.

In proving that $f(x)$ can be factored into linear factors in this way, we shall assume the truth of the following theorem, which is called **the fundamental theorem of algebra:**

Every polynomial equation has at least one root, real or imaginary. The proof of this fundamental theorem is beyond the scope of this book. A proof will be found in any standard textbook on the theory of equations.

Let r_1 be a root of $f(x) = 0$. By the factor theorem (Art. 273), it follows that

$$f(x) = (x - r_1)q_1(x),$$

where $q_1(x)$, the quotient obtained by dividing $f(x)$ by $x - r_1$, is a polynomial of degree $n - 1$ in x, having a_0 as the coefficient of x^{n-1}.

If $n > 1$, it follows from the fundamental theorem of algebra that $q_1(x) = 0$ has a root, r_2. Hence, by the factor theorem,

$$q_1(x) = (x - r_2)q_2(x),$$

and

$$f(x) = (x - r_1)(x - r_2)q_2(x).$$

Continuing in this way through n successive steps, we finally arrive at a quotient which is the constant a_0. Then

$$f(x) = a_0(x - r_1)(x - r_2)(x - r_3) \cdots (x - r_n),$$

which is the required factored form of $f(x)$.

EXAMPLE. Factor into linear factors: $f(x) = 2x^3 - 13x^2 + 26x - 15$.

By inspection, we find that 1 is a root of $f(x) = 0$. It follows by the factor theorem that $x - 1$ must be a factor of $f(x)$. Using synthetic division, we find that

$$2x^3 - 13x^2 + 26x - 15 = (x - 1)(2x^2 - 11x + 15).$$

The equation $2x^2 - 11x + 15 = 0$ has a root 3. We find by division that

$$2x^2 - 11x + 15 = (x - 3)(2x - 5) = 2(x - 3)(x - \tfrac{5}{2}).$$

Hence, $$2x^3 - 13x^2 + 26x - 15 = 2(x - 1)(x - 3)(x - \tfrac{5}{2}).$$

This is the required factored form of $f(x)$.

275. The Number of Roots of a Polynomial Equation. Let

$$f(x) = a_0(x - r_1)(x - r_2)(x - r_3) \cdots (x - r_n) = 0, \quad a_0 \neq 0$$

be a polynomial equation in the factored form.

Each of the numbers $r_1, r_2, r_3, \cdots, r_n$ is a root of $f(x) = 0$. For, if we replace x by any one of these numbers, one of the factors, and hence their product, is equal to zero.

The equation has no other roots. For, let r be any number different from $r_1, r_2, r_3, \cdots, r_n$. If we substitute r for x in the given equation, we have

$$f(r) = a_0(r - r_1)(r - r_2)(r - r_3) \cdots (r - r_n). \quad a_0 \neq 0$$

This product does not equal zero since no one of its factors is equal to zero (Art. 4). Hence $f(r) \neq 0$ and r is not a root of $f(x) = 0$.

It follows that the n numbers $r_1, r_2, r_3, \cdots, r_n$, and no others, are the roots of the polynomial equation of degree n, $f(x) = 0$.

276. Multiple Roots. The n roots of $f(x) = 0$ may not all be distinct numbers. If $r_1 = r_2 = \cdots = r_k$, but none of the other roots is equal to r_1, we say that r_1 is a **k-fold root,** or **root of multiplicity k,** of $f(x) = 0$.

From equation (2), if r_1 is a k-fold root of $f(x) = 0$, then

$$f(x) = a_0(x - r_1)^k(x - r_{k+1}) \cdots (x - r_n).$$

It follows that *the condition that a number r is a k-fold root of $f(x) = 0$ is that $(x - r)^k$ [but not $(x - r)^{k+1}$] is a factor of $f(x)$.*

Thus, the equation $x^4 + 3x^3 - 3x^2 - 7x + 6 = (x - 1)^2(x + 2)(x + 3) = 0$ has 1 as a double root and -2 and -3 as simple roots. The four roots of this equation are 1, 1, -2, and -3.

The equation $x^5 - 4x^4 + 4x^3 = (x - 2)^2x^3 = 0$ has 2 as a double root and 0 as a threefold root. The five roots are 2, 2, 0, 0, 0.

With the aid of this definition of a k-fold root and the theorems of Arts. 274 and 275, we can now state the theorem concerning the number of roots of a polynomial equation more precisely in the following form: *a polynomial equation of degree n has precisely n roots provided each root of multiplicity k is counted as k roots.*

Thus, the equation of degree eight,

$$x^8 - 16x^7 + 77x^6 - 98x^5 = (x - 2)(x - 7)^2x^5 = 0,$$

has precisely eight roots, namely 2, 7, 7, 0, 0, 0, 0, 0. It has 2 as a simple root, 7 as a double root, and 0 as a fivefold root.

Exercises

State the degree of each of the following equations, find all the roots, and state their multiplicities.

1. $x^3(x - 3 + \sqrt{5})(x - 3 - \sqrt{5}) = 0.$

2. $(x + 2 + i)^2(x + 2 - i)^2(2x + 5)^4 = 0.$

3. $(x^2 - 9)(x - 6)^2(x + 1)^3 = 0.$

4. $(x^2 + 1)(x^2 - 3x + 2) = 0.$

5. $(3x - 2)^3(x^2 + 4)^2 = 0.$

6. $(x^2 + 2x - 3)^2(x^2 - 6x + 13) = 0.$

7. $x^6 + 2x^5 - 8x^4 = 0.$

8. $(x^3 + 4x^2 - 3x)(x^2 + 7x + 10) = 0.$

Write $f(x)$ in the factored form, given that:

9. The roots of $f(x) = x^3 - 6x^2 + 11x - 6 = 0$ are 1, 2, and 3.

10. The roots of $f(x) = 2x^3 - 5x^2 + 2x + 21 = 0$ are $-\frac{3}{2}$, $2 + \sqrt{3}i$, $2 - \sqrt{3}i$.

11. The roots of $f(x) = x^4 + 2x^3 - 14x^2 + 2x - 15 = 0$ are 3, -5, i, and $-i$.

12. One root of $f(x) = 6x^3 + x^2 - 31x + 10 = 0$ is 2.

13. 1 is a double root of $f(x) = x^4 + 2x^3 - 8x^2 + 6x - 1 = 0.$

14. 3 is a double root of $f(x) = x^4 - 4x^3 + 2x^2 - 12x + 45 = 0.$

15. -1 is a triple root of $f(x) = x^5 + 3x^4 + 4x^3 + 4x^2 + 3x + 1 = 0.$

16. -1 is a root of $f(x) = 5x^4 + x^3 - 2x^2 + 2x = 0.$

17. $x^2 - 4$ is a factor of $f(x) = 2x^4 + x^3 - 4x - 32.$

18. $f(x) = x^4 - 10x^3 + 37x^2 - 60x + 36 = 0$ has two pairs of equal roots.

277. Formation of an Equation with Given Roots. Let $r_1, r_2, \cdots, r_n$ be any n given numbers. It is required to write a polynomial equation of degrees n having these n numbers as roots.

The equation

$$f(x) = a_0(x - r_1)(x - r_2) \cdots (x - r_n) = 0,$$

where a_0 is any constant different from zero, clearly has the given numbers as roots since, if x is replaced by any of the numbers, one factor of the product in the second number, and hence the entire product, is equal to zero.

If we multiply together the factors in this product, we obtain a polynomial equation in which the term of highest degree in x is a_0x^n. It follows that this equation is the required polynomial equation.

EXAMPLE. Write an equation of degree four, having -1, 3, $\frac{5}{2}$, and $-\frac{2}{3}$ as its roots.

We form the equation

$$a_0(x+1)(x-3)(x-\tfrac{5}{2})(x+\tfrac{2}{3}) = 0,$$

which is clearly satisfied if we substitute for x any one of the given values.

If we put $a_0 = 6$, to avoid fractional coefficients, and multiply out, we obtain

$$6(x+1)(x-3)(x-\tfrac{5}{2})(x+\tfrac{2}{3}) = 6x^4 - 23x^3 - 6x^2 + 53x + 30 = 0.$$

As a check, the student should verify that each of the given numbers satisfies the final equation.

Exercises

Write a polynomial equation with integral coefficients having the given roots and no others.

1. $-2, 1, 3.$

2. $-5, -3, 7.$

3. $3, 1+\sqrt{5}, 1-\sqrt{5}.$

4. $2, -1+\sqrt{3}i, -1-\sqrt{3}i.$

5. $2, -2, 3+i, 3-i.$

6. $1, -1, i, -i.$

7. $1+i, 1+i, 1-i, 1-i.$

8. $2, 2, -\frac{5}{2}, -\frac{5}{2}.$

9. $0, 0, 1, 1, -3, -3.$

10. $0, 0, 0, -3, -3, -5.$

11. $-7, -2, 1, 3, 5.$

12. $1, 1, 1, 1, 2, -2, i, -i.$

278. Equation with Roots Opposite in Sign to Those of a Given Equation. Let the roots of

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0, \quad a_0 \neq 0$$

be

$$r_1, r_2, r_3, \cdots, r_n.$$

It is required to find an equation whose roots are

$$-r_1, -r_2, -r_3, \cdots, -r_n;$$

that is, are opposite in sign to those of $f(x) = 0$.

Write $f(x)$ in the factored form

$$\begin{aligned} f(x) &= a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n \\ &= a_0(x-r_1)(x-r_2)(x-r_3)\cdots(x-r_n) = 0. \end{aligned}$$

Replace x by $-x$ throughout these expressions:

$$\begin{aligned} f(-x) &= a_0(-x)^n + a_1(-x)^{n-1} + a_2(-x)^{n-2} + \cdots + a_n \\ &= a_0(-x-r_1)(-x-r_2)(-x-r_3)\cdots(-x-r_n) \\ &= (-1)^n a_0(x+r_1)(x+r_2)(x+r_3)\cdots(x+r_n) \\ &= 0 \end{aligned}$$

The roots of this equation are $-r_1, -r_2, -r_3, \cdots, -r_n$; that is, they are opposite in sign to the roots of $f(x) = 0$.

Hence, *to find an equation whose roots are opposite in sign to those of a given equation, replace x by $-x$ throughout the equation.*

EXAMPLE. The roots of $x^3 + x^2 - 10x + 8 = 0$ are 1, 2, and -4. Find an equation whose roots are -1, -2, 4.

The equation formed by replacing x by $-x$ in the given equation,

$$(-x)^3 + (-x)^2 - 10(-x) + 8 = 0,$$

or

$$-x^3 + x^2 + 10x + 8 = 0,$$

is satisfied by $-x = 1$, $-x = 2$, and $-x = -4$, or $x = -1$, $x = -2$, and $x = 4$. It is therefore the required equation.

The student should check the roots of both equations.

Exercises

Write an equation whose roots are opposite in sign to those of the given equation. In Ex. 1 to 4, check by finding all the roots of the given, and of the transformed, equation.

1. $x^3 - 7x^2 + 7x + 15 = 0$. One root is -1.
2. $x^3 - 5x^2 - 17x + 21 = 0$. One root is 7.
3. $x^4 - 5x^2 + 10x - 6 = 0$. Two roots are -3 and 1.
4. $x^4 - 8x^3 + 8x^2 + 41x - 30 = 0$. Two roots are -2 and 5.

5. $x^4 + 3x^3 + 5x - 3 = 0$.
6. $2x^4 + 7x^3 - 8x^2 + 2x - 5 = 0$.
7. $x^4 - 13x^2 + 36 = 0$.
8. $3x^7 - 4x^4 - 9x^3 + 2x - 8 = 0$.
9. $2x^3 - 5x^4 - 3x - x^5 = 0$.
10. $5x^3 - 2x^2 + 8x^5 - 11 = 0$.

279. Descartes' Rule of Signs. If a polynomial is arranged in decreasing powers of x, but with the terms having zero coefficients omitted, there is said to be a *variation in signs* whenever two successive terms of the polynomial are opposite in sign.

Thus, in the polynomial $2x^7 + 3x^5 - 6x^4 - 2x^3 + x - 8$, there are three variations in sign: one from $3x^5$ to $-6x^4$, another from $-2x^3$ to x, and the third from x to -8.

The following theorem is proved in advanced mathematics.*

DESCARTES' RULE OF SIGNS. *The number of positive roots of $f(x) = 0$ is either equal to the number of variations of sign in $f(x)$ or is less than that number by a positive, even integer.*

In particular, if there are no variations of sign in $f(x)$, there are no positive roots of $f(x) = 0$ and, if there is just one variation, then $f(x) = 0$ has just one positive root. In the remaining cases, the rule does not

* See Dickson, *First Course in the Theory of Equations*, Art. 67.

determine the precise number of positive roots but only a number that cannot be exceeded by the number of positive roots.

A corresponding limitation on the number of negative roots of $f(x) = 0$ can be obtained by transforming $f(x) = 0$ into an equation whose roots are opposite in sign to those of $f(x) = 0$, as shown in the preceding article. Since every negative root of $f(x) = 0$ is transformed into a positive root of $f(-x) = 0$, and conversely, it follows that *the number of negative roots of* $f(x) = 0$ *is either equal to the number of variations in sign in* $f(-x) = 0$ *or is less than that number by a positive, even integer.*

A real root of $f(x) = 0$ that is neither positive nor negative must be equal to zero. If $a_n \neq 0$, zero is not a root of $f(x) = 0$. If $a_n = 0$, and if x^k is the highest power of x that is a factor of $f(x)$, then, by Art. 267, zero is a k-fold root of $f(x) = 0$.

If the sum of the number of positive, negative, and zero roots of $f(x) = 0$ is less than n, the remaining roots must be imaginary. We shall show in Art. 281 that, if the coefficients of $f(x) = 0$ are real numbers, then *the number of imaginary roots is an even number.*

EXAMPLE. Discuss the number of positive, negative, zero, and imaginary roots of $f(x) = 3x^8 + 2x^6 - x^5 + x^4 + x^3 - 6x^2 = 0$.

Since there are three variations in sign, $f(x) = 0$ has either three positive roots, or just one.

The transformed equation, $f(-x) = 3x^8 + 2x^6 + x^5 + x^4 - x^3 - 6x^2 = 0$, has just one variation in sign. Hence, $f(x) = 0$ has just one negative root.

Since x^2 (but not x^3) is a factor of $f(x)$, zero is a double root of $f(x) = 0$.

There are 8 roots in all. Hence, there are either 3 positive, 1 negative, 2 zero, and 2 imaginary roots or 1 positive, 1 negative, 2 zero, and 4 imaginary roots.

Exercises

Using Descartes' Rule of Signs, tell what you can about the number of positive, negative, zero, and imaginary roots of the following equations. In Ex. 1 to 6, find the roots and compare your results with the roots found.

1. $x^2 + 4x + 8 = 0$.

2. $x^5 - 3x^4 - 5x^3 = 0$.

3. $x^3 - 27 = 0$.

4. $x^6 + 8x^3 = 0$.

5. $x^4 - 13x^2 + 36 = 0$.

6. $x^4 + 17x^2 + 16 = 0$.

7. $x^3 - 5x^2 - 9 = 0$.

8. $2x^6 + 4x^5 - x^2 = 0$.

9. $x^6 + 4x^4 + 3x^2 - 7 = 0$.

10. $x^{10} - 4x^6 + 7x^3 - 3 = 0$.

280. Limits for the Magnitudes of the Real Roots. Descartes' Rule gives a limit for the *number* of positive roots of $f(x) = 0$. The following rule states a limit for the *size* of the *largest* positive root.

The advantage of knowing this limit will be that, when we are searching for the roots of an equation, we need not look for any root larger than this limit.

If R is a positive number such that, when $f(x)$ is divided by $x - R$ by synthetic division, all the numbers on the third line are either positive or zero (or all either negative or zero) then no real root of $f(x) = 0$ is greater than R.

The proof of this theorem, in any given numerical equation, may be arrived at by the reasoning used in the following example.

EXAMPLE 1. Show that 4 is greater than any real root of $x^4 - 3x^3 - 4x^2 + 3x - 7 = 0$.

By synthetic division, we have

$$\begin{array}{r|l} 1 - 3 - 4 + 3 \; - \; 7 & 4 \\ + 4 + 4 + 0 \; + 12 & \\ \hline 1 + 1 + 0 + 3 \; (+ \;\; 5) & \end{array}$$

Hence, identically, by the complete check formula for division,

$$f(x) = x^4 - 3x^3 - 4x^2 + 3x - 7 = (x - 4)(x^3 + x^2 + 3) + 5.$$

Obviously, 4 is not a root since $f(4) = 5$.

Let x be any number greater than 4. Then $x - 4$ is positive and $x^3 + x^2 + 3$ is positive because each of its terms is positive. It follows that, for the chosen value of x, $f(x)$ is greater than 5 and x is not a root of $f(x) = 0$.

To find a lower limit for the real roots of $f(x) = 0$, transform the equation into one whose roots are opposite in sign to those of $f(x) = 0$. Then, if r is a positive number such that no real root of $f(-x) = 0$ is greater than r, it follows that no root of $f(x) = 0$ is less than $-r$.

EXAMPLE 2. Show that -2 is less than any real root of $x^4 - 3x^3 - 4x^2 + 3x - 7 = 0$.

The transformed equation is $x^4 + 3x^3 - 4x^2 - 3x - 7 = 0$.

By synthetic division, we have

$$\begin{array}{r|l} 1 + 3 - \;\; 4 - \;\; 3 \; - \;\; 7 & 2 \\ + 2 + 10 + 12 \; + 18 & \\ \hline 1 + 5 + \;\; 6 + \;\; 9 \; (+ 11) & \end{array}$$

Hence 2 is greater than any real root of $f(-x) = 0$ and -2 is less than any real root of $f(x) = 0$.

Exercises

Find integral upper and lower limits for the magnitudes of the real roots of the following equations.

1. $x^3 + 4x^2 - 4x + 3 = 0$.
2. $3x^3 + 6x^2 - 43 = 0$.
3. $2x^3 - 9x^2 - 8x - 14 = 0$.
4. $x^3 - 6x^2 - 14x + 6 = 0$.
5. $x^4 - 7x^3 + 3x - 15 = 0$.
6. $2x^4 + 9x^3 - 3x^2 - 2x - 14 = 0$.
7. $x^4 - 13x^2 - 35 = 0$.
8. $x^5 - 4x^2 + 6x - 14 = 0$.

281. Equation with Roots m Times Those of a Given Equation. Let m be a given constant and let the roots of the equation

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0, \quad a_0 \neq 0 \qquad (3)$$

be

$$r_1, r_2, r_3, \cdots, r_n.$$

It is required to find an equation whose roots are

$$mr_1, mr_2, mr_3, \cdots, mr_n,$$

that is, are equal to those of $f(x) = 0$ each multiplied by m.

Write $f(x)$ in the factored form

$$\begin{aligned} f(x) &= a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n \\ &= a_0(x - r_1)(x - r_2)(x - r_3) \cdots (x - r_n) = 0. \end{aligned} \qquad (4)$$

Replace x by $\dfrac{x}{m}$ throughout equations (4).

$$\begin{aligned} f\left(\frac{x}{m}\right) &= a_0\left(\frac{x}{m}\right)^n + a_1\left(\frac{x}{m}\right)^{n-1} + a_2\left(\frac{x}{m}\right)^{n-2} + \cdots + a_n \\ &= a_0\left(\frac{x}{m} - r_1\right)\left(\frac{x}{m} - r_2\right)\left(\frac{x}{m} - r_3\right)\cdots\left(\frac{x}{m} - r_n\right) \\ &= a_0m^{-n}(x - mr_1)(x - mr_2)(x - mr_3) \cdots (x - mr_n) = 0. \end{aligned} \qquad (5)$$

It follows that the roots of

$$a_0\left(\frac{x}{m}\right)^n + a_1\left(\frac{x}{m}\right)^{n-1} + a_2\left(\frac{x}{m}\right)^{n-2} + \cdots + a_n = 0 \qquad (6)$$

are

$$mr_1, mr_2, mr_3, \cdots, mr_n,$$

that is, they are equal to those of $f(x) = 0$ each multiplied by m.

If we multiply equation (6) by m^n, and simplify, we have

$$a_0x^n + ma_1x^{n-1} + m^2a_2x^{n-2} + \cdots + m^na_n = 0. \qquad (7)$$

Hence, *to form an equation each of whose roots is m times the corresponding root of a given equation, write the equation in the standard form and multiply the coefficients, beginning with the second, by m, m^2, m^3, $\cdots$, m^n, respectively.*

EXAMPLE. The roots of the equation $11x + x^3 + 21 = 9x^2$ are -1, 3, and 7. Write an equation whose roots are -2, 6, and 14.

Write the given equation in the standard form

$$x^3 - 9x^2 + 11x + 21 = 0,$$

and multiply the coefficients, in order, beginning with the second, by 2, 4, and 8. We obtain, as the required equation,

$$x^3 - 18x^2 + 44x + 168 = 0.$$

The student should verify that the roots of this equation are -2, 6, and 14.

282. Equations in the *b*-form. A polynomial equation is in the *b-form if the first coefficient is unity and all of the other coefficients are integers or zeros.*

If all the coefficients of a polynomial equation are rational numbers, the equation can be transformed into the b-form by the method shown in the following example.

EXAMPLE. Transform the equation $9x^4 - 15x^3 + \frac{x}{2} - \frac{7}{12} = 0$ into the b-form.

Divide by the coefficient of the highest power of x and write the resulting equation in the standard form

$$x^4 - \frac{5}{3}x^3 + 0x^2 + \frac{x}{18} - \frac{7}{108} = 0.$$

Write the equation whose roots are m times the roots of this equation

$$x^4 - \frac{5m}{3}x^3 + 0m^2x^2 + \frac{m^3}{18}x - \frac{7m^4}{108} = 0.$$

Choose for m the smallest positive integer that will make all of the coefficients integers. In this case, $m = 6$ and the required equation in the b-form is

$$x^4 - 10x^3 + 0x^2 + 12x - 84 = 0, \quad \text{or} \quad x^4 - 10x^3 + 12x - 84 = 0.$$

Exercises

Find an equation whose roots are equal to those of the given equation multiplied by the number in parentheses. In Ex. 1 and 2, check your results by finding the roots of the transformed equation.

1. $x^3 - 4x^2 - 11x + 30 = 0$, (2). The roots of the given equation are $2, 5, -3$.

2. $9x^3 - 12x^2 + x + 2 = 0$, (-3). The roots of the given equation are $-\frac{1}{3}, 1, \frac{2}{3}$.

3. $8x^3 - 6x^2 + 5x + 11 = 0$, (-2). **4.** $27x^3 + 36x^2 - 6x + 7 = 0$, (-3).

5. $3x^2 + 20x^3 = 11$, (5). **6.** $6x^3 - 2x^2 + 8x^6 - 9 = 0$, (2).

Write each of the following equations in the b-form, using the smallest possible positive, integral value of m.

7. $25x^3 + 10x^2 + 3x - 2 = 0$. **8.** $36x^3 - 18x^2 - 7x + 23 = 0$.

9. $9x^3 + 6x^2 + \frac{2}{5}x - \frac{4}{3} = 0$. **10.** $15x^3 - 6x^2 + 5x + 2 = 0$.

11. $18x^4 - 5x^2 + 3x - \frac{7}{9} = 0$. **12.** $12x^4 - 10x^3 - \frac{1}{3}x + \frac{1}{2} = 0$.

283. Rational Roots. Let all the coefficients in the equation

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0$$

be rational numbers,* or zeros, and let $a_0 \neq 0$ and $a_n \neq 0$. It is required to find the roots of $f(x) = 0$ that are rational numbers.

* A rational number (Art. 28) is one that can be written in the form a/b, where a and b are *integers*. In the following discussion, we shall suppose that this rational number is in its lowest terms; that is, that a and b have no common factor. A rational number in its lowest terms is an integer if, and only if, $b = \pm 1$.

If the given equation is not already in the b-form, we first transform it into that form

$$x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_n = 0, \tag{8}$$

where all the coefficients are integers and $b_n \neq 0$. Let m be the integer by which it was necessary to multiply the roots of $f(x) = 0$ to obtain equation (8).

We next form the rational roots of equation (8) with the aid of the following theorem:

Every rational root of an equation in the b-form is an integer and a factor of the constant term.

Having found the rational roots of (8), we divide each of them by m to find the required rational roots of $f(x) = 0$.

To prove the theorem just stated, let a/b be a rational root (in its lowest terms) of equation (8). Then

$$\left(\frac{a}{b}\right)^n + b_1\left(\frac{a}{b}\right)^{n-1} + b_2\left(\frac{a}{b}\right)^{n-2} + \cdots + b_n = 0. \tag{9}$$

Multiply this equation by b^{n-1} and write the resulting equation in the form

$$\frac{a^n}{b} = -\,(b_1a^{n-1} + b_2a^{n-2}b + \cdots + b_nb^{n-1}).$$

Since all the letters in the second member represent either integers or zeros, the second member is an integer. Hence, the first member is an integer and, since a and b (and hence a^n and b) have no common factor, $b = \pm 1$. It follows that the required root, a/b is an integer.

Next, in (9), put $b = 1$ and write this equation in the form

$$a(a^{n-1} + b_1a^{n-2} + b_2a^{n-3} + \cdots + b_{n-1}) = -\,b_n.$$

Since a is a factor of the first member of this equation, it is a factor of the second; that is, b_n is divisible by a.

In searching for the rational roots of an equation in the b-form, the number of trials can often be considerably decreased by using the theorems of Arts. 279 and 280. If, for example, there are no variations in sign, there is no need to look for positive roots; neither is it necessary to look for rational roots outside the limits within which the real roots must lie.

EXAMPLE 1. Find the rational roots of $x^4 - 2x^3 - 13x^2 + 14x + 24 = 0$.

By Art. 280, an upper limit for the real roots is 5 and a lower limit is -4. Since the equation is in the b-form, its rational roots are integers and divisors of 24. The only numbers that satisfy these conditions are ± 1, ± 2, ± 3, and 4.

By trial, we find that 2 is a root and that

$$x^4 - 2x^3 - 13x^2 + 14x + 24 = (x - 2)(x^3 - 13x - 12) = 0.$$

We search for the remaining rational roots, not in the given equation, but in the depressed equation,

$$x^3 - 13x - 12 = 0,$$

formed by removing the factor $x - 2$ from the given equation.

By further trial, we find that 4 is a root of this equation, and that

$$x^3 - 13x - 12 = (x - 4)(x^2 + 4x + 3) = 0.$$

The roots of $x^2 + 4x + 3 = 0$ are -3 and -1. Hence the required roots are -3, -1, 2, and 4.

Example 2. The equation $18x^4 - 51x^3 + 47x^2 - 17x + 2 = 0$ has two rational roots. Find all the roots.

We first transform the equation into the b-form by multiplying its roots by 6. We have

$$x^4 - 17x^3 + 94x^2 - 204x + 144 = 0.$$

This equation has no negative roots and an upper limit for the roots is 17. The rational roots, if there are any, must thus be included among the numbers 1, 2, 3, 4, 6, 8, 9, 12, and 16.

By trial, we find that 3 is a root and that

$$x^4 - 17x^3 + 94x^2 - 204x + 144 = (x - 3)(x^3 - 14x^2 + 52x - 48) = 0.$$

One root of $x^3 - 14x^2 + 52x - 18 = 0$ is 4 and

$$x^3 - 14x^2 + 52x - 48 = (x - 4)(x^2 - 10x + 12) = 0.$$

The roots of $x^2 - 10x + 12 = 0$ are the irrational numbers $5 + \sqrt{13}$ and $5 - \sqrt{13}$. The roots of the transformed equation are thus 3, 4, $5 + \sqrt{13}$, and $5 - \sqrt{13}$. The roots of the given equation are 1/6 of these, or 1/2, 2/3, $\dfrac{5 + \sqrt{13}}{6}$ and $\dfrac{5 - \sqrt{13}}{6}$.

Exercises

Find the rational roots of the following equations. If the final reduced equation is a quadratic, find all the roots.

1. $x^3 + x^2 - 17x + 15 = 0.$ **2.** $x^3 - 3x^2 - 7x + 6 = 0.$

3. $3x^3 + 14x^2 + 2x - 4 = 0.$ **4.** $3x^3 + 11x^2 + 8x - 4 = 0.$

5. $2x^3 - 15x^2 + 10x + 12 = 0.$ **6.** $5x^3 - 13x^2 + 31x - 15 = 0.$

7. $x^4 + 9x^3 + 20x^2 - 8x - 40 = 0.$ **8.** $8x^4 - 36x^3 + 18x^2 + 5x - 3 = 0.$

9. $4x^4 + 28x^3 + 41x^2 + 22x + 4 = 0.$

10. $4x^4 - 28x^3 + 61x^2 - 42x + 9 = 0.$

11. $4x^5 - 13x^3 - 6x^2 = 0.$ **12.** $9x^6 + 9x^5 - 16x^4 + 4x^3 = 0.$

Hint. First find the zero roots; then remove the corresponding factor x^k.

13. $x^6 - 12x^4 + 23x^2 + 36 = 0.$ **14.** $8x^4 - 16x^2 - 7x - 15 = 0.$

284. The Graph of the Polynomial Function. The graph of the equation

$$y = f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n \tag{10}$$

may be found by the methods outlined in Chapters 6 and 27. It presents, however, some special features which should be noticed here.

Since to each value of x there corresponds just one value of y, every vertical line meets the curve in just one point. To find the value of y corresponding to any given value of x except $x = 0$, we use synthetic division and the remainder theorem. A horizontal line meets the curve in, at most, n points where n is the degree of $f(x)$. In particular, the abscissas of the points where the curve meets the x-axis are the real roots of $f(x) = 0$ since these are the real values of x which, when substituted in equation (10), make $y = 0$.

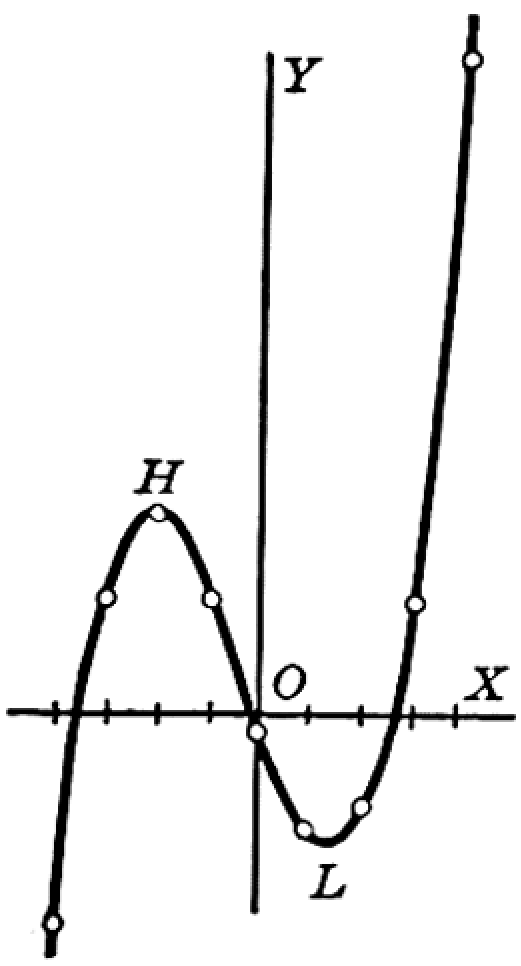


FIG. 173

EXAMPLE 1. Draw the graph of $y = x^3 + x^2 - 9x - 1$.

By synthetic division and the remainder theorem, we compute the following table of pairs of values of x and y. We next plot the corresponding points and draw a smooth curve through them. In Figure 173, the unit on the y-axis is one third as long as the unit on the x-axis.

x	-4	-3	-2	-1	0	1	2	3	4
y	-13	8	13	8	-1	-8	-7	8	43

If a more accurately drawn graph is needed, fractional values, also, should be assigned to x.

EXAMPLE 2. Estimate, from the figure, to one decimal place, the real roots of $x^3 + x^2 - 9x - 1 = 0$.

The required roots are the abscissas of the points of intersection of the graph with the x-axis (Fig. 173). To one decimal place, they are -3.5, -0.1, and 2.6.

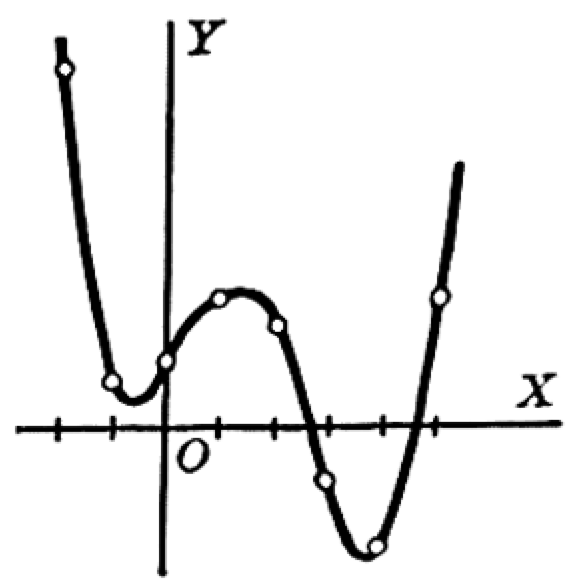


FIG. 174

EXAMPLE 3. Draw the graph of $y = x^4 - 6x^3 + 3x^2 + 12x + 9$.

We compute the following table, plot the corresponding points, and draw a smooth curve through them (Fig. 174). The unit on the y-axis has been taken one tenth as long as on the x-axis.

x	-2	-1	0	1	2	3	4	5
y	61	7	9	19	13	-9	-23	19

The equation $x^4 - 6x^3 + 3x^2 + 12x + 9 = 0$ has only two real roots. From the figure, these are found to be approximately 2.6 and 4.8.

285. Slope of the Tangent; Derived Curve; Maximum and Minimum Points. The slope of the tangent to the curve defined by equation (10) can be found by the methods shown in Chapter 26. We find in this way that *the slope m of the tangent to the curve* (10) *at the point on it whose abscissa is x is*

$$m\,^* = a_0nx^{n-1} + a_1(n-1)x^{n-2} + a_2(n-2)x^{n-3} + \cdots + a_{n-1}. \quad (11)$$

In any interval in which m is positive, y increases as x increases and, in any interval in which m is negative, y decreases as x increases. Whenever $m = 0$, the tangent to the curve is horizontal.

If, in equation (11), we replace m by y, we obtain the equation of a curve

$$y = a_0nx^{n-1} + a_1(n-1)x^{n-2} + a_2(n-2)x^{n-3} + \cdots + a_{n-1}. \quad (12)$$

This curve is called the **derived curve** of the curve defined by equation (10). It has the property that *the ordinate of any point* (x_1, y_1) *on the derived curve equals the slope of the tangent to the curve* (10) *at the point on it whose abscissa is* x_1.

In particular, the abscissa of any point where the derived curve crosses the x-axis from above to below (as x increases) is the abscissa of a **maximum point** on the original curve (H in Fig. 175); that is, of a point which is higher than any near-by point on the curve (10). Similarly, the abscissa of any point where the derived curve crosses the x-axis from below to above is the abscissa of a **minimum point** on the original curve (L in Fig. 175) which is lower than any near-by point on the curve.

Since the second member of equation (12) is of degree $n - 1$, the curve (10) has, *at most*, $n - 1$ maximum and minimum points.

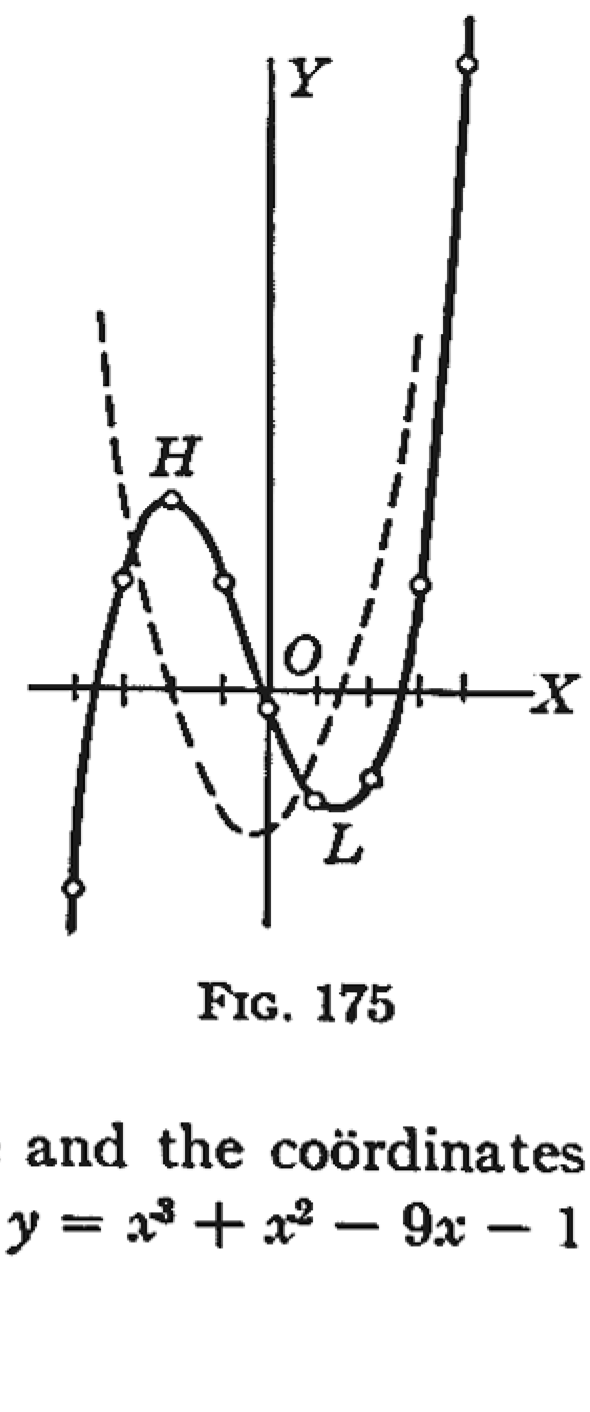


FIG. 175

EXAMPLE 1. Find the equation of the derived curve and the coördinates of the maximum and minimum points of the curve $y = x^3 + x^2 - 9x - 1$ (Ex. 1, Art. 284).

From (12), the equation of the derived curve is

$$y = 3x^2 + 2x - 9.$$

This curve is a parabola (the dotted curve in Fig. 175) which crosses the x-axis at $x = \frac{1}{3}(-1 - 2\sqrt{7}) = -2.1$ and $x = \frac{1}{3}(-1 + 2\sqrt{7}) = 1.4$ (approximately). The ordinates of the corresponding points on the original curve are found, by substituting these values of x in the equation of the given curve, to be 13 and -8.9 approximately. Hence, the coördinates of the maximum point H are $(-2.1, 13)$ and, of the minimum point L are $(1.4, -8.6)$.

* If, as in Figures 173 to 176, a smaller unit is taken on the y-axis than on the x-axis, the slope m, as found from this equation, must be decreased in the same ratio. Thus, in Figure 173, the value of m must be divided by 3 and in Figure 174 it must be divided by 10.

In the figure, to conform with the original curve, the ordinates of all points on the derived curve are divided by three.

EXAMPLE 2. Find the equation of the derived curve and the abscissas of the maximum and minimum points on the curve $y = x^4 - 6x^3 + 3x^2 + 12x + 9$ (Art. 284, Ex. 3).

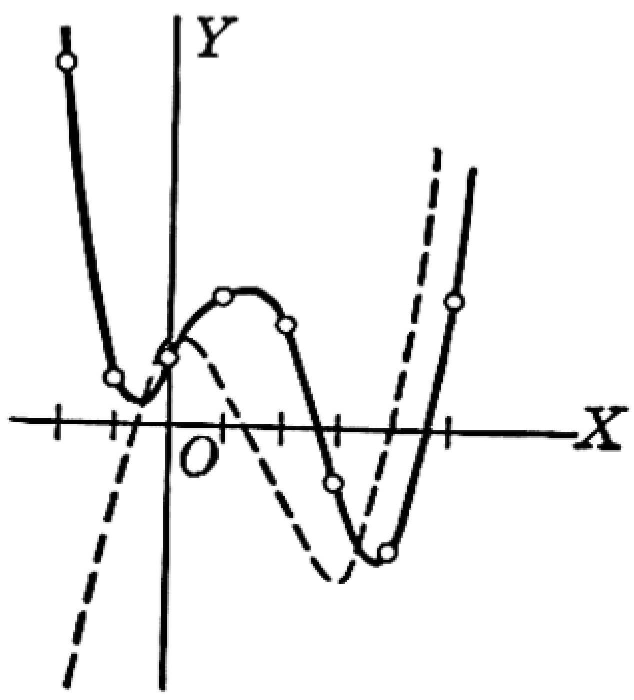


FIG. 176

The equation of the derived curve is

$$y = 4x^3 - 18x^2 + 6x + 12.$$

Its graph is the dotted curve in Figure 176.

This curve crosses the x-axis at $x = -0.6$, $x = 1.2$, and $x = 3.9$, approximately. The first and last of these numbers are the abscissas of minimum points on the original curve; the second is the abscissa of a maximum point.

To conform with the original curve, the ordinates of all points on the derived curve are divided by ten.

Exercises

Draw the graphs of the following equations.

1. $y = x^3 - 9x + 5$.
2. $y = x^3 - 2x^2 + 4x + 1$.
3. $y = 2x^3 + x^2 - 4x + 9$.
4. $y = 3x^3 - 5x^2 - 19x$.
5. $y = x^4 + 2x^3 - 5x^2 - 4$.
6. $y = x^4 - 7x^2 + 10$.
7. $y = x^3$.
8. $y = x^4$.
9. $y = (x - 2)^2(x + 2)^2$.
10. $y = x^3(x - 2)^2$.

Draw the graph of the given function and find the real roots of $f(x) = 0$ to one decimal place.

11. $f(x) = x^3 - 7x$.
12. $f(x) = x^3 + 52$.
13. $f(x) = x^3 + 6x^2 - 8$.
14. $f(x) = x^3 - 3x^2 - 9x + 5$.
15. $f(x) = x^3 - 3x^2 + 3x - 18$.
16. $f(x) = 2x^3 + x^2 + x - 9$.
17. $f(x) = x^4 - 59$.
18. $f(x) = x^4 - 13x^2 + 18x - 5$.

286. Graphical Approximation to the Irrational Roots.* When it is required to find the real roots of $f(x) = 0$, one should first find the rational roots, if there are any, by the method shown in Art. 283 and remove from $f(x)$ the linear factors corresponding to these rational roots. The irrational roots can then be found graphically from the depressed equation, to any desired number of decimal places, by successively enlarging the graph in the neighborhood of each desired root until a sufficiently accurate approximation has been obtained. This process of graphical approximation to the value of a root is illustrated by the following example.

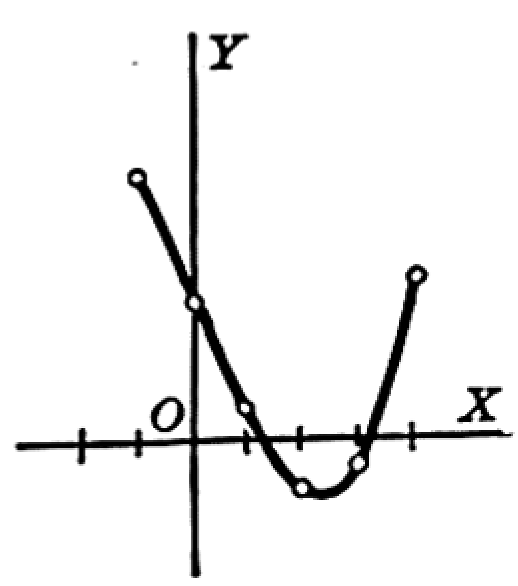


FIG. 177

* This article may be omitted if Horner's Method (Art. 288) is to be presented.

EXAMPLE. Find the smaller positive root of $f(x) = x^3 - x^2 - 11x + 14 = 0$ to three decimal places.

This equation does not have a rational root.

Since a rather small positive root is to be found, we shall draw the graph from $x = -1$ to a value well beyond the required root (Fig. 177).

x	-1	0	1	2	2.5	3	4
y	23	14	3	-4	-4.1	-1	18

From the graph, the required root is found to be 1.3, to one decimal place.

We compute by synthetic division, to two significant figures, the value of $f(1.3) = 0.21$. Since this value of y is positive, it is seen from Figure 177 that $x = 1.3$ is slightly less than the required root. (Why?) We, accordingly, next assume $x = 1.4$ which, if our estimate has been made correctly, should be slightly larger than the root. We find $f(1.4) = -0.62$. Since this result is negative, $x = 1.4$ is slightly larger than the root. On an enlarged scale (Fig. 178), plot the points (1.3, 0.21) and (1.4, $-$ 0.62) which, we have found, lie on the curve. Since we have, in this figure, only two plotted points and since we are drawing only a very small segment of the graph, we shall represent this segment of the curve, in this figure, by the segment of a straight line joining the two plotted points.

From Figure 178, the root is found to lie between $x = 1.32$ and $x = 1.33$. We, accordingly, compute, to two significant figures, $f(1.32) = 0.038$ and $f(1.33) = -0.046$.

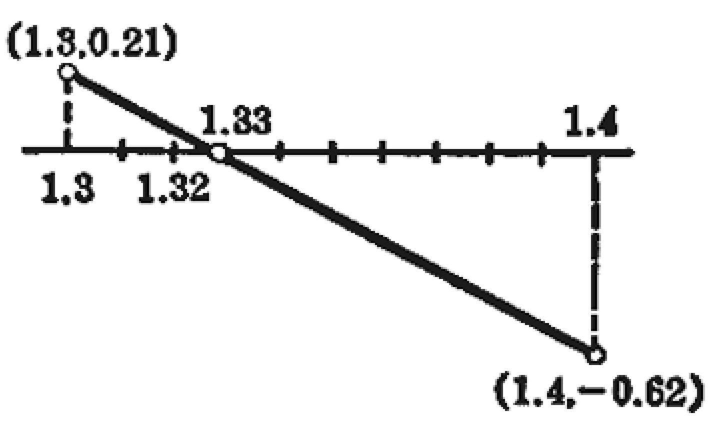


FIG. 178

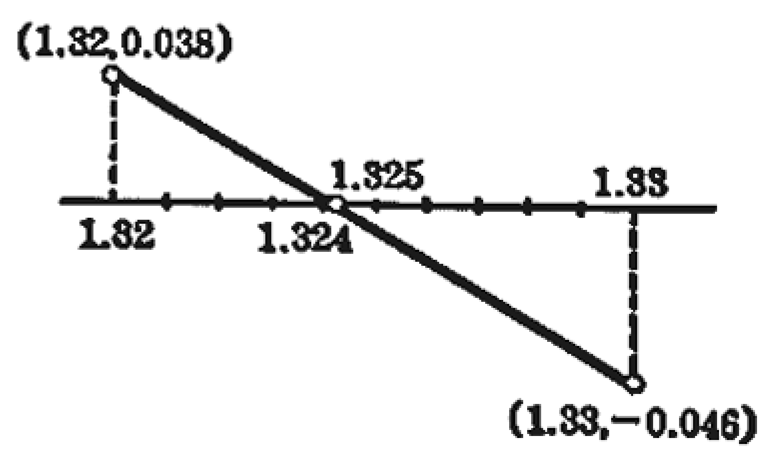


FIG. 179

On a still further enlarged scale (Fig. 179), plot the points (1.32, 0.038) and (1.33, 0.046) and represent the graph by a straight-line segment joining these points.

From Figure 179, the root is found to be approximately $x = 1.324$. The last digit is in doubt. It may be determined definitely, and the next digit approximately, by carrying the computations, and the enlargement of the graph, one step further. We can, in fact, determine the root to any desired number of decimal places by continuing, step by step, the process of successive enlargements.

The other two roots can be found approximately in a similar way. Their values, to three decimal places, are 3.093 and $-$ 3.417.

Exercises

Find the required root graphically, to three decimal places.

1. $x^3 + 3x^2 - 9x - 8 = 0$. The positive root.
2. $x^3 - 4x^2 - 2x + 6 = 0$. The negative root.
3. $x^3 - 2x^2 + 2x - 3 = 0$. The real root.
4. $x^3 - 3x^2 - 6x + 13 = 0$. The larger positive root.
5. $x^3 - 11x - 2 = 0$. The numerically larger negative root.
6. $x^4 + 2x^3 - 10x^2 - 20x - 8 = 0$. The positive root.

Find all the roots to three decimal places.

7. $x^3 - 3x^2 - 5x + 8 = 0$.
8. $x^3 - x^2 - 6x + 4 = 0$.
9. $x^3 + x^2 - 3x + 5 = 0$.
10. $x^3 + 3x^2 + 3x - 14 = 0$.
11. $2x^3 - 5x^2 - 3 = 0$.
12. $3x^3 - 19x - 6 = 0$.
13. $x^3 - 3x + 1 = 0$.
14. $x^3 - 3x^2 - 5x + 9 = 0$.
15. $x^4 + 2x^3 - 11x^2 - 4x + 3 = 0$.
16. $3x^4 - 4x^3 - 7x^2 + 32x - 16 = 0$.

287. Equation with Roots Decreased by h. Let h be a given constant and let the roots of the equation

$$\begin{aligned} f(x) &= a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n, \quad a_0 \neq 0 \\ &= a_0(x - r_1)(x - r_2)(x - r_3) \cdots (x - r_n) = 0, \end{aligned} \tag{13}$$

be

$$r_1, r_2, r_3, \cdots, r_n.$$

We wish to write an equation whose roots are

$$r_1 - h, r_2 - h, r_3 - h, \cdots, r_n - h, \tag{14}$$

that is, are less by h than the roots of $f(x) = 0$.

In equation (13), replace x by $x + h$:

$$\begin{aligned} &f(x + h) \\ &\quad = a_0(x + h)^n + a_1(x + h)^{n-1} + a_2(x + h)^{n-2} + \cdots + a_n \\ &\quad = a_0[(x + h) - r_1][(x + h) - r_2][(x + h) - r_3] \cdots [(x + h) - r_n] \\ &\quad = a_0[x - (r_1 - h)][x - (r_2 - h)][x - (r_3 - h)] \cdots [x - (r_n - h)] \\ &\quad = 0. \end{aligned} \tag{15}$$

The roots of this equation are the required numbers

$$r_1 - h, r_2 - h, r_3 - h, \cdots, r_n - h.$$

Hence, the required equation is obtained from the given one by replacing x by $x + h$.

Before we make this transformation, we shall first write the given equation (13) in powers of $x - h$. After we have done this, if we replace x by $x + h$, we shall have the required equation in powers of x.

Let

$$\begin{aligned} f(x) &= a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n \\ &= a_0(x - h)^n + A_1(x - h)^{n-1} + \cdots + A_n, \end{aligned} \tag{16}$$

where $A_1, A_2, \cdots, A_n$ are constants to be determined.

If both members of the identity (16) are divided by $x - h$, the quotients and remainders must be equal. Denote the quotient and remainder in the first member by $q_{n-1}(x)$ and R_n, respectively. The quotient and remainder in the second member are, by inspection,

$$a_0(x-h)^{n-1} + A_1(x-h)^{n-2} + A_2(x-h)^{n-3} + \cdots + A_{n-1}, \quad \text{and} \quad A_n.$$

Since the remainders and quotients are respectively equal,

$$R_n = A_n,$$

and

$$q_{n-1}(x) \equiv a_0(x-h)^{n-1} + A_1(x-h)^{n-2} + \cdots + A_{n-1}. \qquad (17)$$

Divide both members of the identity (17) by $x - h$ and equate the quotients and remainders. If we denote the quotient and remainder from the first member by $q_{n-2}(x)$ and R_{n-1}, respectively, we have

$$R_{n-1} = A_{n-1},$$

and

$$q_{n-2}(x) = a_0(x-h)^{n-2} + A_1(x-h)^{n-3} + \cdots + A_{n-2}.$$

Continuing in this way, we find that *the required coefficients A_n, A_{n-1}, A_{n-2}, and so on to A_1 are the successive remainders when $f(x)$, $q_{n-1}(x)$, $q_{n-2}(x)$, and so on to $q_1(x)$ are divided by $x - h$.*

If we substitute these values of $A_n, A_{n-1}, A_{n-2}, \cdots, A_1$ in equation (16), replace x by $x + h$, and equate the result to zero, we have the required equation with roots less by h than the roots of $f(x) = 0$.

The successive quotients and remainders should be found by synthetic division. The computation should be arranged according to the form shown in the following examples.

Example 1. Form the equation whose roots are less by 2 than the roots of $f(x) = 2x^3 - 9x^2 + 4x + 15 = 0$.

$$\begin{array}{rrrrl}
2 & -9 & +\ 4 & +15 & \underline{|2} \\
 & +4 & -10 & -12 & \\
\hline
2 & -5 & -\ 6 & (+\ 3) & \\
 & +4 & -\ 2 & & \\
\hline
2 & -1 & (-8) & & \\
 & +4 & & & \\
\hline
2 & (+3) & & &
\end{array}$$

We first express $f(x)$ in powers of $x-2$. Let $f(x) = 2x^3 - 9x^2 + 4x + 15 \equiv 2(x-2)^3 + A_1(x-2)^2 + A_2(x-2) + A_3$, where A_3, A_2, and A_1 are to be determined.

The first three lines of the computation show the division of $f(x)$ by $x - 2$. From the third line, we find that the quotient is $q_2(x) = 2x^2 - 5x - 6$ and the remainder is $R_3 = A_3 = 3$.

Lines 3 to 5 in the computation show the division of $q_2(x) = 2x^2 - 5x - 6$ by $x - 2$. Observe that we do not recopy either the dividend or the divisor which already appear in the computation. Line 5 shows that, on this division, the quotient is $q_1(x) = 2x - 1$ and the remainder is $R_2 = A_2 = -8$.

The last division, in lines 5 to 7, gives a quotient $2 = a_0$ and a remainder $R_1 = A_1 = 3$.

Substitute the values just found for A_1, A_2, and A_3 in the expression for $f(x)$ in powers of $x - 2$. We have

$$f(x) = 2x^3 - 9x^2 + 4x + 15 \equiv 2(x-2)^3 + 3(x-2)^2 - 8(x-2) + 3.$$

In the last member, replace $x - 2$ by x. We have

$$2x^3 + 3x^2 - 8x + 3 = 0.$$

This is the required equation with roots less by 2 than the roots of the given equation.

The student should verify that -1, $\frac{5}{2}$, and 3 are the roots of the given equation and -3, $\frac{1}{2}$, and 1 the roots of the transformed one.

EXAMPLE 2. Find an equation whose roots are greater by 3 than the roots of $4x^3 + 16x^2 + 17x + 5 = 0$.

To *increase* the roots by 3, we *decrease* them by -3.

The successive divisions are shown in the adjoining computation for the reduction of the roots. We have $A_3 = -10$, $A_2 = 29$, and $A_1 = -20$. The transformed equation is, accordingly,

$$4x^3 - 20x^2 + 29x - 10 = 0.$$

$$\begin{array}{rrrrl}
4 & +16 & +17 & +5 & \lfloor -3 \\
 & -12 & -12 & -15 & \\
\hline
4 & +4 & +5 & (-10) & \\
 & -12 & +24 & & \\
\hline
4 & -8 & (+29) & & \\
 & -12 & & & \\
\hline
4 & (-20) & & &
\end{array}$$

The roots of the given equation are $-\frac{5}{2}$, $-\frac{1}{2}$, and -1. Add 3 to each of these numbers and show that the results are roots of the transformed equation.

Exercises

Find an equation whose roots are less than those of the given equation by the number stated in parentheses. In Ex. 1 and 2, the roots are rational. Find them and show that the transformed equation has the required roots.

1. $x^3 - 4x^2 + x + 6 = 0$, (1).
2. $2x^3 + 9x^2 + 10x + 3 = 0$, (-2).
3. $x^3 - 7x^2 + 10x + 8 = 0$, (3).
4. $2x^3 - 11x^2 + 14x - 33 = 0$, (5).
5. $3x^3 + 17x^2 + 29x + 31 = 0$, (-3).
6. $x^4 - 7x^3 + 9x^2 + 6 = 0$, (4).
7. $x^3 - 2.1x^2 + 1.35x - 0.429 = 0$, (0.4).
8. $2x^5 - 5x^4 - 3x^3 + 7x^2 - 6x - 3 = 0$, (2).

288. Irrational Roots by Horner's Method. This method of finding the approximate values of the irrational roots of a polynomial equation consists in reducing the roots by successive steps until the root we are trying to determine has been reduced as near to zero as we please. The total amount by which we have reduced the roots to effect this result is approximately the value of the required root. The process of effecting the successive reductions is illustrated by the following example.

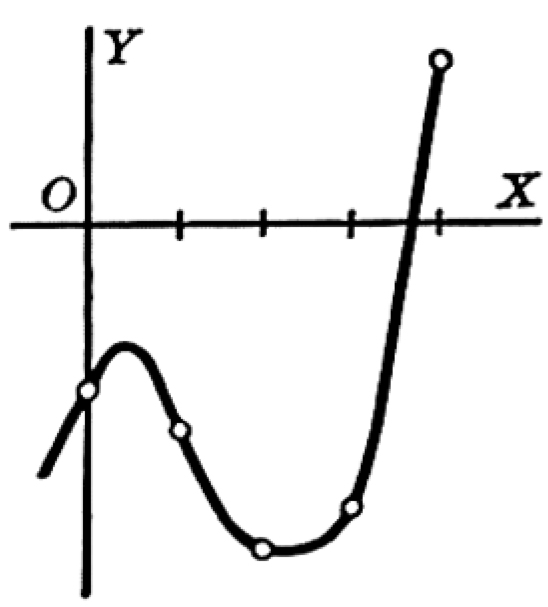


FIG. 180

EXAMPLE. Find the real root of $f(x) = x^3 - 4x^2 + 2x - 4 = 0$ to three decimal places.

We first locate the required root, and estimate its value to one decimal place, by means of the graph (Fig. 180). The real root is approximately 3.7.

x	0	1	2	3	4
y	-4	-5	-8	-7	4

We reduce the roots of the equation by the integer part of the required root; in this case, by 3. The computations for effecting this reduction are shown in computation I, at the right. The transformed equation is

$$f_1(x) = x^3 + 5x^2 + 5x - 7 = 0.$$

Since the required root of the given equation is about 3.7, the corresponding root of the transformed equation is about 0.7. Our estimate of the value of the root may, however, have been inaccurate, so we find the value of $f_1(x)$, for a few tenths near 0.7, until a change of sign of $f_1(x)$ has been located. We find $f_1(0.7) = -0.707$, and $f_1(0.8) = +0.712$.

I

$$\begin{array}{rrrrl}
1 & -4 & +2 & -4 & \lfloor 3 \\
 & +3 & -3 & -3 & \\ \hline
1 & -1 & -1 & (-7) & \\
 & +3 & +6 & & \\ \hline
1 & +2 & +(5) & & \\
 & +3 & & & \\ \hline
1 & (+5) & & &
\end{array}$$

Hence the root lies between 0.7 and 0.8 and we reduce the roots of $f_1(x) = 0$ by 0.7. The transformed equation is, by II,

$$f_2(x) = x^3 + 7.1x^2 + 13.47x - 0.707 = 0.$$

II

$$\begin{array}{rrrrl}
1 & +5.0 & +\ \ 5.00 & -7.000 & \lfloor 0.7 \\
 & +0.7 & +\ \ 3.99 & +6.293 & \\ \hline
1 & +5.7 & +\ \ 8.99 & (-0.707) & \\
 & +0.7 & +\ \ 4.48 & & \\ \hline
1 & +6.4 & (+13.47) & & \\
 & +0.7 & & & \\ \hline
1 & (+7.1) & & &
\end{array}$$

To estimate the first significant digit of the root of $f_2(x) = 0$, we notice that, since the required root is a small number, when we substitute this root into $f_2(x)$ the value of $x^3 + 7.1x^2$ will be small compared with the value of the remaining terms. Neglecting these two terms, we estimate the first digit of the root, by solving $13.47x - 0.707 = 0$, to be 0.05.

We now reduce the roots of $f_2(x) = 0$ by 0.05. From III, the resulting equation is

$$f_3(x) = x^3 + 7.25x^2 + 14.1875x - 0.015625 = 0.$$

III

$$\begin{array}{rrrrl}
1 & +7.10 & +13.4700 & -0.707000 & \lfloor 0.05 \\
 & +0.05 & +\ \ 0.3575 & +0.691375 & \\ \hline
1 & +7.15 & +13.8275 & (-0.015625) & \\
 & +0.05 & +\ \ 0.3600 & & \\ \hline
1 & +7.20 & (+14.1875) & & \\
 & +0.05 & & & \\ \hline
1 & (+7.25) & & &
\end{array}$$

To determine the next digit, we neglect the first two terms of $f_3(x) = 0$ and solve $14.1875x - 0.015625 = 0$ for x. This gives, for the next digit, 0.001. Since this is the last digit to be determined, we do not effect the reduction.

We have reduced the roots of the original equation $f(x) = 0$, in all, by 3.75. We then found that the required root of the last transformed equation was about 0.001. It follows that the required root of $f(x) = 0$ is about 3.751. The error in the final result should not exceed 5 in the fourth decimal place.

The entire computation should be arranged in the following form:

$$
\begin{array}{l}
1 - 4 \ + 2 \ - 4 \ \underline{|3} \\
\underline{\ \ + 3 \ - 3 \ - 3} \\
1 - 1 \ - 1 \ (-7) \\
\underline{\ \ + 3 \ + 6} \\
1 + 2 \ (+5) \\
\underline{\ \ + 3} \\
1 + 5.0 \ + \ 5.00 \ - 7.000 \ \underline{|0.7} \\
\underline{\ \ + 0.7 \ + \ 3.99 \ + 6.293} \\
1 + 5.7 \ + \ 8.99 \ (-0.707) \\
\underline{\ \ + 0.7 \ + \ 4.48} \\
1 + 6.4 \ (+13.47) \qquad\qquad\qquad\qquad 13.47x - 0.707 = 0. \\
\underline{\ \ + 0.7} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x = 0.05^{+}. \\
1 + 7.10 \ + 13.4700 \ - 0.707000 \ \underline{|0.05} \\
\underline{\ \ + 0.05 \ + \ 0.3575 \ + 0.691375} \\
1 + 7.15 \ + 13.8275 \ (-0.015625) \\
\underline{\ \ + 0.05 \ + \ 0.3600} \\
1 + 7.20 \ (+14.1875) \qquad\qquad\qquad 14.1875x - 0.015625 = 0. \\
\underline{\ \ + 0.05} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x = 0.001^{+}. \\
1 + 7.25 \ + 14.1875 \ - 0.015625
\end{array}
$$

$\boldsymbol{x = 3.751^{+}}$, to three decimal places.

When it is required to find a negative root of $f(x) = 0$, one should transform the equation into one whose roots are opposite in sign to those of $f(x) = 0$, find the corresponding positive root of the transformed equation, and change its sign.

Exercises

1–16. Solve Exs. 1 to 16, Art. 286 by Horner's Method.

Find all the real roots of the following equations by Horner's Method.

17. $x^3 - 5x^2 + x + 8 = 0.$ **18.** $x^3 - 4x^2 - 6x + 8 = 0.$

19. $2x^3 - 5x^2 - x + 9 = 0.$ **20.** $2x^4 - x^3 - 17x^2 - 14x + 14 = 0.$

21. The equation $x^3 + x^2 - 10x + 9 = 0$ has two roots between 1 and 2. Find them to three decimal places by first transforming the equation into one with roots ten times those of the given equation.

22. By applying Horner's Method to the equation $x^3 - 43 = 0$, find the real cube root of 43 to three decimal places.

Find the positive real roots to three decimal places by Horner's Method.

23. $\sqrt[3]{843}.$ **24.** $\sqrt[4]{57}.$ **25.** $\sqrt[4]{1725}.$ **26.** $\sqrt[5]{67}.$

27. A sphere of ice of radius one foot, floating in water, will sink to a depth given by the smaller positive root of the equation $x^3 - 3x^2 + 3.644 = 0$. Find this root to three decimal places.

28. An open top box of 320 cubic inches capacity is to be made by cutting

a square of side x from each corner of a rectangular piece of tin of dimensions 15 by 20 inches and turning up the sides. Find x.

29. Solve the simultaneous equations $y^2 - 4xy - 8y + 18x + 20 = 0$, $y = x^2$.

30. Find the real simultaneous solutions of $7x^2 + xy - 10x - 7y + 3 = 0$, $y = 2x^3 - x^2$.

289. Identical Polynomials. *If two polynomials*

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n,$$

and

$$b_0x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_n,$$

neither of which is of degree greater than n, are equal in value for $n + 1$ distinct values of x, then

$$a_0 = b_0,\ a_1 = b_1,\ a_2 = b_2,\ \cdots,\ a_n = b_n,$$

and the two polynomials are identical.

For, any value of x that makes the two polynomials equal, that is, for which

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = b_0x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_n, \quad (18)$$

is a root of the equation

$$(a_0 - b_0)x^n + (a_1 - b_1)x^{n-1} + (a_2 - b_2)x^{n-2} + \cdots + (a_n - b_n) = 0. \quad (19)$$

If any of the coefficients in equation (19) are different from zero, then equation (19) is an equation of degree equal to or less than n, in which the coefficient of the highest power of x is not zero, and which has, by hypothesis, $n + 1$ distinct roots. Since, by the theorem of Art. 275, this is impossible, all of the coefficients in equation (19) are equal to zero, that is,

$$a_0 - b_0 = 0,\ a_1 - b_1 = 0,\ a_2 - b_2 = 0,\ \cdots,\ a_n - b_n = 0,$$

or

$$a_0 = b_0,\ a_1 = b_1,\ a_2 = b_2,\ \cdots,\ a_n = b_n.$$

Hence, the given polynomials are identical.

EXAMPLE 1. Without expanding, show that, identically,

$$(x + 2)^3 - 2(x + 1)^3 = (x - 2)^3 - 2(x - 1)^3 + 12.$$

Each member of this identity represents a polynomial of degree not greater than three. If we put, successively, $x = -2, -1, 1$, and 2, we have $0 + 2 = -64 + 54 + 12$, $1 + 0 = -27 + 16 + 12$, $27 - 16 = -1 + 0 + 12$, and $64 - 54 = 0 - 2 + 12$.

Since $n = 3$, and the polynomials are equal for four distinct values of x, they are identical.

This result may be checked by expanding the two members of the equation.

EXAMPLE 2. Find values of A, B, and C such that, identically,

$$x^2 + 15x - 30 = A(x+2)(x-5) + Bx(x-5) + Cx(x+2).$$

Since $n = 2$, these expressions are identical if they are equal for three values of x. Put $x = 0$, -2, and 5. We have

$$-30 = -10A, \quad -56 = 14B, \quad 70 = 35C.$$

Hence, $\quad A = 3, \quad B = -4, \quad C = 2,$ and we have, identically,

$$x^2 + 15x - 30 = 3(x+2)(x-5) - 4x(x-5) + 2x(x+2).$$

Exercises

Without expanding, show that the following expressions are identical.

1. $5(x+1)^2 - 2(x+3)^2 = 8(x-2)^2 - 5(x-3)^2.$

2. $(x-1)^2 + 8(x-4)^2 = 6(x-3)^2 + 3(x-5)^2.$

3. $(x-1)(x-3)(x-5) = (1-x)^3 + 2(x-2)^3 + 2x.$

4. $5x + 11 = 3(x+5) + 2(x-2).$

5. $x^2 + 4x + 19 = 3(x+1)(x+3) - 4(x-1)(x+3) + 2(x+1)(x-1).$

6. $4x^2 - x - 39 = 3(x+2)(x-5) - (x-1)(x-5) + 2(x-1)(x+2).$

Find the values of the capital letters, given that, identically:

7. $3(3x-4)^2 + 2(2x-1)^2 = A(x-2)^2 + B(x-1)^2.$

8. $7x - 12 = A(4x+1) + B(x+3).$

9. $x^2 + 12x - 4 = A(x-1)(x+2) + Bx(x+2) + Cx(x-1).$

10. $3x^2 + 32x + 42$
$$= A(x-3)(2x+5) + B(2x+5)(x+2) + C(x-3)(x+2).$$

290. Imaginary Roots. *If the coefficients of $f(x) = 0$ are real numbers, and if $a + bi$ $(b \neq 0)$ is a complex root of $f(x) = 0$, then the conjugate complex number $a - bi$ is also a root of $f(x) = 0$.*

To prove this theorem we first form the quadratic expression

$$D(x) = (x - a - bi)(x - a + bi) = x^2 - 2ax + a^2 + b^2,$$

such that the roots of $D(x) = 0$ are $a + bi$ and $a - bi$.

Divide $f(x)$ by $D(x)$. Let the quotient be $q(x)$ and let the remainder be $rx + s$, where r and s are constants. Since the coefficients of $f(x)$ and of $D(x)$ are real numbers, r, s, and the coefficients of $q(x)$ are real numbers.

By the complete check formula for division (Art. 10),

$$f(x) = D(x) \cdot q(x) + rx + s. \tag{20}$$

Since $a + bi$ is a root of $f(x) = 0$ and also of $D(x) = 0$, if we substitute $a + bi$ for x in equation (20), we have

$$0 = 0 \cdot q(a + bi) + r \cdot (a + bi) + s,$$

or

$$ra + s + rbi = 0.$$

If a complex number is equal to zero, its real part equals zero and the coefficient of i is also equal to zero (Art. 262). Hence,

$$ra + s = 0, \quad \text{and} \quad rb = 0.$$

By hypothesis, $b \neq 0$. Hence, $r = 0$. It now follows that $s = 0$. Equation (20) now reduces to

$$f(x) = D(x) \cdot q(x) = (x - a - bi)(x - a + bi)q(x).$$

Since $x - (a - bi)$ is a factor of $f(x)$, it follows that $a - bi$ is a root of $f(x) = 0$.

It can now be shown that *if the coefficients of $f(x)$ are real numbers, the polynomial $f(x)$ can be factored into a product of linear and quadratic factors, with real coefficients, such that the linear factors of each quadratic are imaginary.*

For, we have seen that, if $a + bi$ is an imaginary root of $f(x) = 0$, then $f(x) = D(x) \cdot q(x)$, where

$$D(x) = x^2 - 2ax + a^2 + b^2$$

is quadratic and the coefficients of $D(x)$ and of $q(x)$ are real. Similarly, if $q(x) = 0$ has an imaginary root, then $q(x) = D_1(x) \cdot q_1(x)$, where $D_1(x)$ is quadratic and the coefficients of $D_1(x)$ and $q_1(x)$ are real. Continuing in this way, we obtain, ultimately, a quotient $q_j(x)$ that is either a real constant or has only real roots. If all of the roots of $q_j(x) = 0$ are real, it follows from Art. 274 that $q_j(x)$ can be factored into linear factors in such a way that all of the coefficients of the factors are real numbers.

If the coefficients of $f(x)$ are real numbers, the imaginary roots of $f(x) = 0$ (if there are any) enter in pairs of conjugate imaginary numbers. For, the roots of each of the quadratic equations $D(x) = 0$, $D_1(x) = 0$, and so on, are pairs of conjugate imaginary numbers which are roots of $f(x) = 0$.

As a particular consequence of the preceding theorem we have: *if the coefficients of $f(x)$ are real numbers, the number of imaginary roots of $f(x) = 0$ is zero or an even integer.* For, if $f(x) = 0$ has any imaginary roots, these roots are roots of the equations $D(x) = 0$, $D_1(x) = 0$, and so on. Each of these quadratic equations has precisely two roots.

Exercises

Write an equation of the given degree, with real coefficients, having the given roots.

1. $n = 2, r_1 = 3 - 4i.$ **2.** $n = 3, r_1 = -6, r_2 = -2 + 3i.$

3. $n = 4, r_1 = -5 - 3i, r_2 = -1 + i.$

4. $n = 4, r_1 = r_2 = -1, r_3 = -1 + \sqrt{3}i.$

5. $n = 4, r_1 = r_2 = 4 - i.$

6. $n = 7, r_1 = -3, r_2 = r_3 = r_4 = 1 - i.$

In the following equations, certain roots are given. Find all the roots.

7. $x^4 - 3x^3 - 2x^2 + 2x + 12 = 0$, $r_1 = -1 + i$.

8. $2x^4 - 11x^3 + 35x^2 - 47x + 45 = 0$, $r_1 = 2 - \sqrt{5}i$.

9. $x^4 - 4x^3 + 12x^2 - 16x + 16 = 0$, $r_1 = r_2 = 1 + \sqrt{3}i$.

10. $3x^6 - 32x^5 + 105x^4 - 124x^3 + 206x^2 - 92x + 104 = 0$, $r_1 = 5 - i$, $r_2 = i$.

Factor $f(x)$ into linear and quadratic factors with real coefficients, given that:

11. $f(x) = 2x^3 + 5x^2 - 2x - 15 = 0$ has a root $-2 + i$.

12. $f(x) = x^4 - 20x^2 - 36x + 55 = 0$ has a root $-3 + \sqrt{2}i$.

13. $f(x) = 2x^7 - x^6 + 31x^5 + 88x^4 + 36x^3 + 864x^2 + 432x = 0$ has $1 + \sqrt{11}i$ as a double root.

14. $f(x) = x^8 + x^6 - 21x^4 - 41x^2 - 20 = 0$, has i, i, $2i$ as roots.

15. By a method similar to that used in the text, show that, if the coefficients of $f(x)$ are rational numbers, and if $a + \sqrt{b}$, where a is rational and $\sqrt{b}$ is irrational, is a root of $f(x) = 0$, then $a - \sqrt{b}$ is also a root of $f(x) = 0$.

HINT. If $u + v\sqrt{b} = 0$, where u and v are rational numbers and $\sqrt{b}$ is irrational then $u = 0$ and $v = 0$.

Chapter 35

Partial Fractions

291. Partial Fractions. We learned in elementary algebra how to add two or more fractions by reducing them to a common denominator and adding the numerators.

Thus,

$$\frac{5}{x+7}+\frac{2}{x-4}=\frac{7x-6}{x^2+3x-28};$$

$$\frac{2}{x+1}+\frac{3}{(x+1)^2}+\frac{4}{x-3}=\frac{6x^2+7x-11}{(x+1)^2(x-3)}.$$

In certain types of mathematical problems, it is necessary to perform the operation inverse to this one; that is, we have given the second members of equations such as those shown in the illustration and we are required to find the first members. The process of doing this is called the operation of *resolving a given fraction into* **partial fractions.**

We say that the fraction formed by dividing one polynomial by another polynomial is a **proper fraction** if the numerator is of lower degree than the denominator; otherwise, it is an **improper fraction.** If the fraction we wish to resolve into partial fractions is an improper fraction, we must first reduce the fraction to a mixed expression by dividing the numerator by the denominator until a remainder is obtained which is of lower degree than the denominator.

Thus, if we wish to resolve the improper fraction $\dfrac{2x^3+7x^2-20x-38}{x^2+2x-15}$ into partial fractions, we must first divide the numerator by the denominator and write the fraction as a mixed expression in the form $2x+3+\dfrac{4x+7}{x^2+2x-15}$. The fractional part, $\dfrac{4x+7}{x^2+2x-15}$, can then be resolved by the methods which will be explained in the following articles.

After we have reduced the fraction to be resolved to a proper fraction, we must *next factor the denominator*. As we shall deal only with fractions in which the coefficients are real numbers, we shall use the theorem, proved in Art. 290, that the denominator can be factored into a product of linear and quadratic factors, with real coefficients, such that the linear factors of each quadratic are imaginary. In what follows, we shall suppose that the denominator has been factored in this way.

After the denominator has been factored, the fraction must be put identically equal to a sum of fractions, the forms of which are stated in the following theorem which we shall give without proof.

It is essential that this form for putting down the partial fractions be learned correctly as, otherwise, the result obtained will almost certainly be erroneous.

Theorem. *A proper fraction, in its lowest terms, can be expressed as a sum of partial fractions, as follows:*

1. *To a linear factor,* $ax + b$, *that occurs just once as a factor of the denominator, there corresponds a partial fraction of the form*

$$\frac{A}{ax+b},$$

where A *is a constant, the value of which is to be determined.*

2. *To a linear factor that occurs* k *times as a factor of the denominator,* $(ax + b)^k$, *there corresponds the following sum of* k *partial fractions*

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k},$$

where $A_1, A_2, \cdots, A_k$ *are constants, the values of which are to be determined.*

3. *To a quadratic factor,* $ax^2 + bx + c$, *that occurs just once as a factor of the denominator, there corresponds a partial fraction of the form*

$$\frac{Ax+B}{ax^2+bx+c},$$

where A *and* B *are constants to be determined.*

4. *To a quadratic factor that occurs* k *times as a factor of the denominator,* $(ax^2 + bx + c)^k$, *there corresponds the following sum of* k *partial fractions*

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k},$$

where $A_1, B_1, A_2, B_2, \cdots, A_k, B_k$ *are constants to be determined.*

Thus, we can write, identically,

$$\frac{3x^3-9x^2+2}{(x-5)(2x+1)^3(x^2+x+1)(x^2+1)^2}$$
$$= \frac{A}{x-5} + \frac{B}{2x+1} + \frac{C}{(2x+1)^2} + \frac{D}{(2x+1)^3} + \frac{Ex+F}{x^2+x+1} + \frac{Gx+H}{x^2+1} + \frac{Ix+J}{(x^2+1)^2},$$

where $A, B, C, \ldots, J$ are constants to be determined.

The process of determining the numerical values of the constants appearing in the numerators of the various partial fractions will constitute the subject matter of the remaining articles of this chapter. The solution will depend on the theorem proved in Art. 289, that, if two polynomials are identical, then the coefficients of like powers of x in the two polynomials are equal. By equating like coefficients, we shall set up a system of linear equations from which the required values of the constants may be determined.

Exercises

Write each of the following fractions as a sum of partial fractions without determining the values of the constants.

1. $\dfrac{3x^2 - 2x + 5}{(x - 4)(x + 7)(3x + 1)}$.

2. $\dfrac{6x + 9}{(x^2 - 4)(x^2 + 2x - 4)}$.

3. $\dfrac{x^4 + 5x^3 - 9x}{x^3 + 3x^2 - x - 3}$.

4. $\dfrac{x^4 + 11}{(x + 1)^2(x^2 - 5x + 7)}$.

5. $\dfrac{x^3 - 2x + 5}{(x + 3)^2(x^2 + 1)}$.

6. $\dfrac{2x^3 + 8x + 17}{x^3 + 1}$.

7. $\dfrac{3x^4 - 8x^2 + 2}{x^3(x - 1)^2(x^2 + 2)^2}$.

8. $\dfrac{4x^4 - x^3 + 1}{x(x + 1)^2(x^2 - x + 1)(x^2 + 9)^3}$.

292. Linear Factors; None Repeated. If the factors of the denominator are all linear and distinct, the values of the unknown constants appearing in the numerators of the second member can be found by either of the methods shown in the following example.

EXAMPLE. Resolve into partial fractions: $\dfrac{6x^2 - 25x + 1}{(x - 1)(x - 3)(x + 2)}$.

By the theorem of the preceding article, there exist three constants, A, B, and C, such that we have, identically,

$$\frac{6x^2 - 25x + 1}{(x - 1)(x - 3)(x + 2)} = \frac{A}{x - 1} + \frac{B}{x - 3} + \frac{C}{x + 2}. \tag{1}$$

Clear of fractions by multiplying by the denominator of the fraction appearing in the first member. We then have, identically,

$$6x^2 - 25x + 1 = A(x - 3)(x + 2) + B(x - 1)(x + 2) + C(x - 1)(x - 3). \tag{2}$$

First method. Collect the coefficients of the various powers of x in the second member of equation (2). We have, identically,

$$6x^2 - 25x + 1 = (A + B + C)x^2 + (- A + B - 4C)x + (- 6A - 2B + 3C).$$

Since the second member is just another way of writing the first member, the coefficients of the various powers of x on the two sides of the identity must be equal. Hence,

$$\begin{aligned} A + B + C &= 6, \\ - A + B - 4C &= - 25, \\ - 6A - 2B + 3C &= 1. \end{aligned}$$

By solving these three equations for A, B, and C, we obtain $A = 3$, $B = - 2$, and $C = 5$. On substituting these values of A, B, and C in equation (1), we have, as the required expression for the given fraction as a sum of partial fractions,

$$\frac{6x^2 - 25x + 1}{(x - 1)(x - 3)(x + 2)} = \frac{3}{x - 1} - \frac{2}{x - 3} + \frac{5}{x + 2}. \tag{3}$$

Second method. Since equation (1) is, by hypothesis, an identity, it is true for all values of x except the values $x = 1$, $x = 3$, and $x = -2$ which make the denominators zero and thus make the equation meaningless. It follows that equation (2) is also true except, possibly, for these same values of x. But it now follows, by Art. 289, that equation (2) is true for these values of x, also. By putting, successively, $x = 1$, $x = 3$, and $x = -2$ in equation (2), we obtain

$$-18 = -6A, \qquad -20 = 10B, \qquad 75 = 15C.$$

Hence, $A = 3$, $B = -2$, and $C = 5$. If we substitute these values of A, B, and C in equation (1), we obtain equation (3) which is the required result.

Exercises

Resolve the following fractions into partial fractions. Check by adding the resulting fractions.

1. $\dfrac{x+26}{(x+4)(2x-3)}$.

2. $\dfrac{5x-13}{(3x+2)(2x-1)}$.

3. $\dfrac{10x+35}{2x^2+5x}$.

4. $\dfrac{8x+23}{x^2+3x-28}$.

5. $\dfrac{2x^2+9x-18}{x^2-3x-10}$.

6. $\dfrac{6x^4-13x^3+34x-120}{6x^3-x^2-40x}$.

7. $\dfrac{2x+14}{(x-1)(x-3)(x-5)}$.

8. $\dfrac{83-88x-3x^2}{(x-2)(3x-1)(x+5)}$.

9. $\dfrac{2x+2}{x^2+2x-4}$.

10. $\dfrac{11x^2-24}{x^4-13x^2+36}$.

293. Linear Factors; Some Repeated. If the denominator contains linear factors, some of which appear to powers higher than the first, care must be taken to put down all of the partial fractions that are called for by such a repeated factor. The values of the constants are then found by a process similar to that used in Art. 292.

EXAMPLE. Resolve into partial fractions: $\dfrac{7x^2+25x+24}{(x+1)^3(x+2)}$.

By the theorem of Art. 282,

$$\frac{7x^2+25x+24}{(x+1)^3(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{x+2}. \tag{4}$$

Clear of fractions by multiplying by the denominator of the first member:

$$7x^2+25x+24 = A(x+1)^2(x+2) + B(x+1)(x+2) + C(x+2) + D(x+1)^3. \tag{5}$$

In the second member of equation (5), collect the coefficients of the various powers of x.

$$7x^2 + 25x + 24 = (A + D)x^3 + (4A + B + 3D)x^2 + (5A + 3B + C + 3D)x \\ + 2A + 2B + 2C + D.$$

Equate the coefficients of like powers of x in this identity:

$$\begin{aligned} A + \quad D &= 0, \\ 4A + \quad B + 3D &= 7, \\ 5A + 3B + \quad C + 3D &= 25, \\ 2A + 2B + 2C + \quad D &= 24. \end{aligned} \tag{6}$$

By solving these four linear equations, we obtain $A = 2$, $B = 5$, $C = 6$, and $D = -2$. On substituting these four values of A, B, C, and D in equation (1), we have

$$\frac{7x^2 + 25x + 24}{(x+1)^3(x+2)} = \frac{2}{x+1} + \frac{5}{(x+1)^2} + \frac{6}{(x+1)^3} - \frac{2}{x+2}, \tag{7}$$

which is the required resolution of the given fraction into partial fractions.

The problem of solving equations (6) can be simplified by the reasoning used in the second method of Art. 292.

If, in equation (5), we put $x = -1$, we obtain at once $C = 6$ and, if we put $x = -2$, we get $D = -2$. On substituting the value of D in the first two of equations (6), we find that $A = 2$ and $B = 5$. The remaining two of equations (6) may be used to check these results.

Exercises

Resolve the following fractions into partial fractions.

1. $\dfrac{7x+9}{(x+3)^2}$.
2. $\dfrac{3x^2 - 10x + 3}{(x-2)^2}$.
3. $\dfrac{x^3 - 8x^2 + 17x + 1}{(x-3)^3}$.
4. $\dfrac{4x^2 + 5x + 4}{x(x+2)^2}$.
5. $\dfrac{7x^2 - 4x - 6}{(x+1)^3(2x+7)}$.
6. $\dfrac{x^3 - 18x^2 + 4x + 8}{(x^2-4)^2}$.
7. $\dfrac{5 + 10x - 3x^3}{(x^2+x)^2}$.
8. $\dfrac{4x^2 + 7x + 20}{x^3(2x-5)}$.
9. $\dfrac{8 + 60x - 14x^2 + 4x^3 - 2x^4}{(x^2-1)^3}$.
10. $\dfrac{x^4 + 14x^3 - 4x^2 + 2x - 1}{x^2(x+1)^2(x-1)^2}$.

294. Quadratic Factors; None Repeated. According to the theorem of Art. 291, to a quadratic factor of the denominator of the given fraction corresponds a partial fraction having a linear expression in its numerator.

Example. Resolve into partial fractions: $\dfrac{x^3 + 8x^2 + 9x - 1}{(x+1)(x^2+2x+2)}$.

Since this is an improper fraction, we must first divide the numerator by the denominator and write the fraction in the form

$$\frac{x^3 + 8x^2 + 9x - 1}{(x+1)(x^2+2x+2)} = 1 + \frac{5x^2 + 5x - 3}{(x+1)(x^2+2x+2)}. \tag{8}$$

We now put, identically,

$$\frac{5x^2+5x-3}{(x+1)(x^2+2x+2)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2x+2}. \tag{9}$$

Clear of fractions:

$$5x^2+5x-3 = A(x^2+2x+2) + Bx(x+1) + C(x+1). \tag{10}$$

Collect the coefficients in the second member:

$$5x^2+5x-3 = (A+B)x^2 + (2A+B+C)x + 2A + C.$$

Equate the coefficients of like powers of x:

$$\begin{aligned} A + B &= 5, \\ 2A + B + C &= 5, \\ 2A + C &= -3. \end{aligned} \tag{11}$$

Hence $A = -3$, $B = 8$, and $C = 3$. Substitute these values in equation (9):

$$\frac{5x^2+5x-3}{(x+1)(x^2+2x+2)} = \frac{-3}{x+1} + \frac{8x+3}{x^2+2x+2},$$

or, from equation (8),

$$\frac{x^3+8x^2+9x-1}{(x+1)(x^2+2x+2)} = 1 - \frac{3}{x+1} + \frac{8x+3}{x^2+2x+2}. \tag{12}$$

If, in equation (10), we put $x = -1$, we find at once that $A = -3$. From the first and third of equations (11), we now have $B = 8$ and $C = 3$. The second equation may be used as a check.

Exercises

Resolve the following fractions into partial fractions.

1. $\dfrac{2x^4-9x^3-4x+4}{x^3+x}$.
2. $\dfrac{8x^2-11x+6}{x^3+8}$.
3. $\dfrac{x^2-2x+9}{x^4-1}$.
4. $\dfrac{4x^2+17x}{(x-1)(2x^2+5)}$.
5. $\dfrac{2x^2+12x-9}{x(x^2-x+3)}$.
6. $\dfrac{3x^3+3x^2-12x+2}{(x-1)^2(2x^2+x+1)}$.
7. $\dfrac{x^3+7x^2+25x}{(x^2+4)(2x^2+1)}$.
8. $\dfrac{3x^3-2x^2-5x+2}{(x^2+5)(x^2+1)}$.
9. $\dfrac{7x^3+4x^2+2x-4}{x^3(x^2+2x+2)}$.
10. $\dfrac{6x^4+22x^3+33x^2+11x}{(x+1)(x+2)^2(x^2+3x+5)}$.

295. Quadratic Factors; Some Repeated.

EXAMPLE. Resolve into partial fractions: $\dfrac{49}{(x-2)(x^2+3)^2}$.

By the theorem of Art. 291, we have, identically,

$$\frac{49}{(x-2)(x^2+3)^2} = \frac{A}{x-2} + \frac{Bx+C}{x^2+3} + \frac{Dx+E}{(x^2+3)^2}. \tag{13}$$

Clear of fractions:

$$49 = A(x^2+3)^2 + (Bx+C)(x-2)(x^2+3) + (Dx+E)(x-2). \qquad (14)$$

Collect the coefficients in the second member:

$$49 = (A+B)x^4 + (C-2B)x^3 + (6A+3B-2C+D)x^2 + (-6B+3C-2D+E)x + (9A-6C-2E).$$

Equate coefficients of like powers of x:

$$\begin{aligned} A+B&=0, & 6A+3B-2C+D&=0,\\ C-2B&=0, & 6B-3C+2D-E&=0,\\ & & 9A-6C-2E\quad&=49. \end{aligned} \qquad (15)$$

Moreover, by putting $x = 2$ in (14), we find that $A = 1$. Hence we have $A = 1$, $B = -1$, $C = -2$, $D = -7$, and $E = -14$. The required equation is, accordingly,

$$\frac{49}{(x-2)(x^2+3)^2} = \frac{1}{x-2} - \frac{x+2}{x^2+3} - \frac{7x+14}{(x^2+3)^2}. \qquad (16)$$

Exercises

Resolve the following fractions into partial fractions.

1. $\dfrac{6x^3+x^2-8}{(2x^2+x+3)^2}$.

2. $\dfrac{3x^4+5x^2+7}{x^4+2x^2+1}$.

3. $\dfrac{2x^3-17x^2+28x-61}{(x-1)(x^2-2x+5)^2}$.

4. $\dfrac{6x^4+4x+1}{x^2(x^2+x+1)^2}$.

5. $\dfrac{x^4-3x^3-7x^2+2x-4}{(x^2+1)^3}$.

6. $\dfrac{3x^4-x^3+19x^2-x+23}{(2x^2+3)(x^2+2)^2}$.

7. $\dfrac{4x^4-7x^3+6x^2+9}{x^3(x^2+3)^2}$.

8. $\dfrac{9}{(x^3-1)^2}$.

Chapter 36

Coördinates in Space

296. Rectangular Coördinates. Through a fixed point O, the **origin,** in space, let there be given three directed lines, the **x-axis,** the **y-axis,** and the **z-axis,** each perpendicular to both of the others. The three planes, each of which contains two of the axes, are the **coördinate planes.** They are named, from the two axes that they contain, the **xy-plane,** the **yz-plane,** and the **zx-plane,** respectively.

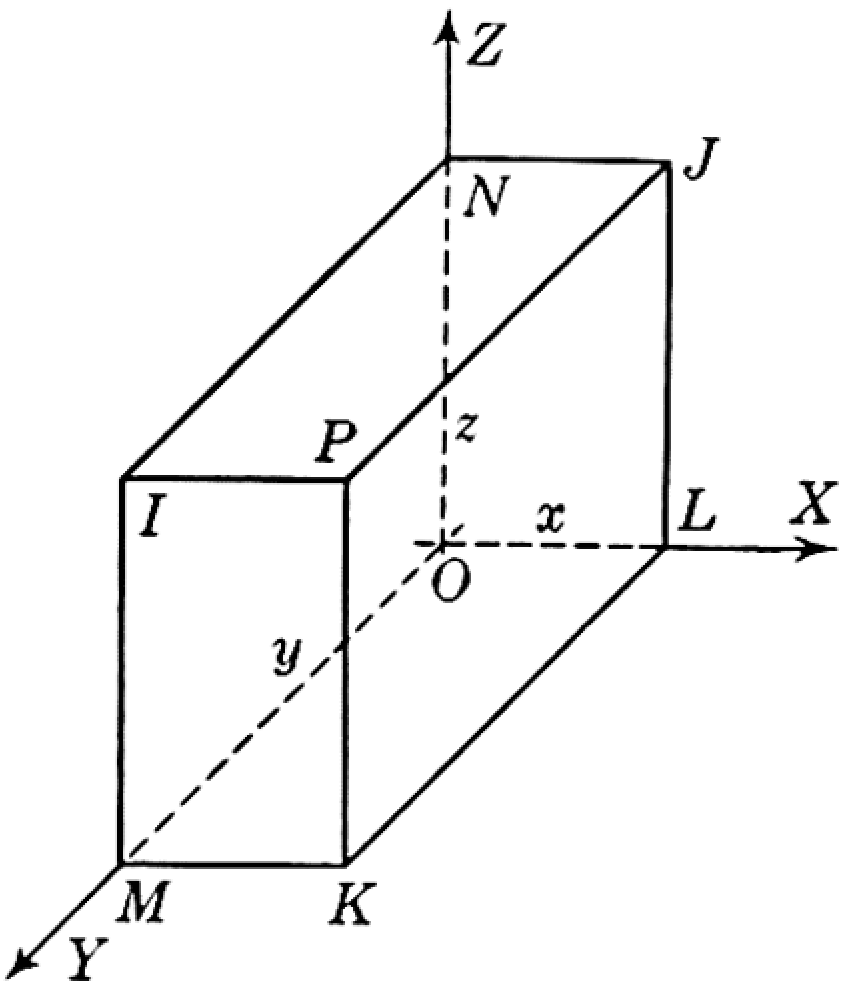


FIG. 181

Let P be any given point in space. To define the coördinates of P, we pass planes through P parallel to the three coördinate planes and denote the points of intersection of these planes with the x-, y-, and z-axes by L, M, and N, respectively. Then the directed lengths

$$x = OL, \quad y = OM, \quad z = ON, \tag{1}$$

are the **coördinates of the point P.**

Conversely, if the coördinates (x, y, z) of the point P are given, we can locate this point P in the following way: measure off from the origin, on the x-axis, the directed distance $OL = x$; from L measure off, on a line through L parallel to the y-axis, a distance $LK = y$; and finally, from K, on a line parallel to the z-axis, lay off $KP = z$. The point P so determined is the point whose coördinates are (x, y, z).

The three coördinate planes divide space into eight parts, called **octants,** which may be distinguished by the signs of the coördinates of the points in them. In particular, the octant in which all the coördinates of a point are positive is known as the *first octant.*

297. Figures. To represent a figure in space on a plane, we shall use what is known as a parallel projection. We represent the x- and z-axes by two mutually perpendicular lines and the y-axis by a line that bisects one pair of vertical angles formed by the other two (Fig. 181). Distances parallel to the x-axis and to the z-axis will be represented correctly to scale but distances parallel to the y-axis will be shortened by dividing them by $\sqrt{2}$. As in plane analytic geometry, *the first step in the solution of an exercise should consist in drawing an accurate figure.*

One serious difficulty the student will encounter throughout the study

of analytic geometry of space will be the visualization of the figure as it actually is in space. The representation of this figure on a plane will usually be necessarily somewhat distorted and the student must be certain that he understands clearly the properties of the figure as it actually is in space. For this purpose, he will often find it convenient to visualize this figure with reference to the floor and two adjacent walls of the room in which he is sitting, as coördinate planes.

Exercises

Plot the following points.

1. $(-3, 0, 0)$, $(5, 4, 0)$, $(0, 3, -2)$, $(3, 1, 4)$, $(-2, -7, -5)$.

2. $(0, 0, -5)$, $(3, -4, 0)$, $(4, 0, -3)$, $(-8, 2, 3)$, $(2, -6, 1)$, $(-6, -4, 5)$.

3. Find the coördinates of the feet of the perpendiculars from the point (x, y, z) to (a) the coördinate planes and (b) the coördinate axes.

4. Show that the figure O–$LJIM$–P (Fig. 181) is a rectangular parallelepiped (that is, a box-shaped figure) and find the lengths of all of its edges.

5. Find the lengths of the segments LP, MP, and NP (Fig. 181).

6. Find the length of the segment OP (Fig. 181).

7. What is the locus of a point for which (a) $z = 0$, (b) $z = 5$?

8. What is the locus of a point for which $y = 0$, $z = 0$?

9. What is the locus of a point for which $x = 5$, $y = 3$?

10. A cube of side a has one vertex at the origin and three of its edges extending in the positive directions along the axes. Find the coördinates of its vertices.

11. Solve Ex. 10 if the center of the cube is at the origin and its edges are parallel to the coördinate axes.

12. Describe the position in space of the octant for which the signs of the coördinates are $(-, -, +)$.

13. Two points are *symmetric* with respect to a plane if the line segment joining them is perpendicular to the plane and is bisected by the plane. Find the coördinates of the point symmetric to $P(x, y, z)$ with respect to each of the coördinate planes.

14. Find the coördinates of the point symmetric to $P(x, y, z)$ with respect to (a) each of the coördinate axes and (b) with respect to the origin.

298. Distance between Two Points. To find the distance between two given points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, we construct a box-shaped figure by passing planes through P_1 and P_2 parallel to the coördinate planes (Fig. 182). The required distance $d = P_1P_2$ is the length of the diagonal of this box and the lengths of the sides of the box are given by the numerical values of P_1U, P_1V, and P_1W.

From Figure 182, we have, using directed segments

$$P_1U = L_1L_2 = OL_2 - OL_1 = x_2 - x_1. \tag{2}$$

Similarly $\quad P_1V = y_2 - y_1 \quad$ and $\quad P_1W = z_2 - z_1.$

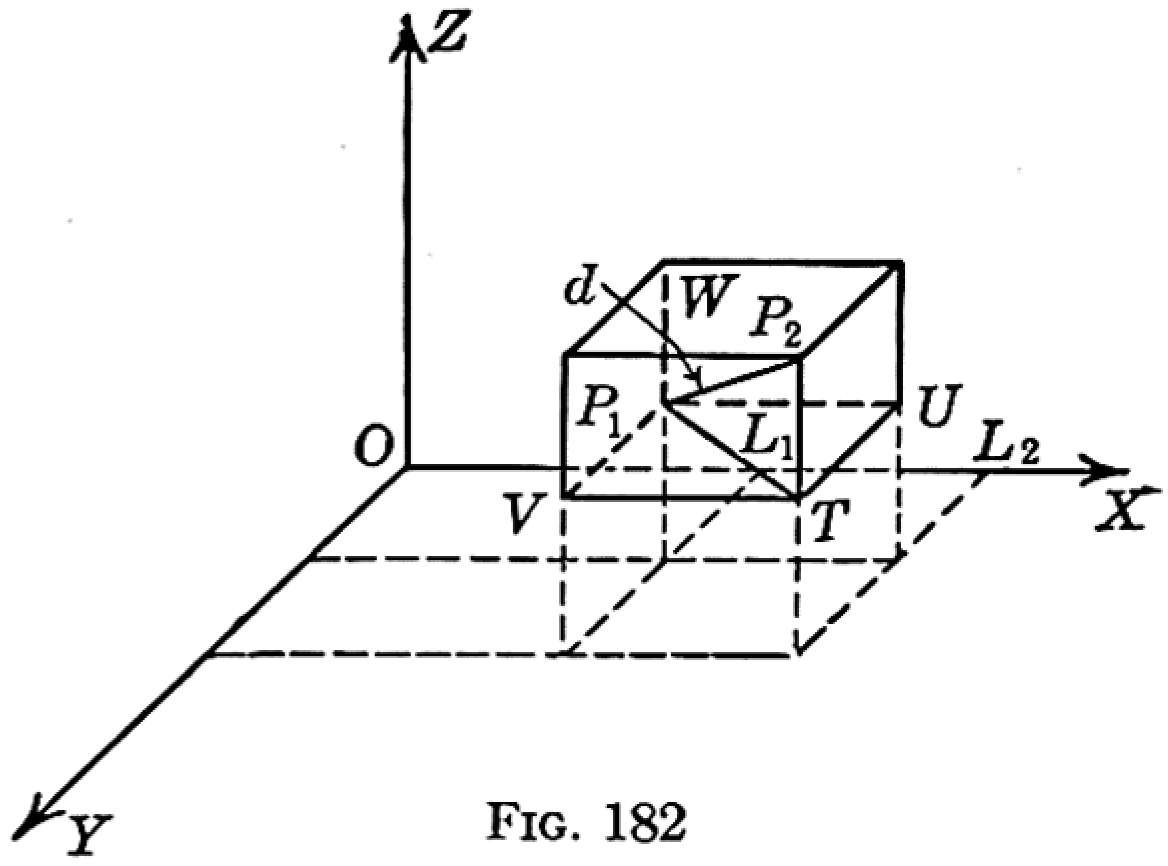


FIG. 182

By elementary geometry, since the triangles P_1TP_2 and P_1UT are right triangles,

$$P_1P_2^2 = P_1T^2 + TP_2^2 = P_1U^2 + UT^2 + TP_2^2 = P_1U^2 + P_1V^2 + P_1W^2.$$

If we put $P_1P_2 = d$, we have, from (2),

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$$

Hence

$$\boldsymbol{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.} \tag{3}$$

Exercises

Find the distance between the two given points.

1. (0, 0, 0), (2, − 11, 10).
2. (0, 0, 0), (− 4, − 1, 8).
3. (3, 2, − 7), (− 6, 4, − 1).
4. (− 5, 7, − 1), (− 9, − 13, 4).
5. (5, − 8, 2), (− 5, 7, 8).
6. (11, 3, 1), (4, 7, − 3).
7. (− 1, 6, 3), (8, 5, 7).
8. (− 5, 3, 8), (2, 5, 4).

9. Show that (1, 4, − 2), (7, 2, 3), and (4, 3, − 6) are the vertices of a right triangle and find the lengths of its three sides.

10. Show that (7, 3, − 2), (3, 5, 4), and (4, − 1, 2) are the vertices of an isosceles triangle and find the lengths of its three sides.

11. Show that (6, 1, 3), (4, 5, 5), and (2, 3, 1) are the vertices of an equilateral triangle and find the lengths of its sides.

12. Show that (2, 1, 8), (1, 2, 4), (2, − 2, 5), and (5, 1, 5) are the vertices of a regular tetrahedron (or triangular pyramid); that is, of a tetrahedron whose edges are all equal in length.

13. Find the equation of the locus of a point whose distance from (− 7, 3, − 2) equals 5. What locus is defined by this equation?

14. What locus is defined by the equation $(x - 3)^2 + (y + 5)^2 + (z - 2)^2 = 9$?

15. What locus is defined by the equation $x^2 + y^2 + z^2 = 36$?

16. Find the equation of the locus of a point whose distances from (8, − 3, 5) and (4, 1, 3) are equal. What locus is defined by this equation?

299. Direction Cosines of a Line. Let $P_1(x_1, y_1, z_1)$ be any point on a given directed line l in space. Through P_1 draw the lines P_1A, P_1B,

and P_1C, having the same directions as the x-, y-, and z-axes, respectively. Then the angles α, β, and γ, which the positive direction on l makes with the positive direction on P_1A, P_1B, and P_1C, respectively, are called the **direction angles** of l.

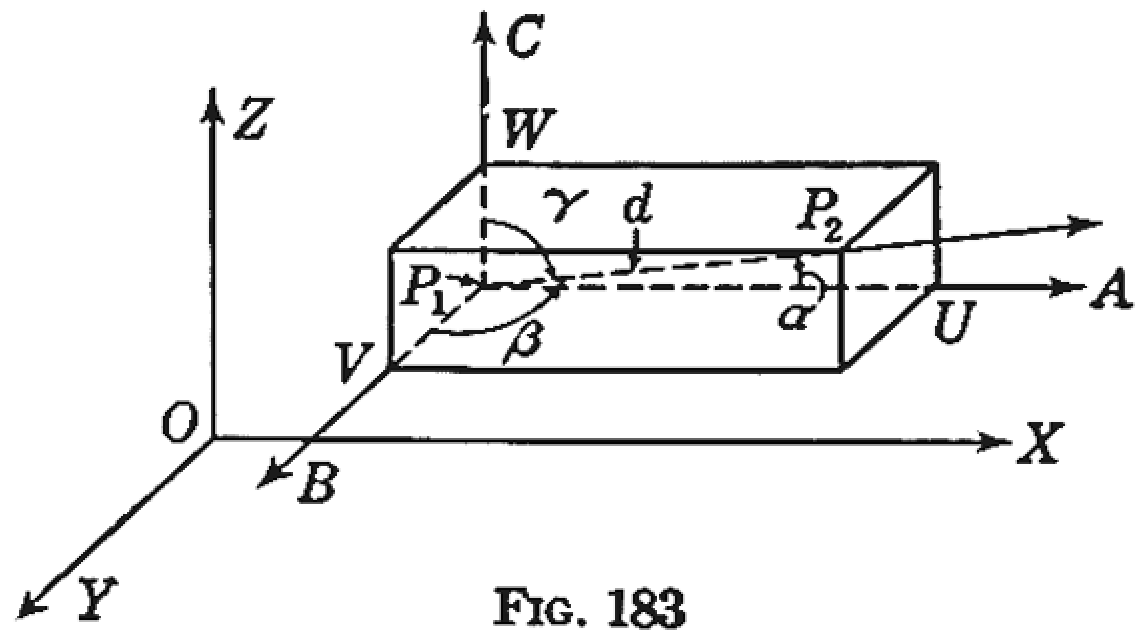


FIG. 183

We shall usually deal, not with the direction angles α, β, and γ themselves, but with their cosines. These three cosines, $\cos \alpha$, $\cos \beta$, and $\cos \gamma$, are the **direction cosines** of the line l.

Let $P_2(x_2, y_2, z_2)$ be any point on l in the positive direction from P_1 and let U, V, and W be the points in which the planes through P_2 perpendicular to the x-, y-, and z-axes intersect P_1A, P_1B, and P_1C, respectively. Since the triangles P_1UP_2, P_1VP_2, and P_1WP_2 are right triangles, we now have, from the definition of the cosine of an angle,

$$\cos \alpha = \frac{P_1U}{P_1P_2}, \quad \cos \beta = \frac{P_1V}{P_1P_2}, \quad \text{and} \quad \cos \gamma = \frac{P_1W}{P_1P_2}.$$

If we now put $P_1P_2 = d$, and substitute for P_1U, P_1V, and P_1W their values from (2), we have

$$\cos \alpha = \frac{x_2 - x_1}{d}, \quad \cos \beta = \frac{y_2 - y_1}{d}, \quad \text{and} \quad \cos \gamma = \frac{z_2 - z_1}{d}, \tag{4}$$

wherein

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

If we square the members of equations (4), add, and substitute for d its value, we find that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \tag{5}$$

that is, *the sum of the squares of the direction cosines of any line is equal to unity.* This relation will be found to be of importance whenever we shall deal with the direction cosines of a line.

If, in (4), we let P_1 be the origin and let P_2 be any other point $P(x, y, z)$ in space, and if we further denote the distance OP by ρ, we find that

$$\cos \alpha = \frac{x}{\rho}, \qquad \cos \beta = \frac{y}{\rho}, \qquad \cos \gamma = \frac{z}{\rho}, \tag{6}$$

are the direction cosines of the line through the origin and the point P and directed from O toward P.

300. Direction Numbers of a Line. Any three real numbers a, b, and c, not all zero, are called the **direction numbers** of a line l if they are proportional to the direction cosines of l, that is, if

$$\frac{a}{\cos \alpha} = \frac{b}{\cos \beta} = \frac{c}{\cos \gamma}, \tag{7}$$

wherein $\cos \alpha$, $\cos \beta$, and $\cos \gamma$ are the direction cosines of l.

To find the direction cosines of a line when its direction numbers a, b, and c are given, we set each of the above fractions equal to k and solve for a, b, and c. The resulting equations are

$$a = k \cos \alpha, \quad b = k \cos \beta, \quad \text{and} \quad c = k \cos \gamma. \tag{8}$$

By squaring the members of these equations, adding, and simplifying by means of equation (5), we obtain

$$a^2 + b^2 + c^2 = k^2(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma) = k^2.$$

Hence,
$$k = \pm \sqrt{a^2 + b^2 + c^2}.$$

If we substitute this expression for k in equations (8), and solve, we obtain, as the direction cosines of a line whose direction numbers are a, b, and c,

$$\cos \alpha = \frac{a}{\pm \sqrt{a^2 + b^2 + c^2}}, \quad \cos \beta = \frac{b}{\pm \sqrt{a^2 + b^2 + c^2}},$$
$$\cos \gamma = \frac{c}{\pm \sqrt{a^2 + b^2 + c^2}}. \tag{9}$$

The sign in the denominator is to be taken as positive throughout, or as negative throughout, according as one direction on the line, or the other, is to be taken as the positive direction on the line (see example 2).

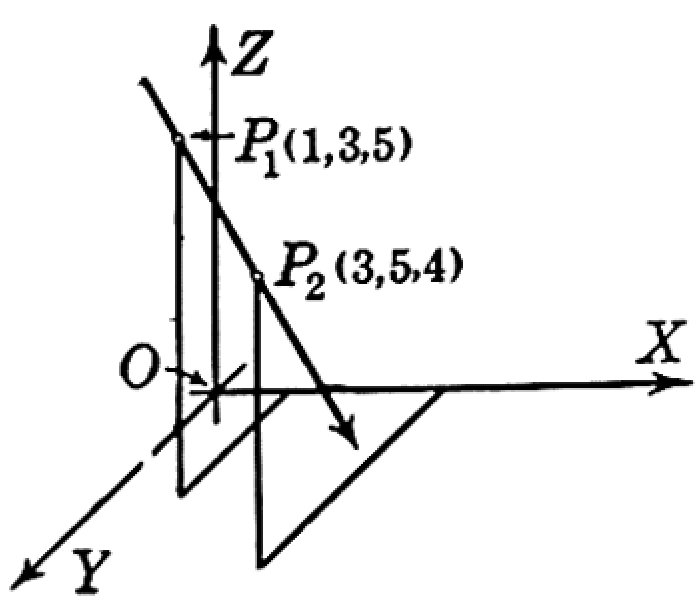


FIG. 184

EXAMPLE 1. Find the direction cosines of the line through $P_1(1, 3, 5)$ and $P_2(3, 5, 4)$ and directed from P_1 toward P_2.

The distance between these points is

$$d = \sqrt{(3-1)^2 + (5-3)^2 + (4-5)^2} = 3.$$

Hence, from (4), the required direction cosines of this line are

$$\cos \alpha = \tfrac{2}{3}, \quad \cos \beta = \tfrac{2}{3}, \quad \cos \gamma = -\tfrac{1}{3}.$$

EXAMPLE 2. The direction numbers of a line are 6, 2, − 3 and the positive direction is chosen on the line so that the angle γ is acute. Find the direction cosines of the line.

On substituting these values of a, b, and c in equations (9), we have

$$\cos \alpha = \frac{6}{\pm \sqrt{36 + 4 + 9}} = \frac{6}{\pm 7}, \quad \cos \beta = \frac{2}{\pm 7}, \quad \cos \gamma = \frac{-3}{\pm 7}.$$

Since the angle γ is acute, its cosine is positive. Hence, from the last of the above equations, the sign in the denominator is negative and we have

$$\cos \alpha = -\tfrac{6}{7}, \quad \cos \beta = -\tfrac{2}{7}, \quad \cos \gamma = \tfrac{3}{7}.$$

These are the required direction cosines of the line.

Exercises

Find the direction cosines of the line through the two given points and directed from the first point toward the second.

1. (0, 0, 0), (3, − 4, 12). **2.** (7, 4, − 8), (− 1, 5, − 4).
3. (− 6, − 4, 7), (4, 7, 9). **4.** (6, 1, − 3), (2, 5, − 1).
5. (3, − 7, 2), (5, 1, − 6). **6.** (4, 6, 1), (5, 3, − 4).

Find the direction cosines of a line, given that γ is acute and that its direction numbers are:

7. 4, − 7, 4. **8.** − 9, 2, − 6. **9.** 23, − 2, − 14.
10. − 15, − 6, 10. **11.** − 3, − 5, − 2. **12.** 4, 6, − 7.

Find, with the aid of equation (5), the direction cosines of the following lines, given that the angles not specified are acute.

13. $\alpha = 60°$, $\beta = 120°$. **14.** $\beta = 60°$, $\gamma = 135°$.
15. $\alpha = 2\pi/3$, $\gamma = \pi/4$. **16.** $\alpha = \pi/6$, $\beta = \pi/2$.

17. Find the direction cosines of each of the coördinate axes.

18. Find the coördinates of P_2, given that the coördinates of P_1 are (− 2, 6, − 5), the direction cosines of the line directed from P_1 toward P_2 are $\frac{3}{7}$, $-\frac{2}{7}$, and $\frac{6}{7}$, and that the length of the segment P_1P_2 is 14.

19. Show, using direction cosines, that the points (− 6, 7, − 9), (1, 3, − 5), and (15, − 5, 3) lie on a line.

20. Show, using direction cosines and the distance formula, that $[\frac{1}{2}(x_1 + x_2)$, $\frac{1}{2}(y_1 + y_2)$, $\frac{1}{2}(z_1 + z_2)]$ are the coördinates of the mid-point of the segment joining $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$.

21. Show that any three real numbers a, b, and c (not all zero) are the direction numbers of the line through the origin and the point (a, b, c).

301. The Angle between Two Directed Lines. Two lines drawn at random in space, will usually not intersect. In order that we may speak of the angle between two such lines, we make the following definition: *The angle between two directed lines in space that do not meet is equal to the angle between the positive directions of two intersecting lines having the same directions as the given lines.*

In particular, if the given lines are parallel, the angle between them is zero or π according as their positive directions are the same or opposite.

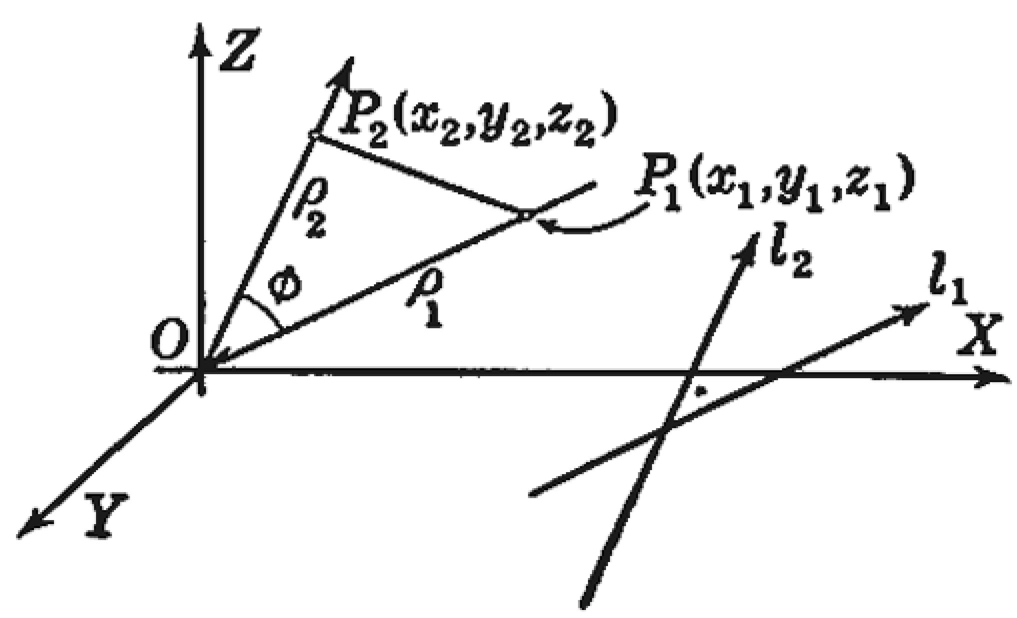


FIG. 185

Let l_1 and l_2 (Fig. 185) be two given directed lines and let ϕ be the angle between them. It is required to express $\cos \phi$ in terms of the direction cosines of l_1 and l_2.

Through the origin O, draw the lines OP_1 and OP_2, having the same directions as l_1, and l_2, respectively. Then, from the above definition of the angle between two directed lines, we have

$$\text{angle } P_1OP_2 = \phi.$$

Let the coördinates of P_1 be (x_1, y_1, z_1) and of P_2 be (x_2, y_2, z_2). Let the length of the segment $OP_1 = \rho_1$ and of $OP_2 = \rho_2$. Draw P_1P_2 and apply the law of cosines to the triangle P_1OP_2. We have

$$P_1P_2^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos \phi$$

or

$$\cos \phi = \frac{\rho_1^2 + \rho_2^2 - P_1P_2^2}{2\rho_1\rho_2}. \tag{10}$$

But $\quad \rho_1^2 = x_1^2 + y_1^2 + z_1^2 \quad \text{and} \quad \rho_2^2 = x_2^2 + y_2^2 + z_2^2$

and $\quad P_1P_2^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$

On making these substitutions in the numerator of (10), and simplifying, we obtain

$$\cos \phi = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\rho_1\rho_2}. \tag{11}$$

From (6), we have

$$\cos \alpha_1 = \frac{x_1}{\rho_1}, \qquad \cos \beta_1 = \frac{y_1}{\rho_1}, \qquad \cos \gamma_1 = \frac{z_1}{\rho_1},$$

and

$$\cos \alpha_2 = \frac{x_2}{\rho_2}, \qquad \cos \beta_2 = \frac{y_2}{\rho_2}, \qquad \cos \gamma_2 = \frac{z_2}{\rho_2}.$$

On making these substitutions in (11), we have the equation,

$$\cos \phi = \cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2, \tag{12}$$

which expresses *the cosine of* ϕ, *the angle between* l_1 *and* l_2, *in terms of the direction cosines of* l_1 *and* l_2.

In particular, the condition that l_1 and l_2 are perpendicular to each other is that $\phi = \pi/2$ so that $\cos \phi = 0$. On substituting this value of $\cos \phi$ in (12), we obtain

$$\cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2 = 0 \tag{13}$$

as *the condition that the lines* l_1 *and* l_2 *are perpendicular*.

If, instead of the direction cosines of l_1 and l_2, we have their direction numbers a_1, b_1, c_1 and a_2, b_2, c_2, respectively, we first find the direction cosines of l_1 and l_2 from (9), then substitute these values in (12). This gives

$$\cos \phi = \pm \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} \tag{14}$$

as *the value of* $\cos \phi$ *in terms of the direction numbers of* l_1 *and* l_2.

Since l_1 and l_2 are perpendicular if, and only if, $\cos\phi = 0$, it follows that

$$a_1a_2 + b_1b_2 + c_1c_2 = 0 \tag{15}$$

is the condition that l_1 and l_2 are perpendicular.

EXAMPLE 1. Find the angle between the line through $P_1(1, -2, 4)$ and $P_2(3, 8, -7)$ and the line through $P_1'(1, 5, -2)$ and $P_2'(7, -2, 4)$.

From (4), the direction cosines of the first of these lines are $\frac{2}{15}$, $\frac{10}{15}$, and $-\frac{11}{15}$; and those of the second are $\frac{6}{11}$, $-\frac{7}{11}$, and $\frac{6}{11}$.

On substituting these values of the direction cosines of the given lines in (12), we have

$$\cos\phi = \frac{2\cdot 6 + 10\cdot(-7) - 11\cdot 6}{15\cdot 11} = \frac{-124}{165} = -0.7515.$$

Hence $\phi = 138° 43'$.

EXAMPLE 2. Find direction numbers of a line that is perpendicular to each of two lines for which the direction numbers are 4, 1, 3 and 6, 3, 5, respectively.

Denote the required direction numbers by a, b, c. We have, from (15),

$$4a + b + 3c = 0$$

and

$$6a + 3b + 5c = 0.$$

If we solve these equations for a and b in terms of c, we have $a = -2c/3$ and $b = -c/3$. Since only the ratios of these numbers are significant, we may give c any value, except zero, that we please. If we put $c = -3$, we obtain $2, 1, -3$ as the required direction numbers.

Exercises

Find the angle between two lines whose direction cosines are:

1. $\frac{6}{11}, \frac{-2}{11}, \frac{9}{11}; \frac{7}{9}, \frac{-4}{9}, \frac{-4}{9}.$
2. $\frac{-14}{15}, \frac{-5}{15}, \frac{2}{15}; \frac{10}{15}, \frac{-2}{15}, \frac{11}{15}.$
3. $\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}; \frac{-3}{7}, \frac{6}{7}, \frac{-2}{7}.$
4. $\frac{5}{\sqrt{35}}, \frac{3}{\sqrt{35}}, \frac{-1}{\sqrt{35}}; \frac{3}{\sqrt{29}}, \frac{2}{\sqrt{29}}, \frac{-4}{\sqrt{29}}.$

Find the acute angle between two lines whose direction numbers are:

5. 12, 3, − 4; 6, − 6, 7.
6. 4, 20, 5; 7, − 4, − 4.
7. 4, − 8, 1; 2, − 1, − 2.
8. 1, 2, 1; − 3, 2, 5.

Using direction cosines, show that the three given points are vertices of a right triangle. Find also, for each triangle, the direction numbers of a line perpendicular to the sides of the triangle.

9. (5, 6, 4), (3, 2, 6), (2, 3, − 5).
10. (7, 3, 5), (1, 4, 1), (4, 1, 9).

11. Show that the points (3, 7, − 5), (2, 2, 4), (− 2, 4, − 3), and (7, 5, 2) are vertices of a rectangle.

12. Show that the points $(7, -5, 4)$, $(3, -1, 2)$, $(5, 3, 6)$, and $(9, -1, 8)$ are vertices of a square and find its area.

13. Show that the points $(2, 2, 4)$, $(3, 4, 5)$, $(5, 1, 2)$, and $(6, 3, 3)$ are the vertices of a parallelogram and find its acute angle.

14. Show that the points $(3, -1, 4)$, $(9, -4, 2)$, $(1, 5, 1)$, and $(7, 2, -1)$ are the vertices of a parallelogram and find the lengths of its diagonals.

15. Show that the three pairs of opposite edges of the tetrahedron $(2, 3, -1)$, $(3, 2, 1)$, $(5, 3, 2)$, and $(-2, 4, 6)$ are respectively perpendicular to each other.

302. Cylindrical Coördinates. In this article and the following one, we shall describe two systems of coördinates in space, each of which bears some resemblance to polar coördinates in the plane. Both of these systems are useful in the applications of analytic geometry.

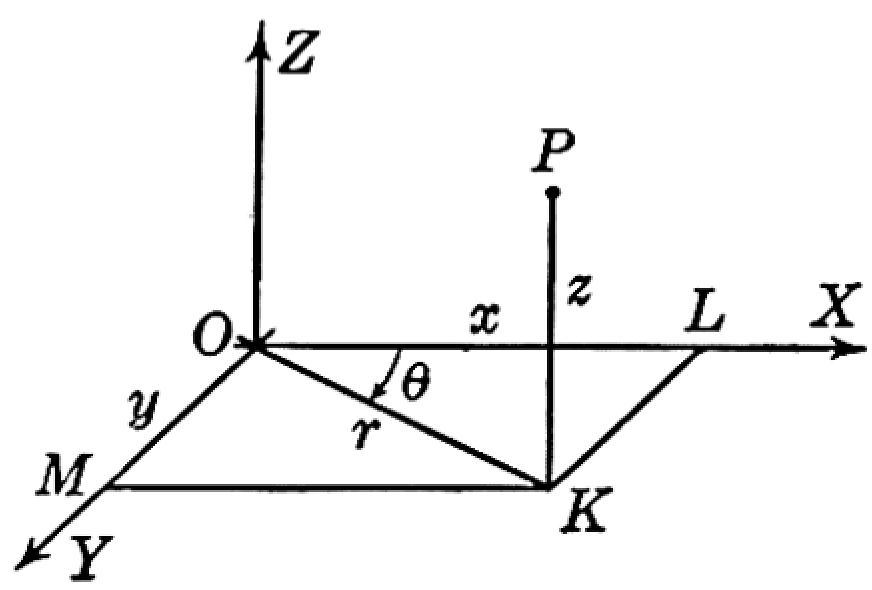


FIG. 186

Let P be any point in space with rectangular coördinates (x, y, z) and let $K(x, y, 0)$ be the foot of the perpendicular from P on the xy-plane. Let (r, θ) be the polar coördinates in the xy-plane of the point K when O is taken as the origin and OX is the initial line. Then the three numbers (r, θ, z) are called the **cylindrical coördinates** of P.

From Art. 176, we have at once for the values of x, y, and z in terms of the cylindrical coördinates

$$x = r \cos \theta, \qquad y = r \sin \theta, \qquad z = z. \tag{16}$$

Similarly, for the values of r, θ, and z in terms of the rectangular coördinates of P, we have

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1} \frac{y}{x}, \qquad z = z, \tag{17}$$

wherein the quadrant in which the angle θ lies is to be determined by plotting the given point on the figure, as in Art. 176.

303. Spherical Coördinates. If the distance ρ of a point P from the origin is known, then P lies on a sphere with center at the origin and radius ρ. We have learned from the study of geography that the position of a point on the surface of a sphere can be determined by two angles (its longitude and latitude). The spherical coördinates of a point consist, accordingly, of a distance and two angles, which we shall define in the following way.

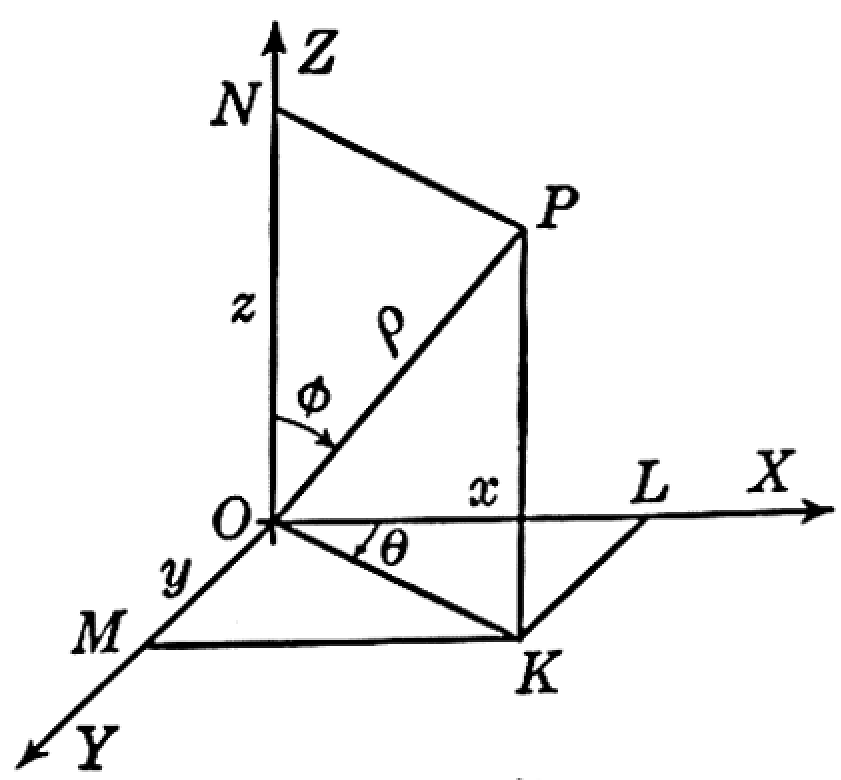


FIG. 187

Let $P(x, y, z)$ be any point in space and let $K(x, y, 0)$ be the foot of the perpendicular from P on the xy-plane. Draw OP, OK, and KP. Let

$$OP = \rho \quad \text{angle } XOK = \theta \quad \text{and} \quad \text{angle } ZOP = \phi.$$

Then (ρ, θ, ϕ) are the **spherical coördinates** of P. We call ρ the **radius vector,** θ, the **longitude,** and ϕ, the **co-latitude,** of P.

To find the values of (x, y, z) in terms of (ρ, θ, ϕ), we note that angle $KOP = 90° - \phi$, so that

$$OK = \rho \cos KOP = \rho \sin \phi.$$

From the right triangles OLK, OKP, and ONP, we now have

$$x = \rho \sin \phi \cos \theta, \qquad y = \rho \sin \phi \sin \theta, \qquad z = \rho \cos \phi, \tag{18}$$

as the equations expressing x, y, and z in terms of ρ, θ, and ϕ.

If we solve these equations for ρ, θ, and ϕ, we obtain

$$\rho = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \tan^{-1} \frac{y}{x}, \quad \phi = \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}} \tag{19}$$

as the equations expressing ρ, θ, and ϕ in terms of x, y, and z.

Exercises

Plot the point and find its rectangular coördinates, given that its cylindrical coördinates are:

1. $(6, 60°, 2)$. **2.** $(4, 150°, -2)$. **3.** $(8, \pi/6, -3)$. **4.** $(2, 3\pi/4, 7)$.

Plot the point and find its cylindrical coördinates, given that its rectangular coördinates are:

5. $(-3, 3, 5)$. **6.** $(2, 2\sqrt{3}, -1)$. **7.** $(5, 0, 2)$. **8.** $(3, -\sqrt{3}, -4)$.

Plot the point and find its rectangular coördinates, given that its spherical coördinates are:

9. $(8, 30°, 45°)$. **10.** $(6, 90°, 60°)$.
11. $(4, -\pi/4, \pi/6)$. **12.** $(12, \pi/3, 2\pi/3)$.

Plot the point and find its spherical coördinates, given that its rectangular coördinates are:

13. $(0, 5, 0)$. **14.** $(0, 1, -\sqrt{3})$. **15.** $(2, -2, -1)$. **16.** $(1, 1, 1)$.

Sketch the surfaces defined by the following equations in cylindrical coördinates and find their equations in rectangular coördinates.

17. $r = 6$. **18.** $\theta = 120°$. **19.** $r = 3 \sin \theta$. **20.** $r = 2z$.

Sketch the surfaces defined by the following equations in spherical coördinates and find their equations in rectangular coördinates.

21. $\rho = 3$. **22.** $\phi = 45°$. **23.** $\rho \sin \theta \sin \phi = 2$. **24.** $\rho = 2 \cos \phi$.

Write each of the following equations in cylindrical and in spherical coördinates.

25. $x^2 + y^2 + z^2 = 25.$

26. $x^2 + y^2 = 9.$

27. $9(x^2 + y^2) + 25z^2 = 225.$

28. $x = 7.$

29. Find the equations expressing the cylindrical coördinates of a point in terms of the spherical coördinates.

30. Find the direction cosines of the line from the origin to the point whose spherical coördinates are (ρ, θ, ϕ).

Chapter 37

Planes and Lines in Space

304. Normal Form of the Equation of a Plane. Let ABC (Fig. 188) be the given plane. It is required to find an equation which is satisfied by the coördinates of those points (and no others) that lie in the plane.

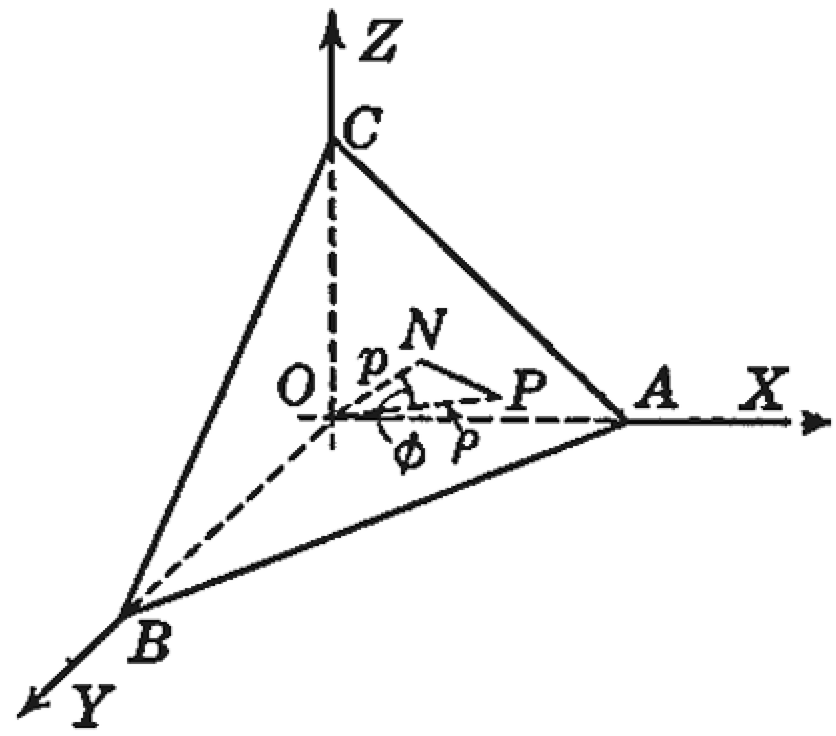


FIG. 188

Let N be the foot of the perpendicular from the origin to the plane. Draw the directed line segment ON, denote its length by p and its direction cosines by $\cos \alpha$, $\cos \beta$, and $\cos \gamma$. Let $P(x, y, z)$ be any point in the plane. Draw the directed segment OP, denote its length by ρ and its direction cosines by $\cos \alpha'$, $\cos \beta'$, and $\cos \gamma'$. Denote also the angle NOP by ϕ.

Since ON is perpendicular to the plane and N and P lie in the plane, the angle ONP is a right angle. Hence $\cos \phi = p/\rho$ or

$$p = \rho \cos \phi. \tag{1}$$

Replace $\cos \phi$ by its value from equation 12, Art. 301. We have

$$p = \rho \cos \alpha' \cos \alpha + \rho \cos \beta' \cos \beta + \rho \cos \gamma' \cos \gamma. \tag{2}$$

From equations (6), Art. 299, we have

$$\rho \cos \alpha' = x, \quad \rho \cos \beta' = y, \quad \rho \cos \gamma' = z.$$

On making these substitutions in (2), we have

$$\boldsymbol{p = x \cos \alpha + y \cos \beta + z \cos \gamma.} \tag{3}$$

Equation (3) is satisfied by the coördinates of every point P in the plane. It is not satisfied by the coördinates of any point P not lying in the plane. For, if P does not lie in the plane, the foot of the perpendicular from P to the line ON is a point N' distinct from N so that $ON' = p' \neq p$ and equation (3) is not satisfied.

Equation (3) is the **normal form** of the equation of a plane. In this equation, *the coefficients of x, y, and z are the direction cosines of the normal to the plane and p is the directed distance from the origin to the plane.*

305. General Form of the Equation of a Plane. The normal form (3) of the equation of a plane is of the first degree in x, y, and z with real

coefficients. We shall now show, conversely, that *the locus of any equation of the first degree in x, y, and z with real coefficients*

$$Ax + By + Cz + D = 0, \tag{4}$$

wherein A, B, and C are not all zero, is a plane.

The locus of equation (4) is not changed if we divide each of its terms by the non-zero constant $\pm \sqrt{A^2 + B^2 + C^2}$. We thus obtain

$$\frac{A}{\pm \sqrt{A^2 + B^2 + C^2}}x + \frac{B}{\pm \sqrt{A^2 + B^2 + C^2}}y + \frac{C}{\pm \sqrt{A^2 + B^2 + C^2}}z + \frac{D}{\pm \sqrt{A^2 + B^2 + C^2}} = 0. \tag{5}$$

By Art. 299, the coefficients of x, y, and z in (5) are the direction cosines of a line, so that we may put

$$\left.\begin{aligned} \frac{A}{\pm \sqrt{A^2 + B^2 + C^2}} &= \cos \alpha \\ \frac{B}{\pm \sqrt{A^2 + B^2 + C^2}} &= \cos \beta \\ \frac{C}{\pm \sqrt{A^2 + B^2 + C^2}} &= \cos \gamma \end{aligned}\right. \tag{6}$$

If we substitute these values of the coefficients of x, y, and z in (5), and compare the result with equation (3), we find that equation (5), and hence equation (4) which has the same locus, is the equation of a plane. It follows, moreover, from the comparison with (3), that this plane is perpendicular to the line whose direction cosines are given by (6) and that it lies at a distance from the origin equal to

$$p = \frac{-D}{\pm \sqrt{A^2 + B^2 + C^2}}. \tag{7}$$

Equation (4) is called the **general form** of the equation of a plane. To reduce it to the normal form (5), we divide each of its terms by $\pm \sqrt{A^2 + B^2 + C^2}$. In order to fix the sign of the radical by which we divide each term of (4) to get (5), *we shall take the sign to agree with that of C if $C \neq 0$, to agree with that of B if $C = 0$, and to agree with that of A if $B = 0$* and *$C = 0$.*

Of frequent importance in the applications is the following theorem which follows at once from the foregoing discussion: *the coefficients of x, y, and z in the equation of a plane are the direction numbers of a line perpendicular to the plane.*

306. The Traces of a Plane. The lines in which a given plane intersects the coördinate planes are called its **traces** on those planes. A plane

is usually represented on the figure by means of its traces on the coördinate planes, as in Figure 188. If, however, it passes through (or very near to) the origin, or if it is parallel to one of the coördinate axes, it should be represented, instead, by a parallelogram having two of its sides extending along two of its traces on the coördinate planes.

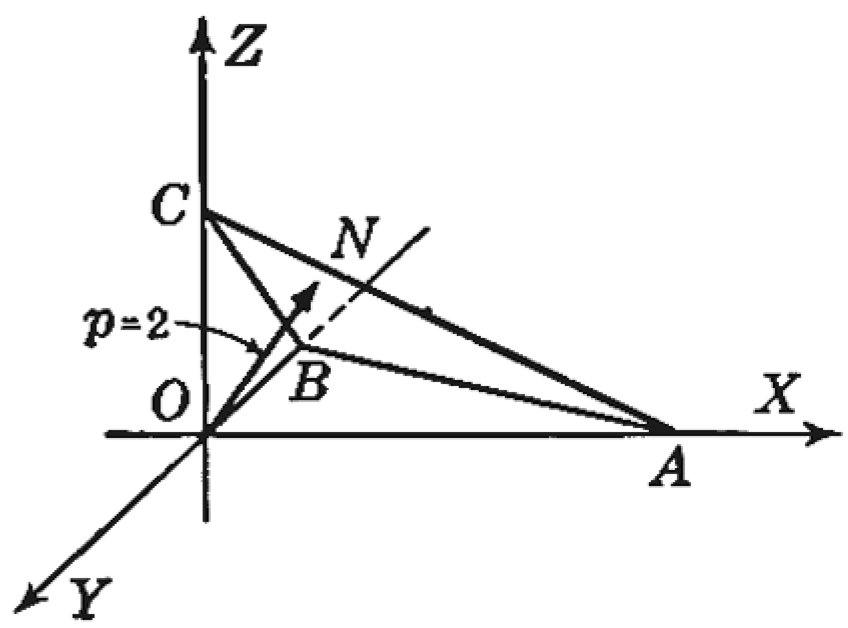


FIG. 189

EXAMPLE. Reduce the equation of the plane $x - 2y + 2z - 6 = 0$ to the normal form. Find the direction cosines of the normal and the distance of the plane from the origin. Determine its traces on the coördinate planes.

To reduce the equation of the plane to the normal form, we divide through by $\sqrt{1^2 + (-2)^2 + 2^2} = 3$. The result is $\frac{1}{3}x - \frac{2}{3}y + \frac{2}{3}z - 2 = 0$.

The direction cosines of the normal to the plane are found, by comparing this equation with (3), to be $\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}$ and the distance of the plane from the origin is similarly found to be 2.

The equations of the traces of the given plane on the coördinate planes are:

On the xy-plane	$x - 2y - 6 = 0,$	$z = 0;$
On the xz-plane	$x + 2z - 6 = 0,$	$y = 0;$
On the yz-plane	$-2y + 2z - 6 = 0,$	$x = 0.$

Exercises

Find the normal form of the equation of a plane, given:

1. $\alpha = 120°, \beta = 135°, \gamma = 60°, p = 3.$
2. $\alpha = 135°, \beta = 120°, \gamma = 60°, p = 5.$
3. $\alpha = 90°, \beta = 135°, \gamma = 45°, p = -7.$
4. $\alpha = 120°, \beta = 30°, \gamma = 90°, p = -4.$

Find the equation of a plane, given that the direction numbers of its normal and its distance from the origin are:

5. $2, -2, 1; p = 7.$
6. $8, 4, -1; p = 3.$
7. $2, -10, -11; p = 4.$
8. $3, -6, 2; p = 5.$
9. $5, 3, 7; p = -6.$
10. $-2, -7, 3; p = -4.$

Find the equation of a plane, given that the coördinates of the foot of the perpendicular from the origin to the plane are:

11. $(2, -3, 6)$.
12. $(-4, 4, 2)$.
13. $(-7, 6, 6)$.
14. $(3, 1, 4)$.

HINT. The direction numbers of the line joining the origin to the point (a, b, c) are a, b, and c.

Write the equations of the following planes in the normal form. Find the direction cosines of the normal and the distance from the origin to the plane.

Write two equations which are satisfied by the coördinates of the points on each trace of the plane on the coördinate planes.

15. $4x - 7y + 4z - 18 = 0$.
16. $x - 2y - 2z + 12 = 0$.
17. $3x + 12y - 4z - 39 = 0$.
18. $6x - 2y + 9z - 30 = 0$.
19. $5x + 12y - 26 = 0$.
20. $x + 7 = 0$.

21. Find two values of k such that the distance of the plane $8x - 9y + 12z - k = 0$ from the origin is numerically equal to 4.

22. Show analytically that the locus of a point whose distances from $(3, 7, -2)$ and $(5, -2, 4)$ are equal is a plane. Show also that this plane is perpendicular to the line joining the given points.

23. Write the equation of the plane through $(-2, 3, 1)$ perpendicular to a line whose direction numbers are (*a*) $9, -2, 6$, (*b*) $3, 1, 4$.

24. Write the equation of the plane through $(-5, -2, 7)$ perpendicular to the line through $(8, 1, 5)$ and $(3, 7, 2)$.

25. Find the coördinates of the point of intersection of the planes $2x - 3y - z = 4$, $x + 2y + 6z = 1$, $3x - 2y + 3z = 3$.

307. Distance from a Plane to a Point. Let $P_1(x_1, y_1, z_1)$ be the given point and let the equation of the given plane be

$$Ax + By + Cz + D = 0. \tag{8}$$

The plane

$$Ax + By + Cz - (Ax_1 + By_1 + Cz_1) = 0$$

passes through P_1 since the coördinates of P_1 satisfy the equation. It is parallel to the plane (8) because the direction numbers of the normals to the two planes are equal.

The directed distances from the origin to the two planes are, by equation (7),

$$p_1 = \frac{-D}{\pm\sqrt{A^2 + B^2 + C^2}} \quad \text{and} \quad p_2 = \frac{Ax_1 + By_1 + Cz_1}{\pm\sqrt{A^2 + B^2 + C^2}}.$$

The difference

$$d = p_2 - p_1$$

between these distances is equal to the required distance from the plane (8) to the given point P_1, that is,

$$d = \frac{Ax_1 + By_1 + Cz_1 + D}{\pm\sqrt{A^2 + B^2 + C^2}}, \tag{9}$$

wherein the sign in the denominator is fixed by the rule given in Art. 305.

The value of d, as found from this equation, is a directed distance. It is positive or negative according as the segment of the perpendicular, measured from the plane to the point P_1, is in the positive or the negative direction along the normal.

308. Angle between Two Planes. It is proved in elementary geometry that the magnitudes of the four dihedral angles formed by two inter-

secting planes are numerically equal, respectively, to the four corresponding angles formed by the two lines that can be drawn through any point in space perpendicular to the given planes (Fig. 190).

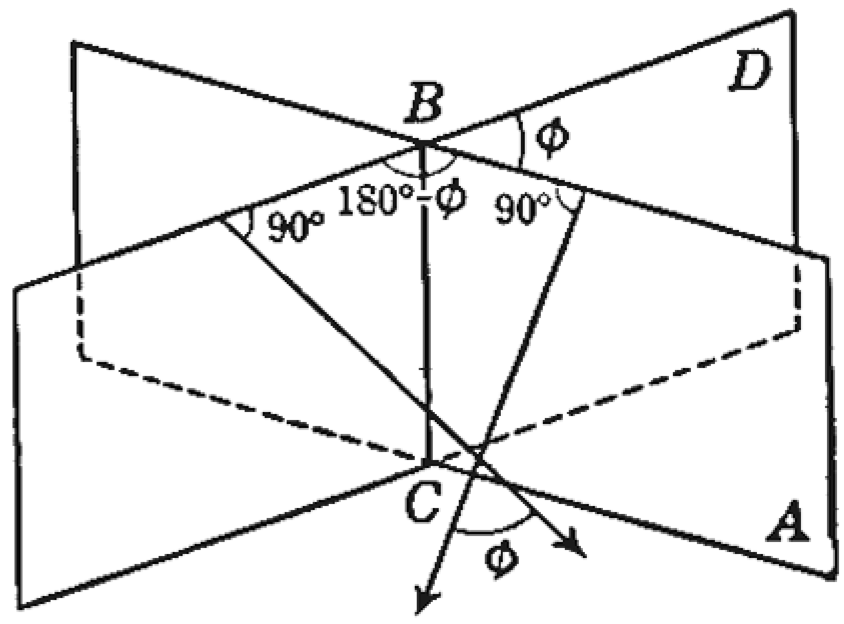


FIG. 190

If positive directions are assigned to the two perpendiculars, we shall choose, as **the angle between the planes,** a dihedral angle formed by them that is equal in magnitude to *the angle between the positive directions of these perpendiculars.*

Thus, if the equations of the planes are given in the normal form

$$x \cos \alpha_1 + y \cos \beta_1 + z \cos \gamma_1 - p_1 = 0,$$

and
$$x \cos \alpha_2 + y \cos \beta_2 + z \cos \gamma_2 - p_2 = 0,$$

then, by Art. 304, the direction cosines of the normals to the planes are $\cos \alpha_1$, $\cos \beta_1$, $\cos \gamma_1$, and $\cos \alpha_2$, $\cos \beta_2$, $\cos \gamma_2$, respectively, and the angle between the planes, being equal in magnitude to the angle between the positive directions of the normals, is found, from equation (12), Art. 301, to be

$$\cos \phi = \cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2, \quad (10)$$

wherein ϕ is the angle between the planes and α_1, β_1, γ_1 and α_2, β_2, γ_2 are the direction angles of the normals to the planes.

Similarly, if the equations of the planes are given in the general form

$$A_1x + B_1y + C_1z + D_1 = 0$$

and
$$A_2x + B_2y + C_2z + D_2 = 0,$$

then A_1, B_1, C_1 and A_2, B_2, C_2 are the direction numbers of the normals to the planes and the angle between the planes is found, from equation (14), Art. 301, to be

$$\cos \phi = \pm \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}. \quad (11)$$

In particular, the condition that the planes are perpendicular is that $\cos \phi = 0$, so that

$$A_1A_2 + B_1B_2 + C_1C_2 = 0. \quad (12)$$

If the given planes are parallel, they are both perpendicular to the same line, and we have

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}. \quad (13)$$

Exercises

Find the angle between the two planes.

1. $6x - 2y + 3z - 12 = 0$,
$4x + 7y + 4z + 9 = 0$.

2. $5x + 4y - 20z - 40 = 0$,
$2x + 2y - z - 7 = 0$.

3. $x - 2y + 3z + 6 = 0$,
$2x + y - z + 9 = 0$.

4. $3x + 4y - 9 = 0$,
$5x - 12y + 8 = 0$.

5. Write the equations of the planes through $(3, -2, -1)$ parallel to the planes in Ex. 1.

6. Find the distance from each of the planes in Ex. 1 to the point $(2, 3, -5)$.

7. Which of the points $(6, -3, 2)$ and $(-1, 3, 3)$ lies on the same side of the plane $2x - y + 3z - 8 = 0$ as the point $(4, -1, -3)$.

Find the distance between the parallel planes:

8. $4x - 2y + 4z + 3 = 0$,
$2x - y + 2z + 5 = 0$.

9. $8x - 4y - z - 8 = 0$,
$8x - 4y - z + 10 = 0$.

10. Show that the equation $6x - 9y - 2z + k = 0$, in which k is a parameter, defines a family of parallel planes. Find the direction cosines of a normal to this family of planes.

11. Find two planes parallel to $6x + 2y - 3z + 5 = 0$ whose distances from $(2, -5, 7)$ are numerically equal to 3.

12. Find k, given that the plane $(k + 3)x + (1 - k)y + 7z - 6 = 0$ is perpendicular to the plane $x - 2y + 2z + 2 = 0$.

13. Show that the condition that the plane $Ax + By + Cz + D = 0$ is perpendicular to the plane $z = 0$ is $C = 0$. Find the condition that it is perpendicular to (*a*) $y = 0$, (*b*) $x = 0$.

309. Planes Satisfying Three Conditions. The position of a plane is usually fixed by assigning three conditions that it must satisfy. For example, we may require it to pass through three given points, or to pass through a given point and be perpendicular to each of two given planes, and so forth.

The method of determining the equation of a plane that satisfies three such conditions is illustrated by the following examples.

Example 1. Find the equation of the plane that passes through the points $(3, 2, -3)$, $(-1, 3, 5)$, and $(5, 4, -2)$.

The condition that any one of these points lies in the plane

$$Ax + By + Cz + D = 0$$

is that its coördinates satisfy the equation of the plane. If we substitute the coördinates of the given points successively in the equation of the plane, we obtain the three equations

$$\begin{aligned} 3A + 2B - 3C + D &= 0 \\ -A + 3B + 5C + D &= 0 \\ 5A + 4B - 2C + D &= 0. \end{aligned}$$

If we solve these three equations for B, C, and D in terms of A, we obtain $B = -\frac{4}{3}A$, $C = \frac{2}{3}A$, $D = \frac{5}{3}A$.

We may now assign to A *any value we please except zero*. To avoid fractions, we put $A = 3$; then $B = -4$, $C = 2$, and $D = 5$. The required equation of the plane is, accordingly,

$$3x - 4y + 2z + 5 = 0.$$

As a check, the student should verify that the coördinates of each of the given points satisfy this equation.

EXAMPLE 2. Find the equation of the plane that passes through the points (6, 1, 2) and (3, 4, 4) and is perpendicular to the plane $x + 3y + 2z - 7 = 0$.

The conditions that the plane $Ax + By + Cz + D = 0$ passes through the given points are found, by substituting the coördinates of the points in the equation of the plane, to be

$$\begin{aligned} 6A + B + 2C + D &= 0 \\ 3A + 4B + 4C + D &= 0. \end{aligned}$$

The condition that it is perpendicular to the given plane is

$$A + 3B + 2C = 0.$$

We cannot solve these equations for B, C, and D in terms of A since the resulting equations are inconsistent. We can, however, solve for A, C, and D in terms of B. The results are $A = 0$, $C = -\frac{3}{2}B$, $D = 2B$.

If we put $B = 2$ we have $C = -3$ and $D = 4$. The required equation is, accordingly, $2y - 3z + 4 = 0$.

310. Intercept Equation of a Plane. The directed distances from the origin to the points of intersection of a plane with the coördinate axes are the **intercepts** of the plane on those axes.

Let a, b, and c (which, we shall here suppose, are all different from zero) be the intercepts of the plane

$$Ax + By + Cz + D = 0 \qquad (14)$$

on the x-, y-, and z-axis, respectively.

Since the plane (14) passes through the points $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$, we have

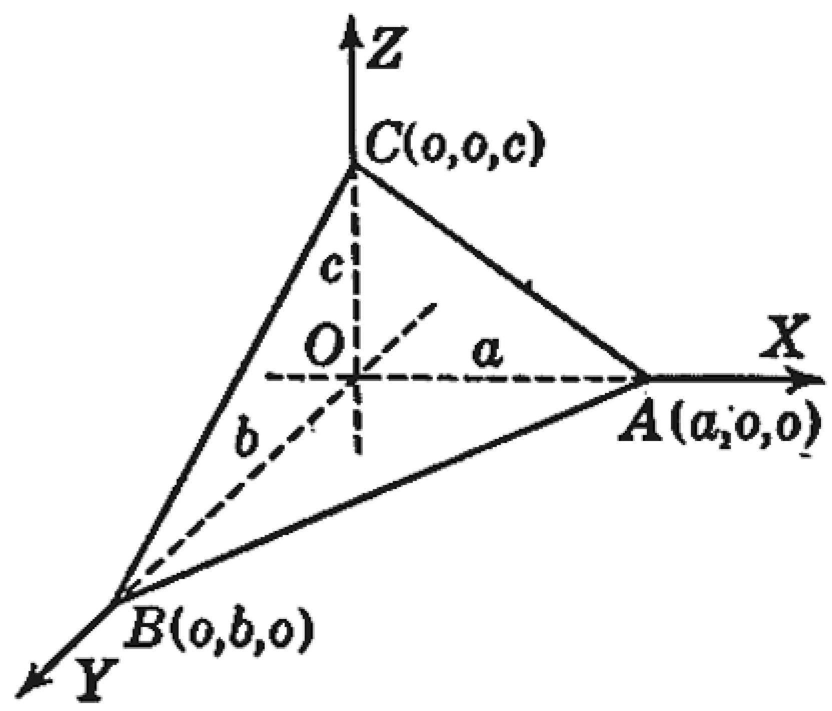


FIG. 191

$$Aa + D = 0, \quad Bb + D = 0, \quad \text{and} \quad Cc + D = 0.$$

Put $D = -1$, solve for A, B, and C, and substitute in (14). We have

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

This is the **intercept form** of the equation of a plane.

Exercises

Write the equations of the given planes in the intercept form.

1. $x + 2y + 5z = 10$.
2. $3x - 4y + 2z = 12$.
3. $6x - 2y - 3z + 18 = 0$.
4. $5x + 3y - 8z + 9 = 0$.

5. Write the equation of the plane that passes through the points $(-2, 4, -1)$ and $(3, 5, 4)$ and has its y-intercept equal to 3.

6. Write the equation of the plane that passes through $(3, 2, 1)$ and $(8, 2, 3)$ and has its y- and z-intercepts equal.

Find the equation of the plane that passes through the given points.

7. $(3, 0, 0)$, $(0, -2, 0)$, $(0, 0, 6)$.
8. $(0, 0, 0)$, $(1, 4, 2)$, $(-3, 2, 4)$.

9. $(2, -4, -1)$, $(3, -8, -2)$, $(4, 4, 2)$.
10. $(4, 2, 3)$, $(-2, 4, 1)$, $(1, -2, -3)$.

Find the equation of the plane that passes through the given points and is perpendicular to the given plane.

11. $(5, 1, 3)$, $(1, 7, -2)$, $2x - y + 2z + 7 = 0$.
12. $(-1, 2, 4)$, $(5, -1, 3)$, $2x + y + 3z - 6 = 0$.
13. $(2, 1, 5)$, $(4, -2, 3)$, $4x + y + 3z - 4 = 0$.

Find the equation of the plane that passes through the given point and is perpendicular to each of the given planes.

14. $(1, -5, -2)$, $3x + 2y + 5z - 8 = 0$, $2x - y + 3z = 0$.
15. $(3, -2, 4)$, $7x - 3y + z - 5 = 0$, $4x - y - z + 9 = 0$.
16. $(5, 4, 2)$, $3x + 4y + z + 1 = 0$, $x + 3y + z - 7 = 0$.

17. Find the equation of the plane that passes through $(2, 5, 9)$, is perpendicular to the plane $x + 4y + 6z - 3 = 0$, and has its x-intercept equal to 1.

18. Find the equations of two planes through $(2, -1, 3)$ and $(7, 4, -2)$ each of which makes an angle of 60° with the plane $x - 3y - 2z + 1 = 0$.

The Line in Space

311. Surfaces and Curves. We have seen, in Art. 305, that a single linear equation in x, y, and z, with real coefficients,

$$Ax + By + Cz + D = 0,$$

defines a plane. When we wish to fix the position of a *line* in space, we shall take simultaneously the two equations

$$A_1x + B_1y + C_1z + D_1 = 0 \qquad A_2x + B_2y + C_2z + D_2 = 0$$

of two planes that have this line as their line of intersection. The condition that a point lies on this line is, then, that its coördinates satisfy *both* of these equations.

The discussion in the preceding paragraph is of importance in that it constitutes an elementary illustration of a very general and fundamental principle in the analytic geometry of space. When we wish to study analytically a *surface* in space, we shall think of its position as fixed by a single equation,

$$f(x, y, z) = 0,$$

just as, when we studied the plane, we fixed the position of our plane in space by a single linear equation. When, on the other hand, we wish to fix the position of a *curve* in space, we shall use two equations

$$f(x, y, z) = 0, \qquad F(x, y, z) = 0,$$

such that each of these equations, taken by itself, is the equation of a surface that contains the curve under consideration. The points that lie on this curve will then possess the property that their coördinates will satisfy the equations of both of these surfaces.

In the articles that follow, we shall deal with the two equations of a line. We shall write these equations in several forms depending on the information that is given us about the line or on the uses to which we intend to put these equations.

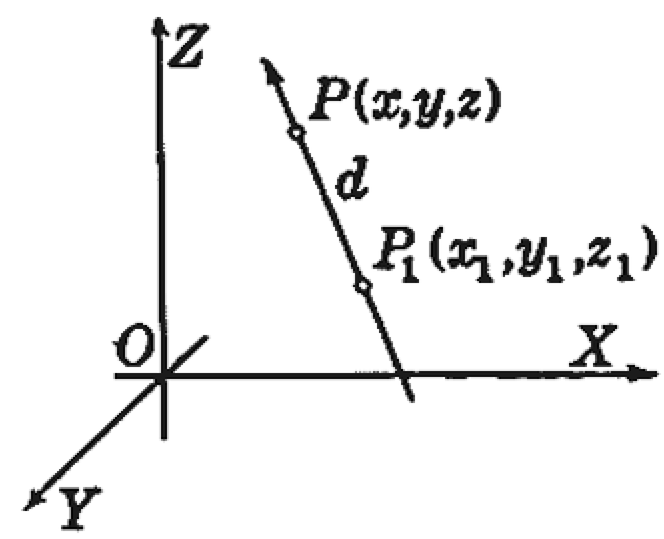


FIG. 192

312. Line through a Given Point Having a Given Direction. The Symmetric Form. Let $P_1(x_1, y_1, z_1)$ be a given point on the given line l and let $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ be the direction cosines of l. Let $P(x, y, z)$ be any point on l and let d be the length of the directed line segment P_1P (Fig. 192). From equations (4), Art. 299, we have

$$x - x_1 = d\cos\alpha, \quad y - y_1 = d\cos\beta, \quad z - z_1 = d\cos\gamma. \qquad (15)$$

If l is not perpendicular to any one of the coördinate axes, so that none of its direction cosines is zero, we can solve each of these equations for d. If we equate the three values of d so obtained, we have, as the equations of the line

$$\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\cos\beta} = \frac{z - z_1}{\cos\gamma}. \qquad (16)$$

These equations constitute the **symmetric form** of the equations of a line in space.

If, in equations (16), we multiply all the denominators by any nonzero constant we please, the resulting equations will still be equal. The new denominators are direction numbers of l. If we denote them by a, b, and c, we may write the resulting equations in the form

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}. \qquad (17)$$

These equations are frequently more convenient to use than equations (16).

If the given line l is perpendicular to any one of the coördinate axes, equations (16) fail. If, for example, l is perpendicular to the x-axis but not to the y- or z-axis, $\cos \alpha = 0$ and equations (15) may be reduced to

$$x - x_1 = 0, \qquad \frac{y - y_1}{\cos \beta} = \frac{z - z_1}{\cos \gamma}, \tag{18}$$

and if l is perpendicular to both the x- and the y-axis, then $\cos \alpha = 0$ and $\cos \beta = 0$ and we have, from (15), as the equations of l,

$$x - x_1 = 0, \qquad y - y_1 = 0. \tag{19}$$

313. The Two-Point Form. Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be any two fixed points on the line l. Since, by equations (4), Art. 299, the numbers $x_2 - x_1$, $y_2 - y_1$, and $z_2 - z_1$ are proportional to the direction cosines of l, we may use them as the direction numbers a, b, and c in equations (17). The resulting equations are

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}. \tag{20}$$

These equations constitute the **two-point form** of the equations of the line.

314. The Parametric Form. If, in equations (17), we equate each of the equal fractions to k and solve for x, y, and z, we obtain

$$x = x_1 + ak \qquad y = y_1 + bk \qquad z = z_1 + ck. \tag{21}$$

These are **parametric equations** of the line in terms of the parameter k. The point determined by assigning to k any value we please lies on the line.

315. The General Form. As the equations of a line, we may take simultaneously the equations,

$$A_1x + B_1y + C_1z + D_1 = 0$$

and

$$A_2x + B_2y + C_2z + D_2 = 0, \tag{22}$$

of any two planes whatever that have this line as their line of intersection. Any point whose coördinates satisfy both of these equations lies in both planes and thus lies on the given line.

To reduce the general form (22) of the equations of a line to the two-point form and to the symmetric form, we may proceed as in the following example.

EXAMPLE. Write the equations of the line of intersection of the planes $3x + 3y - 4z + 7 = 0$ and $x + 6y + 2z - 6 = 0$ in the two-point form and in the symmetric form and find its direction cosines.

To fix a point on the line, we may assume for one of its coördinates any value we please and determine its other two coördinates by means of the given

equations. For example, if we put $z = 1$, the equations to determine x and y are

$$3x + 3y + 3 = 0, \quad x + 6y - 4 = 0.$$

On solving these equations, we find that $x = -2$, $y = 1$. Hence, $(-2, 1, 1)$ is a point that lies on the line. Similarly, by putting $z = 4$, we find that $(4, -1, 4)$ is a second point on the line.

Since we now know the coördinates of two points on the line, we may substitute these in equation (20) and obtain

$$\frac{x+2}{4-(-2)} = \frac{y-1}{-1-1} = \frac{z-1}{4-1}$$

as the two-point form of the equation of this line.

It was pointed out in Art. 313 that the values 6, -2, and 3 of the denominators in these equations are the direction numbers of the line. We may, accordingly, write at once equations (17) for this line. The results are

$$\frac{x+2}{6} = \frac{y-1}{-2} = \frac{z-1}{3}.$$

To reduce these equations to the symmetric form (16), we first determine the direction cosines of the line by dividing each of the direction numbers by $\pm \sqrt{6^2 + (-2)^2 + 3^2} = \pm 7$, the sign being chosen according as one direction on the line, or the other, is taken as the positive direction. The symmetric equations of the line are, accordingly,

$$\frac{x+2}{\pm \frac{6}{7}} = \frac{y-1}{\mp \frac{2}{7}} = \frac{z-1}{\pm \frac{3}{7}}.$$

From the denominators of these expressions, it is seen at once that the direction cosines of the line are $\pm \frac{6}{7}$, $\mp \frac{2}{7}$, and $\pm \frac{3}{7}$.

316. Family of Planes through a Line. Projecting Planes. All of the planes of the family

$$A_1x + B_1y + C_1z + D_1 + k(A_2x + B_2y + C_2z + D_2) = 0, \qquad (23)$$

wherein k is the parameter, contain the line defined by equations (22). For, if $P_1(x_1, y_1, z_1)$ is any point on this line, its coördinates satisfy both equations (22) and thus, when substituted in (23), reduce this equation to $0 + k(0) = 0$, which is true for all values of k.

By assigning a suitable value to k in equation (23), we can determine a plane that passes through the line (22) and satisfies one additional condition; for example, we can make it pass through a given point not on the line or we can make it be perpendicular to a given plane.

The planes through a line that are perpendicular to the xy-, yz-, and zx-planes, respectively, are called the **projecting planes** of the line on these coördinate planes. The equations of the projecting planes of a given line may be found as in the following Example 1.

EXAMPLE 1. Find the projecting planes of the line $3x + y + z - 11 = 0$, $x + 3y - z - 9 = 0$ on the coördinate planes.

The equation of the family of planes through this line is, by (23),

$$3x + y + z - 11 + k(x + 3y - z - 9) = 0. \tag{24}$$

Collect the coefficients of x, y, and z in this equation:

$$(3 + k)x + (1 + 3k)y + (1 - k)z - (11 + 9k) = 0. \tag{25}$$

The condition that this plane is perpendicular to the xy-plane (that is, to the plane $z = 0$) is, by equation (12), $1 - k = 0$, or $k = 1$. If we put $k = 1$ in equation (25) and divide by 4, we have $x + y - 5 = 0$ which is the required equation of the projection plane of the given line on the xy-plane.

The projecting planes of this line on the xz-plane and the yz-plane are found, in a similar way, by equating to zero successively the coefficients of y and of x in equation (25), solving for k, and substituting in equation (25), to be $2x + z - 6 = 0$ and $2y - z - 4 = 0$, respectively.

EXAMPLE 2. Find the plane through the line in Ex. 1 and the point $(3, -1, -5)$.

Since the required plane must pass through the given point, the value of k must be chosen so that the coördinates of this point satisfy equation (24). On substituting the coördinates $(3, -1, -5)$ in (24), we find $k = -2$. Substitute this value of k in (25) and simplify. The result is $x - 5y + 3z + 7 = 0$ which is the equation of the required plane.

EXAMPLE 3. Find the points in which the line in Ex. 1 pierces the coördinate planes.

The z-coördinate of the point in which the line intersects the xy-plane is zero. Put $z = 0$ in the equations of the line and solve for x and y. The results are $x = 3$, $y = 2$. Hence, the required point is $(3, 2, 0)$.

Similarly, by putting $y = 0$ we locate $(5, 0, -4)$, and by putting $x = 0$ we locate $(0, 5, 6)$, as the intersection of the line with the xz-plane and with the yz-plane, respectively.

Exercises

Write the equations of the following lines and find their direction cosines, given that γ is acute.

1. Through $(2, 1, -5)$; direction numbers $6, -2, 9$.

2. Through $(-3, 4, -9)$; direction numbers $9, 11, 1$.

3. Through $(4, -1, -7)$ and $(10, 2, -1)$.

4. Through $(1, -5, 3)$ and $(4, 8, -1)$.

5. Through $(2, 5, -8)$, perpendicular to the plane $3x + 4y + 12z - 6 = 0$.

6. Through $(-4, -2, 8)$, parallel to the line through $(-1, 2, 4)$ and $(4, -8, -6)$.

7. Through $(-5, 1, -3)$, perpendicular to the lines whose direction numbers are 4, 10, 3 and 4, 2, -5.

8. Through $(1, 4, -2)$, parallel to the line of intersection of the planes $6x + 2y + 2z + 3 = 0$ and $3x - 5y - 2z - 1 = 0$.

Write the equations of the projecting planes of each of the following lines on the coördinate planes. Find the points in which each line intersects each of the coördinate planes. Find the direction cosines of each line, given that γ is acute.

9. $2x + 2y + z - 8 = 0,$
$x + 6y - 2z - 9 = 0.$

10. $6x - 5y - 2z + 11 = 0,$
$6x - 4y - 3z + 6 = 0.$

11. $3x + 3y - 2z - 3 = 0,$
$5x + 6y - 3z - 9 = 0.$

12. $5x + 2y - 3z + 26 = 0,$
$4x + y - 2z + 16 = 0.$

13. Find the acute angle between the lines in Ex. 9 and 10.

14. Find the equation of the plane through $(7, -3, -2)$ perpendicular to the line in Ex. 11.

15. Find the equation of the plane through the line in Ex. 11 and the point $(-1, 2, 6)$.

16. Find the equation of the plane through the line in Ex. 12 that is perpendicular to the plane $3x - 5y + 2z + 6 = 0$.

17. Show that the points $(1, 3, 5)$, $(5, 1, 9)$, and $(-5, 6, -1)$ lie on a line and find the equations of this line.

18. Show that the line $\dfrac{x+5}{3} = \dfrac{y+4}{-2} = \dfrac{z-2}{5}$ lies in the plane $x - 6y - 3z - 13 = 0$.

HINT. A line lies in a plane if two points on it lie in the plane.

19. If the planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ intersect in a line, show that $B_1C_2 - C_1B_2$, $C_1A_2 - A_1C_2$, and $A_1B_2 - B_1A_2$ are direction numbers of this line.

Chapter 38

Types of Surfaces

317. Cylinders. A surface generated by a line which moves so that it is always parallel to a fixed line and always intersects a fixed curve is called a **cylinder.** Any position of the generating line is an *element* of the cylinder and the fixed curve which all of these elements intersect is the *directrix curve.*

In elementary solid geometry, special attention is given to the circular cylinders; that is, to cylinders that have circles as directrix curves. Although the circular cylinders are included among the surfaces we shall study, most of the cylinders that will be considered in this course are not circular cylinders.

Consider, for example, the surface in space defined by the equation

$$y^2 = 2px. \tag{1}$$

The section of this surface by the xy-plane is the parabola

$$y^2 = 2px, \qquad z = 0. \tag{2}$$

Let $P'(x, y, 0)$ be any point on this parabola. Draw the line through P' parallel to the z-axis and let $P(x, y, z)$ be any point on this line. Then the x- and y-coördinates of P are equal, respectively, to those of P'. By hypothesis, the coördinates of P' satisfy equations (2) and, since the first of these equations does not contain z, it follows that the coördinates of P will satisfy (1); that is, every point P on the line through P' parallel to the z-axis lies on the surface (1).

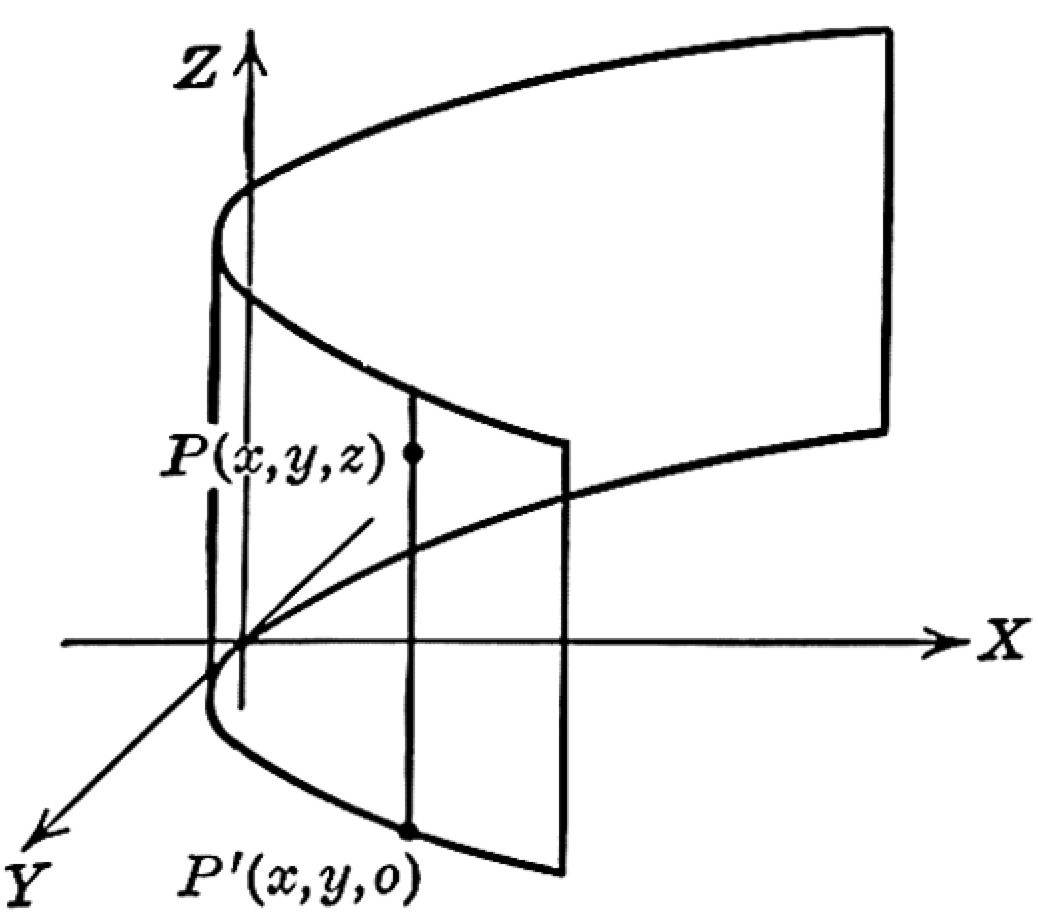


FIG. 193

If we now let P' describe the parabola (2), the line $P'P$ will describe a cylinder having the parabola (2) as directrix curve and having its elements parallel to the z-axis. The coördinates of every point on this cylinder (and no others) satisfy equation (1). Hence this cylinder is the required locus of equation (1). It is called a **parabolic cylinder.**

By extending the reasoning used in the foregoing discussion, we are led to the following theorem: *If the equation of a surface does not contain the variable z, then the surface is a cylinder with elements parallel to the*

z-axis and having the curve of section by the plane $z = 0$ as directrix curve. Similarly, if y, or x, is absent from the equation, then the surface is a cylinder with elements parallel to the y-axis, or the x-axis, respectively.

Exercises

Sketch the cylindrical surfaces defined by the following equations.

1. $x^2 + y^2 = 25$. **2.** $4x^2 + 9y^2 = 36$. **3.** $3y + 5z = 15$.
4. $y^2 = 18z$. **5.** $yz = 12$. **6.** $9x^2 - 4z^2 = 36$.
7. $z = x^3$. **8.** $y = x^3 - 8x$. **9.** $x^2 + 2x - 15 = 0$.
10. $z^2 = x^3$. **11.** $z = \cos x$. **12.** $z = e^x$.

Sketch the surfaces whose equations in cylindrical coördinates (Art. 302) are.

13. $r = a \sin \theta$. **14.** $r^2 \cos 2\theta = a^2$. **15.** $r = a\theta$.
16. $r = a \sin 2\theta$. **17.** $r = a(1 - \sin \theta)$. **18.** $r = e^{a\theta}$.

318. Surfaces of Revolution. The surface generated by revolving a plane curve about a line in its plane is a **surface of revolution.** The line about which this curve revolves is the *axis of revolution* and any position of the revolving curve is a *meridian section.*

Let us find, for example, the equation of the right circular cone generated by revolving the line defined by the equations

$$x = cz, \qquad y = 0, \tag{3}$$

around the z-axis.

Let $P_1(x_1, 0, z_1)$ be any point on the given line and let $N_1(0, 0, z_1)$ be the foot of the perpendicular from P_1 to the z-axis. As the given line revolves around the z-axis, P_1 describes a circle with center at N_1, radius N_1P_1, and lying in a plane perpendicular to the z-axis (Fig. 194).

Let $P(x, y, z)$ be any point on this circle. Since it lies in a plane through P_1 parallel to the xy-plane, we have

$$z = z_1.$$

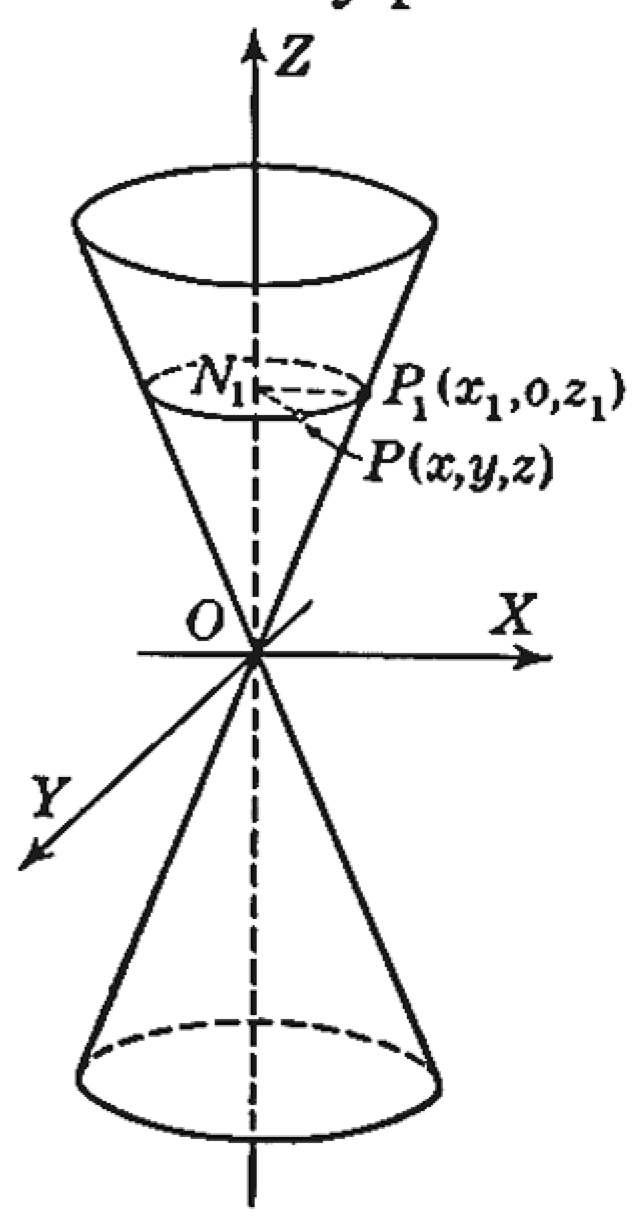


FIG. 194

Since it also lies on a circle with center at N_1 and radius $N_1P = N_1P_1 = x_1$, we have further

$$\sqrt{x^2 + y^2} = x_1.$$

Since P_1 lies on the line (3), it follows that

$$x_1 = cz_1$$

and, on substituting in this equation the values already found for x_1 and z_1, we obtain

$$\sqrt{x^2 + y^2} = cz, \quad \text{or} \quad x^2 + y^2 = c^2z^2,$$

which is the required equation of the right circular cone.

By the same reasoning, we find that, if

$$f(x, z) = 0, \qquad y = 0,$$

are the equations of any curve in the xz-plane, the equation of the surface formed by revolving this curve around the z-axis is

$$f(\sqrt{x^2 + y^2}, z) = 0,$$

and the equation of the surface formed by revolving it around the x-axis is

$$f(x, \sqrt{y^2 + z^2}) = 0.$$

Similarly, if we have given a curve in the xy-plane,

$$f(x, y) = 0, \qquad z = 0,$$

the equations of the surfaces formed by revolving it around the y-axis and around the x-axis, respectively, are

$$f(\sqrt{x^2 + z^2}, y) = 0 \quad \text{and} \quad f(x, \sqrt{y^2 + z^2}) = 0.$$

If the given curve lies in the yz-plane, so that its equations are:

$$f(y, z) = 0, \qquad x = 0,$$

the equations of the surfaces formed by revolving it around the z-axis and around the y-axis are, respectively,

$$f(\sqrt{x^2 + y^2}, z) = 0 \quad \text{and} \quad f(y, \sqrt{x^2 + z^2}) = 0.$$

Exercises

Find the equation of the surface of revolution formed by revolving the given curve around the axis indicated.

1. $x^2 + z^2 = a^2$, $y = 0$; z-axis.

2. $bx + ay = ab$, $z = 0$; x-axis.

3. $x^2 = az$, $y = 0$; z-axis.

4. $b^2x^2 + a^2z^2 = a^2b^2$, $y = 0$; x-axis.

5. $x^2 = az$, $y = 0$; x-axis.

6. $b^2x^2 + a^2z^2 = a^2b^2$, $y = 0$; z-axis.

7. $y = x^3$, $z = 0$; x-axis.

8. $b^2x^2 - a^2z^2 = a^2b^2$, $y = 0$; x-axis.

9. $x^2 = z^3 + z$, $y = 0$; z-axis.

10. $y^2 = x^3$, $z = 0$; x-axis.

11. $z = e^y$, $x = 0$; y-axis.

12. $z = \cos y$, $x = 0$; y-axis.

13. $(x - b)^2 + z^2 = a^2$, $y = 0$; z-axis.

319. The Sphere. A sphere is the locus of a point in space whose distance from a fixed point, the center, is equal to a constant, the radius.

It follows from this definition that the equation of a sphere with center at the point (h, k, l) and radius a is

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = a^2. \quad (4)$$

For, the first member of this equation is the square of the distance of the point (x, y, z) on the locus from the center (h, k, l) and this is equal to the square of the radius.

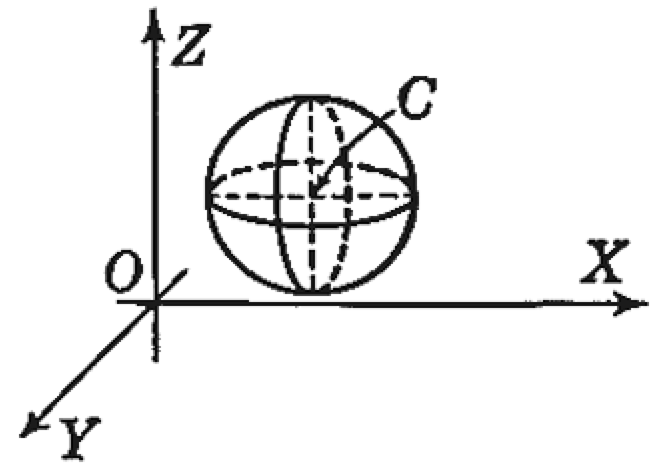


FIG. 195

In particular, if the center of the sphere is at the origin, equation (4) reduces to the simple form

$$x^2 + y^2 + z^2 = a^2. \quad (5)$$

The equation

$$x^2 + y^2 + z^2 + Gx + Hy + Iz + K = 0 \quad (6)$$

is called the **general form** of the equation of the sphere.

To determine the center and radius of the sphere defined by equation (6), we complete the square of the terms in x, y, and z, separately, and write the equation in the form

$$\left(x + \frac{G}{2}\right)^2 + \left(y + \frac{H}{2}\right)^2 + \left(z + \frac{I}{2}\right)^2 = \frac{G^2 + H^2 + I^2 - 4K}{4}$$

By comparing this equation with (4), we find that its locus is a sphere with

center $$\left(-\frac{G}{2}, -\frac{H}{2}, -\frac{I}{2}\right),$$

and radius $$a = \tfrac{1}{2}\sqrt{G^2 + H^2 + I^2 - 4K}.$$

The sphere defined by equation (6) is thus a *real sphere*, a *point sphere*, or an *imaginary sphere*, according as

$$G^2 + H^2 + I^2 - 4K \gtreqqless 0.$$

Exercises

Find the equations of the following spheres.

1. Center $(6, -2, -9)$, $a = 11$. **2.** Center $(-1, 8, -4)$, $a = 9$.
3. Center $(5, 4, -3)$, $a = 8$. **4.** Center $(-4, 7, 6)$, $a = 7$.

Find the center and the radius and draw the sphere if it exists, given:

5. $x^2 + y^2 + z^2 + 6x - 2y - 10z + 19 = 0$.
6. $x^2 + y^2 + z^2 - 7x - 8y + 5z + 10 = 0$.
7. $x^2 + y^2 + z^2 + 6x - 4y + 8z + 29 = 0$.
8. $x^2 + y^2 + z^2 - 11x - 7y + 3z + 51 = 0$.

Find the equation of the sphere:

9. Lying in the first octant, tangent to all the coördinate planes, radius 3.

10. Lying in the first octant, tangent to all the coördinate planes, center in the plane $x - 5y + 7z - 12 = 0$.

11. Center at $(-6, 1, 3)$, tangent to the plane $2x - 2y - z - 10 = 0$.

12. Passing through the points $(7, 9, 1)$, $(-2, -3, 2)$, $(1, 5, 5)$, and $(-6, 2, 5)$.

13. Passing through the origin, center on the line through $(0, 0, 0)$ and $(-8, -4, 1)$, radius 18.

14. Tangent to the plane $2x - 2y - z - 8 = 0$ at $(3, 1, -4)$, radius 6.

15. Show that, for all values of θ and ϕ, the point

$$x = a \sin\phi \cos\theta, \qquad y = a \sin\phi \sin\theta, \qquad z = a \cos\phi,$$

lies on the sphere $x^2 + y^2 + z^2 = a^2$.

NOTE. The above three equations are *parametric equations* of the sphere in terms of the parameters θ and ϕ.

320. Quadric Surfaces. The locus of an equation of the second degree in x, y, and z, that is, an equation of the form

$$Ax^2 + By^2 + Cz^2 + Dyz + Ezx + Fxy + Gx + Hy + Iz + K = 0,$$

wherein A, B, C, D, E, and F are not all zero, is called a *quadric surface.*

It is seen at once from equation (4) that a sphere is a quadric surface. In the following articles, we shall state the standard forms of the equations of the most important quadric surfaces other than the sphere, and point out a few of the outstanding properties of these surfaces.

321. The Ellipsoid. The locus of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is an *ellipsoid.*

This surface is symmetric with respect to each of the coördinate planes since, if we change the sign of any one of the coördinates, we do not change the equation. These planes are called the *principal planes* of the ellipsoid and their point of intersection, the origin, is its *center.*

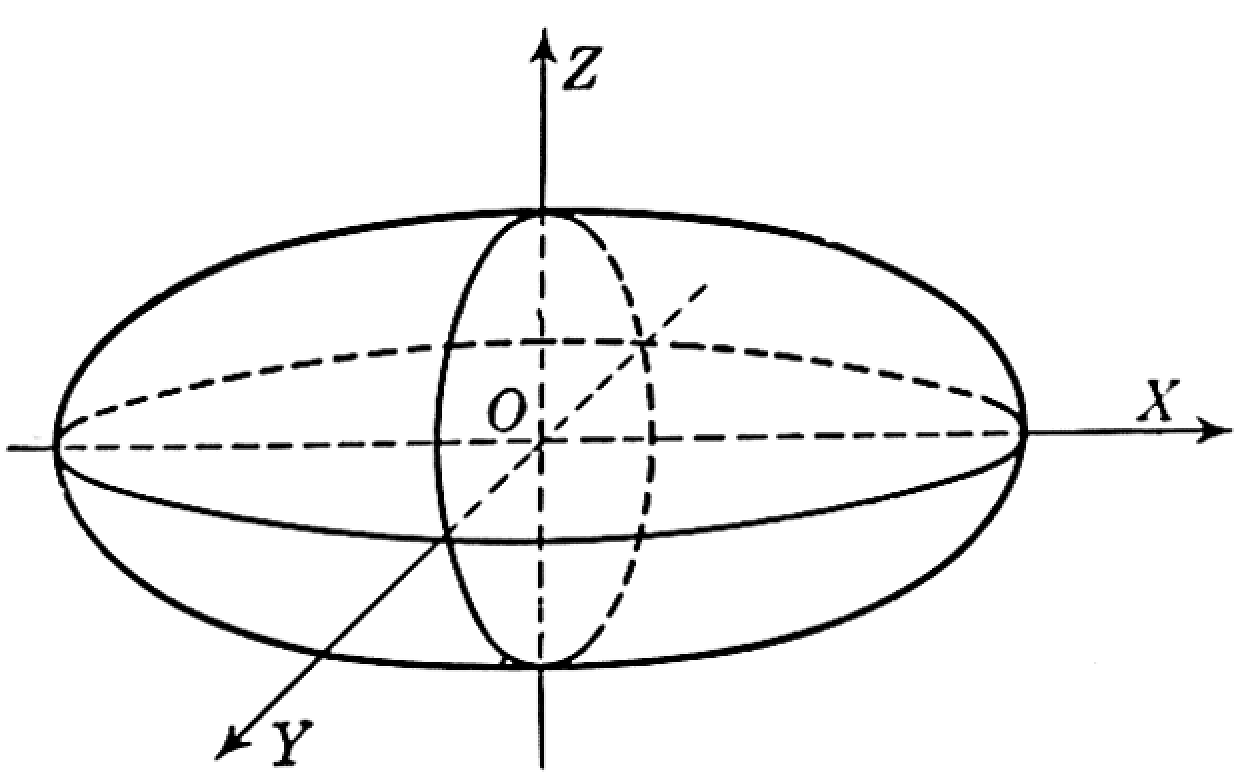


FIG. 196

The segments of the coördinate axes that lie inside the surface are the *axes* of the ellipsoid. By solving the equations of the axes as simultaneous with that of the surface, we find that the intercepts on the x-, y-, and z-axis are, respectively, $\pm a$, $\pm b$, and $\pm c$. If $a > b > c > 0$,

these numbers are called the lengths of the *semi-major*, the *semi-mean*, and the *semi-minor axis*, respectively, of the ellipsoid.

The usual way to determine the form of a surface from its equation is to study the curves of section of the surface by a family of parallel planes. For the ellipsoid, we shall use the sections by planes perpendicular to the z-axis.

The equations of the section of the given ellipsoid by a plane $z = k$ are found, by putting $z = k$ in the equation and simplifying, to be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2}, \quad z = k,$$

or, if $k \neq \pm c$,

$$\frac{x^2}{\frac{a^2}{c^2}(c^2 - k^2)} + \frac{y^2}{\frac{b^2}{c^2}(c^2 - k^2)} = 1, \quad z = k.$$

If $k^2 < c^2$, these are the equations of an ellipse of semi-axes $\frac{a}{c}\sqrt{c^2 - k^2}$ and $\frac{b}{c}\sqrt{c^2 - k^2}$. The largest ellipse of section is thus in the plane $z = 0$. As k increases in numerical value, the ellipse of section becomes smaller and shrinks to a point when $k^2 = c^2$. If $k^2 > c^2$, the ellipse is imaginary; that is, there are no points on the surface in any plane defined by such a value of k.

If $a = b > c$, the ellipsoid is called an **oblate spheroid** and, if $a > b = c$, it is a **prolate spheroid.**

322. The Hyperboloid of One Sheet. The surface defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

is a *hyperboloid of one sheet.*

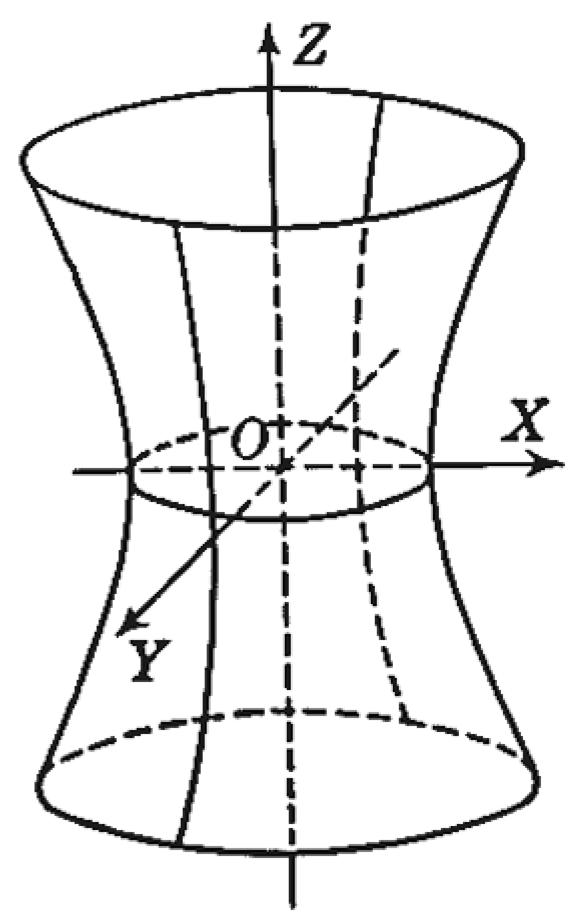


FIG. 197

This surface also has the coördinate planes as planes of symmetry, or *principal planes*, and the origin as *center*. It intersects the x-axis at $(\pm a, 0, 0)$ and the y-axis at $(0, \pm b, 0)$ but it has no point in common with the z-axis.

The section of this hyperboloid by the plane $z = k$ is the ellipse defined by the equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{k^2}{c^2}, \quad z = k.$$

This ellipse is smallest for $k = 0$ and increases indefinitely in size as the numerical value of k increases. The surface thus extends indefinitely far from the origin.

323. The Hyperboloid of Two Sheets. This name is given to the locus of the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

This surface has the coördinate planes as *principal planes* and the origin as *center*. Its x-intercepts are $\pm a$ but it does not meet either of the other coördinate axes.

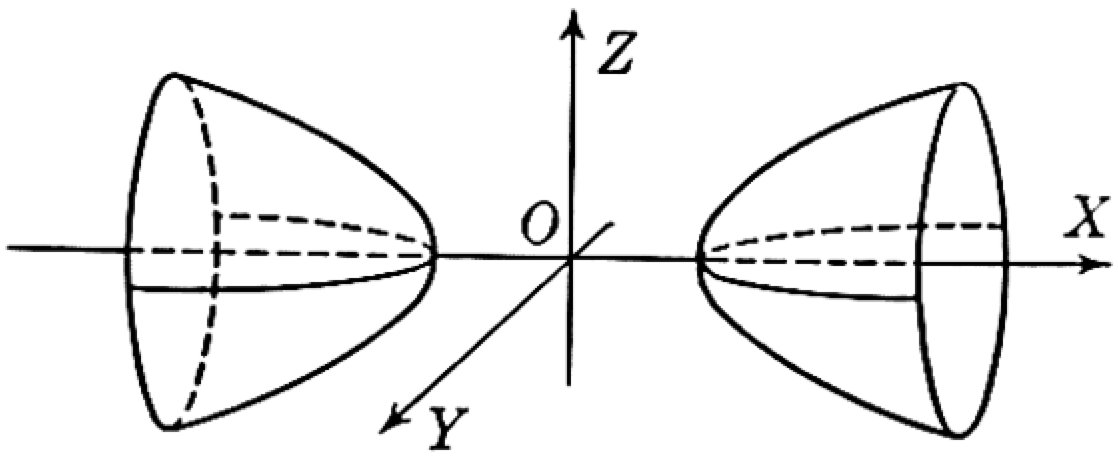


FIG. 198

The equation of its curve of section by the plane $x = k$ are

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{k^2}{a^2} - 1, \quad x = k.$$

If $k^2 < a^2$, this curve is an imaginary ellipse and has no points on it. If $k^2 = a^2$, the curve is a point ellipse and, if $k^2 > a^2$, the curve is a real ellipse which increases indefinitely in size as k^2 increases indefinitely. The surface thus consists of two distinct parts which extend indefinitely far away from the yz-plane.

324. The Elliptic Paraboloid. The locus of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z$$

is an *elliptic paraboloid.*

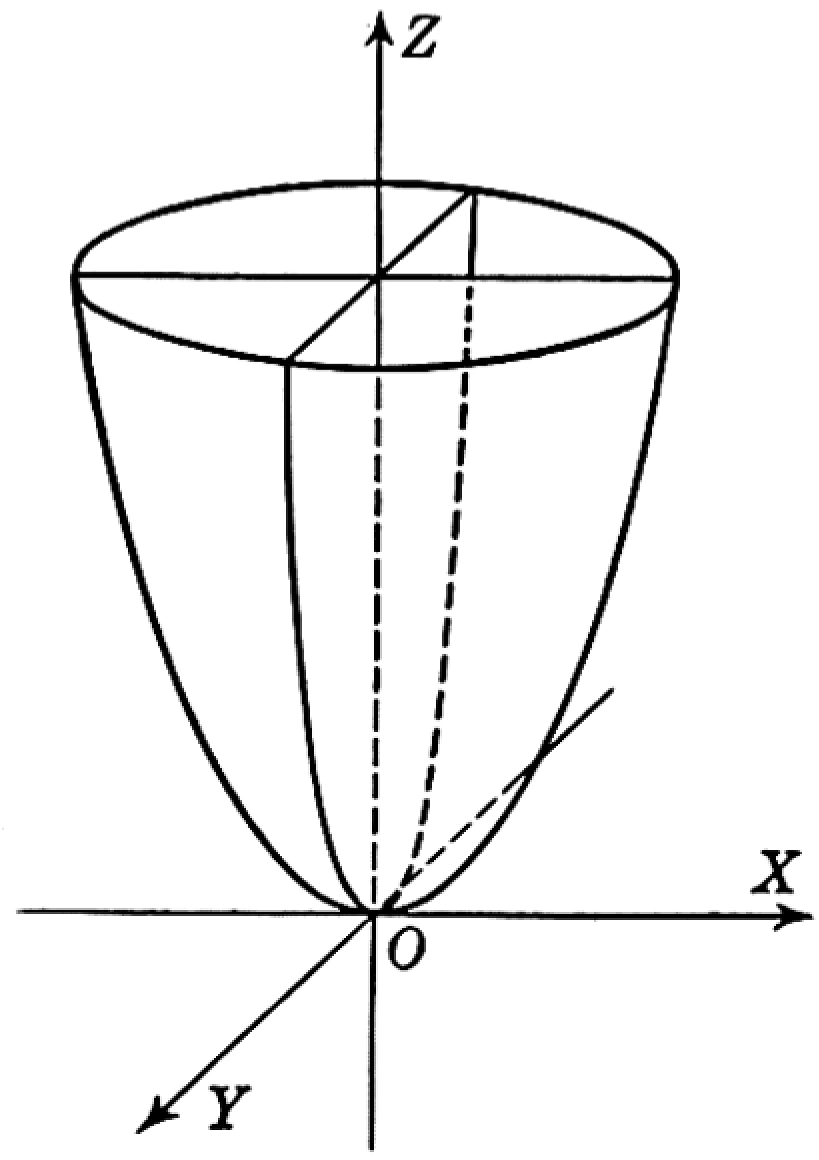


FIG. 199

The surface is symmetric with respect to the xz- and yz-planes but not with respect to the xy-plane. It has no center. It touches the xy-plane at the origin but does not extend below it.

The section of this surface by the plane $z = k$, when $k > 0$, is an ellipse whose semi-axes are $a\sqrt{k}$ and $b\sqrt{k}$. This ellipse thus increases indefinitely in size as k increases. The sections of the surface by planes perpendicular to the x-axis, or to the y-axis, are parabolas.

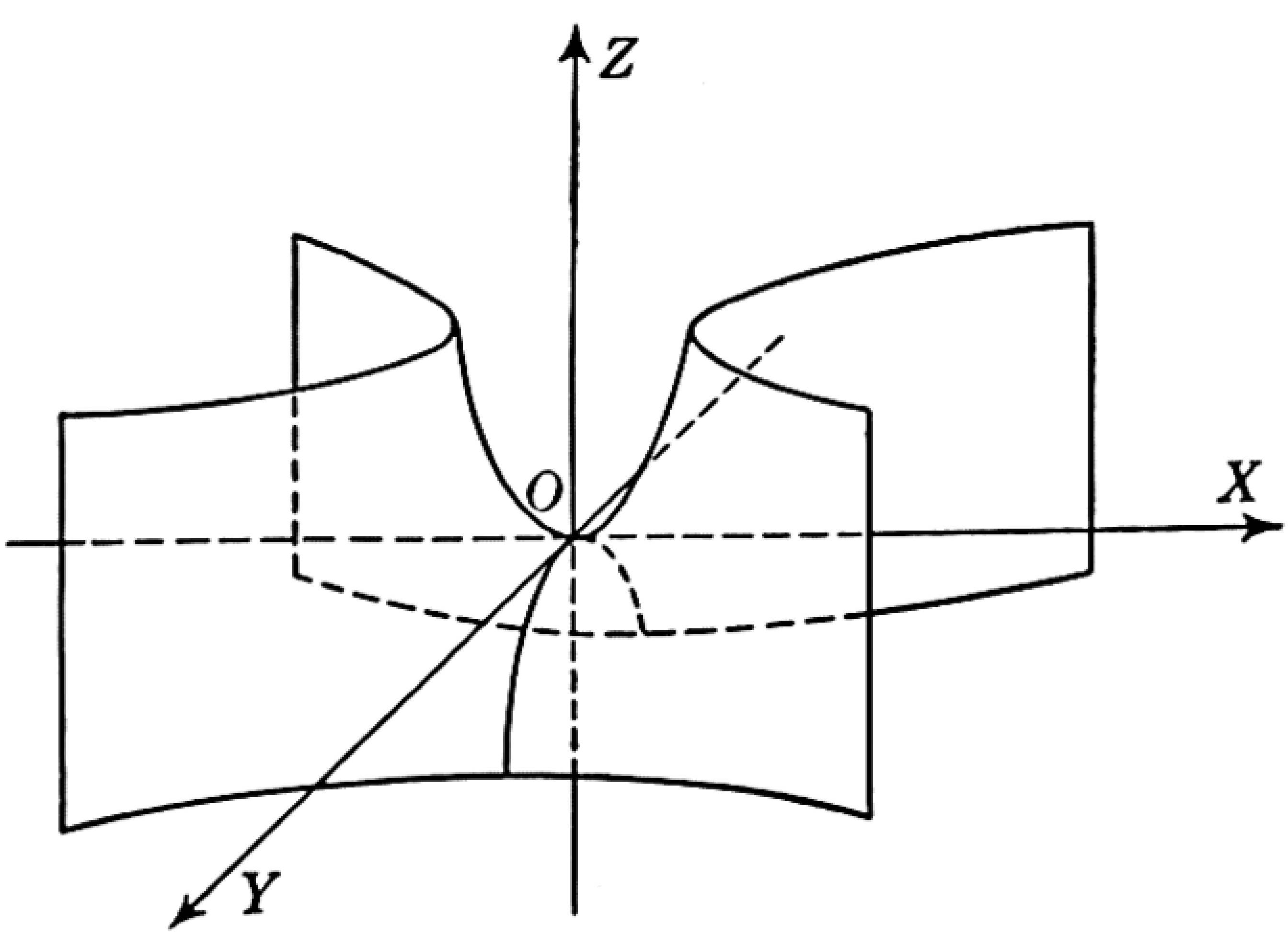


FIG. 200

325. The Hyperbolic Paraboloid. The surface

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = z$$

is a *hyperbolic paraboloid.*

It has the xz- and yz-planes as principal planes, passes through the origin, and has no other points in common with any of the coördinate axes. It has no center.

Its section by the xy-plane is composed of the two lines $y = \pm bx/a$,

$z = 0$. The planes $z = k$ parallel to the xy-plane intersect it in hyperbolas that have their transverse axes parallel to the x-axis if $k > 0$ and parallel to the y-axis if $k < 0$. The planes $y = k$ intersect the surface in parabolas which are concave upward; the planes $x = k$, in parabolas which are concave downward.

If $a = b$, the surface is said to be a **rectangular hyperbolic paraboloid.** In this special case, the equation of the surface may be written in the form

$$x^2 - y^2 = a^2z. \tag{7}$$

If we now rotate the x- and y-axes, in their own plane, through an angle of $-45°$ by means of the equations for a rotation of axes (Art. 196); that is, if we apply to x and y the transformation

$$x = \frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}, \qquad y = \frac{-x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}$$

equation (7) reduces to

$$2x'y' = a^2z'. \tag{8}$$

In the applications, the equation of the rectangular hyperbolic paraboloid is frequently encountered in this form.

326. The Quadric Cone. The surface defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

is symmetric with respect to each of the coördinate planes. Its trace in the yz-plane consists of the two lines $y = \pm bz/c$, $x = 0$ and, in the xz-plane, of the two lines $x = \pm az/c$, $y = 0$. Its intersection with the xy-plane is a single point, the origin. The section of the surface by any plane $z = k$, parallel to the xy-plane is an ellipse the lengths of whose semi-axes, ak/c and bk/c, are proportional to the distance of the plane from the xy-plane. This surface is a cone with vertex at the origin and axis coinciding with the z-axis. It is called a **quadric cone.**

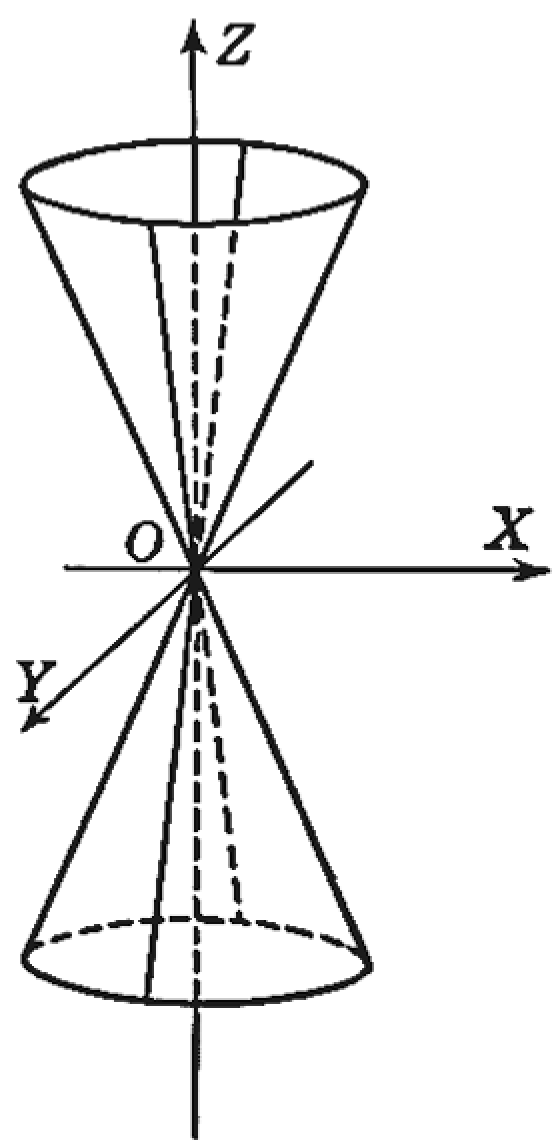


FIG. 201

If $a \neq b$, the quadric cone is also called an *oblique circular cone* and if $a = b$, so that the sections perpendicular to the z-axis are circles, it is a *right circular cone.*

Exercises

Sketch the following surfaces and state the name of each surface.

1. $\frac{x^2}{49} + \frac{y^2}{16} + \frac{z^2}{9} = 1.$
2. $\frac{x^2}{81} - \frac{y^2}{25} - \frac{z^2}{16} = 1.$
3. $9y^2 - 16x^2 + 36z^2 = 144.$
4. $9x^2 + 25y^2 = 225z^2.$

5. $9x^2 + 9y^2 = 25z^2$.

6. $xy = 5z$.

7. $3y^2 - 8x^2 = 24z$.

8. $3x^2 + 5y^2 + z^2 = 15$.

9. $2x^2 + 5y^2 - 3z^2 + 30 = 0$.

10. $x^2 - 5y^2 = 7z^2$.

11. $x^2 = 4y - 9z^2$.

12. $5x^2 - 9y^2 - 9z^2 + 15 = 0$.

13. $4x^2 + 4y^2 + 9z^2 = 36$.

14. $3y^2 + 7z^2 = 84x$.

Chapter 39

Definitions and Theorems in Spherical Trigonometry

327. Circles on a Sphere. If a plane intersects a sphere, the curve of section is a circle. If the cutting plane passes through the center of the sphere, its circle of section is a **great circle** and its radius equals the radius of the sphere; otherwise, it is a **small circle.** Except as otherwise indicated, we shall suppose in what follows that all the circles under consideration are great circles.

Every circle, great or small, has two **poles**; the points where the diameter of the sphere perpendicular to the plane of the circle intersects the sphere.

Thus, the equator of the earth is a great circle, having its radius equal to the radius of the earth. The parallels of latitude, lying in planes parallel to the equator, are small circles. All of these circles have, as poles, the north and south poles of the earth.

328. Spherical Distances. Through any two points A and B, on the sphere but not at the ends of a diameter, there passes just one great circle, since A and B, together with O, the center of the sphere, determine a plane in which this great circle must lie. The points A and B divide this great circle into two unequal arcs. The shorter of these arcs is the shortest path connecting A and B that can be drawn on the surface of the sphere. We define the length of this shortest arc as the **spherical distance** from A to B. We shall usually express this spherical distance as an angle; namely, as the angle AOB formed by the radii joining the center O to A and B.

Thus, the spherical distance from Jacksonville, Fla. (30° 20′ N, 81° 40′ W) to Cleveland, O., (41° 30′ N, 81° 40′ W) is measured along the meridian of 81° 40′ W. This spherical distance is equal to 11° 10′ which is the difference in latitude of the two places. If we consider the earth to be a sphere of radius 3959 miles, we find from formula (2) of Art. 90 that this distance is about 772 statute miles.

The spherical distance from any point on a great circle to either of its poles is 90°. Conversely, if the spherical distance between two points is 90°, then each of these points lies on a great circle having the other point as its pole.

Thus, the spherical distance, measured along a meridian, from any point on the earth's equator to either the north or south pole is 90°. Conversely, if the spherical distance of any point from either pole is 90°, then the point lies on the equator.

329. Spherical Angles. The arcs of two great circles extending from a point A on a sphere form a **spherical angle** at A. To measure this angle, draw the tangents AT and AT' to these circles at A (Fig. 202). Since the lines AT and AT' are perpendicular to OA, the line of intersection of the planes of the circles, *the measure of the spherical angle at A is also the measure of the dihedral angle formed by the planes of the circles.*

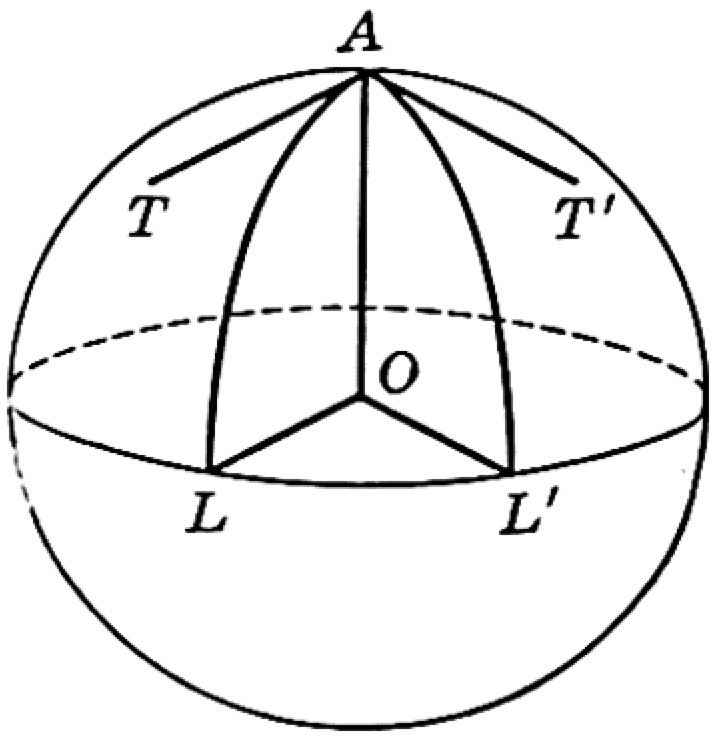


FIG. 202

Extend the circular arcs until they intersect, at L and L', the great circle having A as its pole and draw the radii OL and OL'. Since OL and OL' are parallel to AT and AT', respectively, *the spherical angle at A is also measured by the arc L′L on the great circle having A as its pole.*

For example, Quito, Ecuador, and Macapa (a small town at the mouth of the Amazon) both lie nearly on the equator in longitude 78° 30′ W and 50° 55′ W, respectively. The meridians through these places make, at either pole, an angle of $78° 30' - 50° 55' = 27° 35'$. But this angle is also the spherical distance, measured along the equator, between the two towns.

330. Spherical Triangles. If three points A, B, and C, lying on a sphere but not all lying on one great circle, are joined, in pairs, by arcs of *great* circles, the resulting figure is a **spherical triangle.** We shall denote the spherical angles at A, B, and C by α, β, and γ, and the sides opposite these angles by a, b, and c, respectively. The six quantities α, β, γ, a, b, and c are the six **parts** of the spherical triangle.

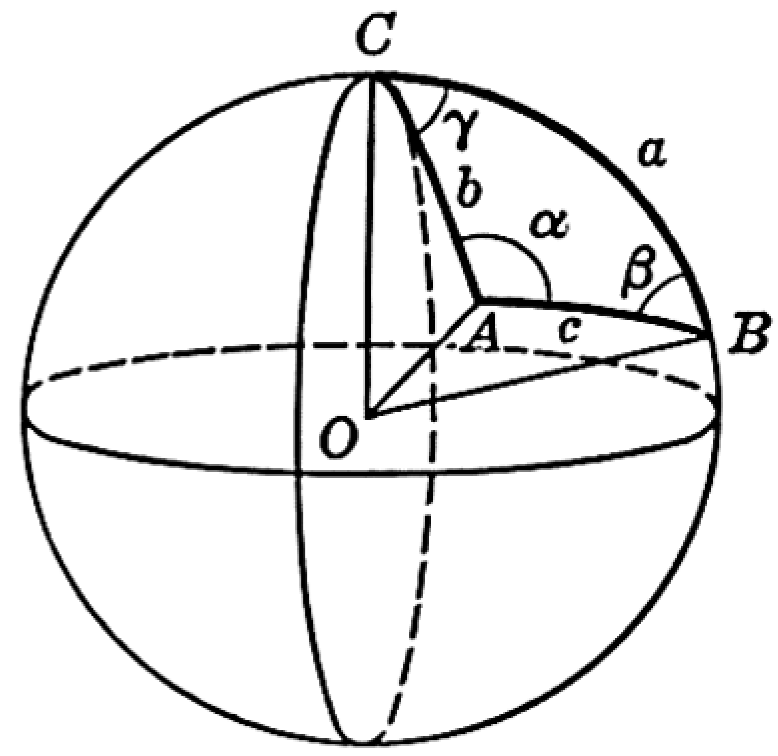


FIG. 203

It should be observed that the sides of a spherical triangle must be arcs of great circles. On the surface of the earth, for example, an arc of a parallel of latitude (not coincident with the equator) cannot be a side of a spherical triangle because it is not an arc of a great circle.

Spherical triangles exist that have one or more parts greater than 180° but we shall consider only those triangles for which each part is less than 180°. For such triangles, it is proved in solid geometry that

$$a + b + c < 360°, \quad \text{and} \quad 180° < \alpha + \beta + \gamma < 540°.$$

331. The Spherical Excess. The amount by which $\alpha + \beta + \gamma$ exceeds 180° is called the **spherical excess** of the triangle; that is, if E is the spherical excess, then

$$\alpha + \beta + \gamma - 180° = E.$$

It is proved in geometry that the area of a spherical triangle on a sphere of radius R is

$$\text{area} = R^2E\frac{\pi}{180}. \qquad (E \text{ in degrees})$$

If E' is the spherical excess in radians, then, since $E' = E\pi/180$, this formula may be written as

$$\text{area} = R^2E'. \qquad (E' \text{ in radians})$$

EXAMPLE. Assuming the earth to be a sphere of radius 3959 miles, find the area of a spherical triangle on it for which $\alpha = 103° 21.6'$, $\beta = 82° 47.7'$ and $\gamma = 73° 42.4'$.

We have

$$E = 103° 21.6' + 82° 47.7' + 73° 42.4' - 180° = 79° 51.7' = 79.862°.$$

$$\begin{aligned} \log \pi/180 &= 8.24188 - 10 \\ \log E &= 1.90234 \\ 2 \log R &= \underline{7.19518} \\ \log \text{area} &= 7.33940 \\ \text{area} &= 21{,}848{,}000 \text{ sq. mi.} \end{aligned}$$

332. Polar Triangles. Let ABC be a spherical triangle. Of the two poles of the side BC (Art. 327), let A' be the one that lies on the same side of BC that A does; that is, such that the spherical distance AA' is less than 90°. Similarly, let B' and C' be the poles of AC and AB, respectively, such that BB' and CC' are each less than 90°. Draw the great circle arcs $A'B'$, $B'C'$, and $C'A'$. Then the spherical triangle $A'B'C'$ is the **polar triangle** of ABC. We shall denote its angles by α', β', and γ' and its sides by a', b' and c'.

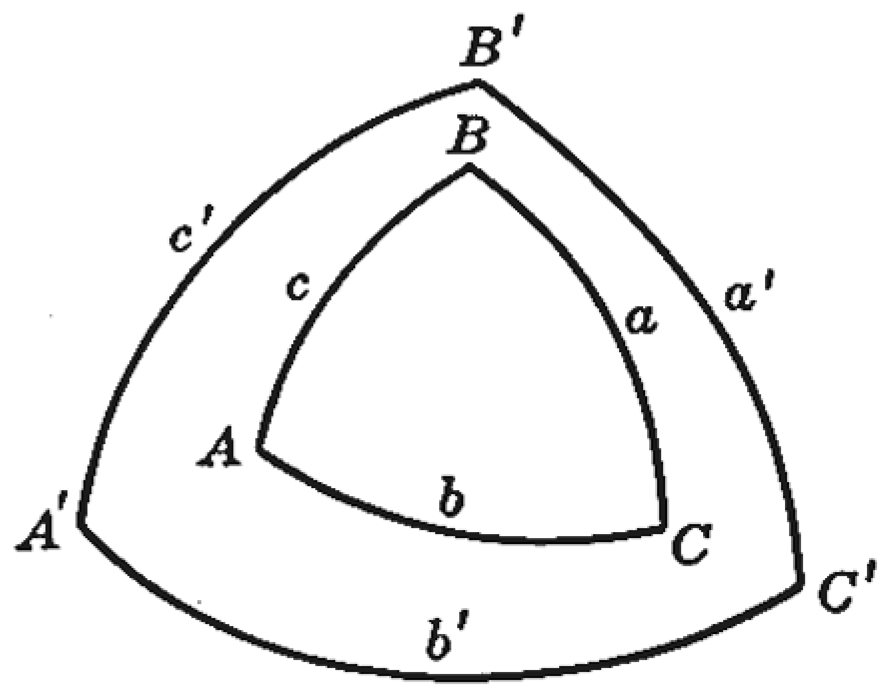


FIG. 204

It is proved in solid geometry that ABC is the polar triangle of $A'B'C'$ and, further, that

$$\begin{aligned} \alpha' &= 180° - a, & \beta' &= 180° - b, & \gamma' &= 180° - c, \\ a' &= 180° - \alpha, & b' &= 180° - \beta, & c' &= 180° - \gamma. \end{aligned} \qquad (1)$$

We shall find these six equations useful in deriving several formulas which we shall need in solving spherical triangles.

Exercises

1. On a sphere of radius 30 inches, the sides of a spherical triangle are 78°, 107°, and 124°. Find the lengths of these sides in inches.

2. Find the angles of the polar triangle of the triangle in Ex. 1.

3. The angles of a spherical triangle are 112°, 129°, and 86°. Find the sides of the polar triangle.

4. If the radius of the sphere in Ex. 3 is 185 feet, find the area of the given triangle.

5. Assuming that the earth is a sphere of radius 3959 miles, find the distance, in statute miles, from Greenwich (Lat. 51° 29′ N) to the north pole.

6. If all the sides of a spherical triangle are 90°, show (*a*) that the triangle is self-polar and (*b*) that all its angles are 90°.

333. The Terrestrial Sphere. In applying spherical trigonometry to the earth's surface, we shall consider the earth to be a sphere of radius 3959 miles. A nautical mile is an arc of 1′ on a great circle of the earth. Since the earth is, in fact, not precisely a sphere, different determinations of the length of a nautical mile are possible. The U.S. nautical mile is taken as 6080.27 feet while the British nautical mile is taken to be 6080 feet.

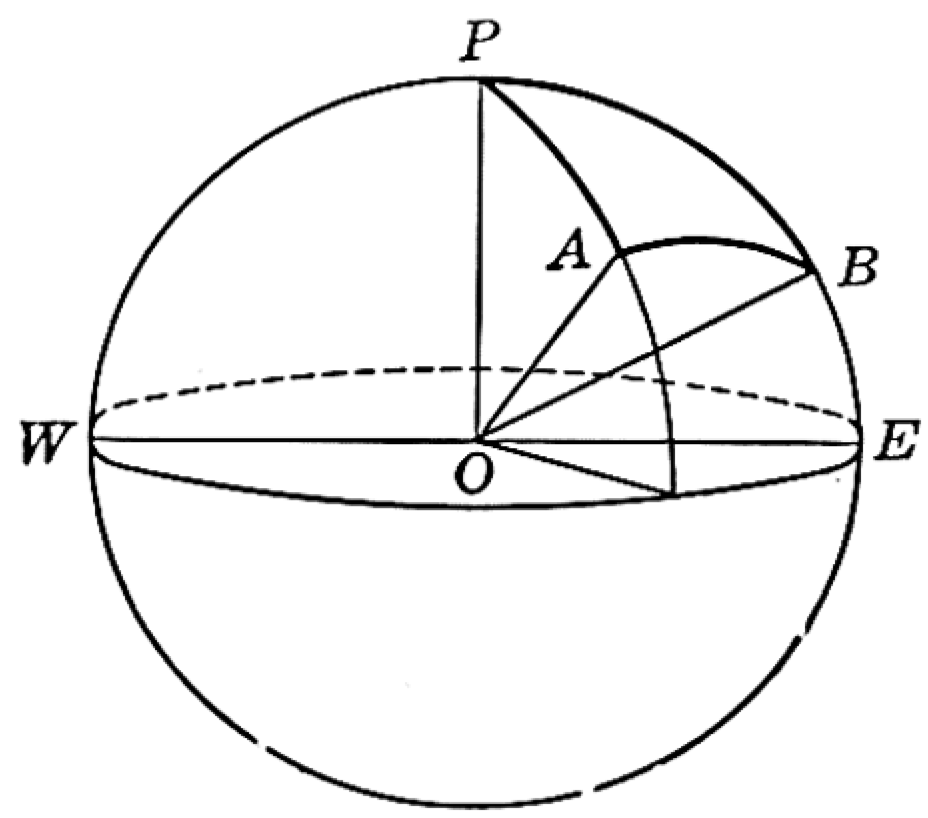


FIG. 205

The terrestrial spherical triangle we shall most often use has one vertex at the north pole P and its other two vertices at two points A and B on the surface of the earth. The sides AP and BP are arcs of meridians such that

$$AP = \text{co-latitude of } A = 90° - \text{latitude of } A.$$
$$BP = \text{co-latitude of } B = 90° - \text{latitude of } B.$$
$$AB = \text{the spherical distance between } A \text{ and } B.$$

In the first two of these equations, we count north latitudes as positive and south latitudes as negative.

For the angles of this triangle, we have

P = either difference of longitudes of A and B or 360° − this difference (whichever is less than 180°).

$$A = \textbf{bearing} \text{ of } B \text{ from } A.$$
$$B = \textbf{bearing} \text{ of } A \text{ from } B.$$

In the first of these three equations, we count west longitudes as positive and east longitudes as negative. In the next to the last equation, we read the bearing as N A° E (or W); and similarly at B.

334. The Celestial Sphere. We shall think of the heavenly bodies as lying on a sphere of indefinitely large radius having the center of the earth as its center. Since the earth is very small compared to this sphere,

we shall consider the lines joining any two points of the earth to any one point of this sphere as parallel.

The points where the line of the earth's axis intersects the celestial sphere are the **celestial poles** P and P'. The circle $E'E$ in which the plane of the earth's equator intersects the celestial sphere is the **celestial equator.** The great circles passing through the celestial poles (corresponding to the earth's meridians) are called **hour circles.** Since the earth turns through 360° in 24 hours (or 15° an hour) the angles between the hour circles may be measured in degrees or in hours (15° = 1 hour).

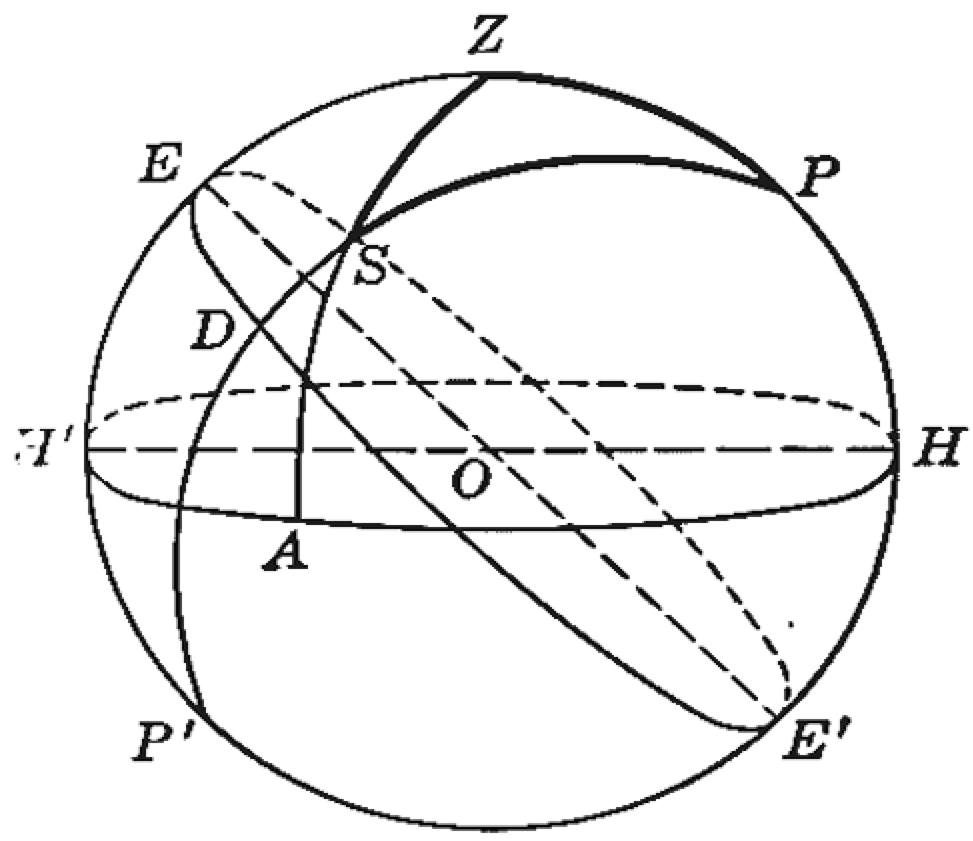


FIG. 206

The point Z of the celestial sphere directly above the observer (at any instant) is his **zenith** and the point directly below him is his **nadir.** The great circle $H'H$ having the zenith and nadir as its poles is his **horizon.** The great circles passing through the zenith and nadir are called **vertical circles.** In particular, the vertical circle ZP passing through the celestial poles (or, in other words, the hour circle passing through the zenith and nadir) is the observer's **meridian circle.**

The position of a point S on the celestial sphere may be fixed by two angles in either of the following two systems.

I. *The equatorial system.* The **declination** of S is the spherical distance DS, measured along an hour circle, from the equator to S. It is positive for points north of the equator and negative for points south. The **hour angle** of S is the angle at P that the observer's meridian makes with the hour circle through S. We shall measure it eastward or westward from the meridian.

II. *The horizon system.* The **altitude** of S is its spherical distance AS, measured along a vertical circle, from the horizon. It is positive for points above the horizon and negative for points below. The **azimuth** of S is the angle at Z that the vertical circle through S makes with the meridian circle. We shall measure it from the northerly direction at Z either to the east or west.

The spherical triangle whose vertices are the celestial pole P, the zenith Z, and a point S on the celestial sphere is the **astronomical triangle.** Five of the parts of this triangle are:

SZ = co-altitude of S.
SP = co-declination of S.
ZP = co-latitude of observer.
Z = azimuth of S.
P = hour angle of S.

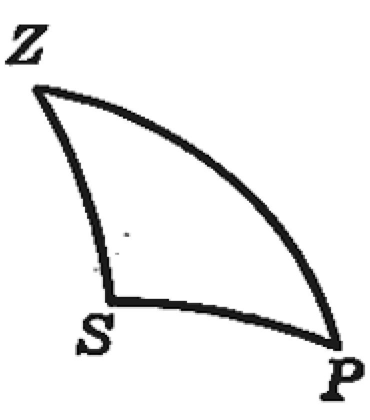


FIG. 207

Exercises

1. Find the distance, in statute miles, from Washington, D.C. (38° 55′ N, 77° 4′ W) to Lima, Peru (12° 3′ S, 77° 4′ W).

2. Show that the length, in statute miles, of 1°, measured along the parallel of latitude ϕ, is $\frac{\pi}{180}$ 3959 cos ϕ.

3. An airplane flew directly west from Chicago, Ill. (41° 50′ N, 87° 37′ W) until it reached the meridian of 96° W. How far did it fly? Is this the shortest distance it could have flown between the starting and landing points?

4. A man traveled 400 miles directly east from Kansas City, Mo. (39° 5′ N, 94° 35′ W). Find the latitude and longitude of the place at which he arrived.

5. The declination of the sun on June 22 is 23° 27′. Find its altitude at noon, solar time, as seen from Seattle (47° 40′ N).

Chapter 40

The Right Spherical Triangle

335. Formulas for the Right Spherical Triangle. A spherical triangle is a right spherical triangle if at least one of its angles is a right angle. A right spherical triangle may have two, or all three of its angles right angles. If, however, at least two of its angles are right angles, the sides opposite these right angles are both 90° and the third side has the same measure as its opposite angle. In this chapter we shall, accordingly, suppose that only one angle, which we shall take to be the angle γ, is a right angle.

For the solution of such a right spherical triangle, we have the following formulas.

$$\begin{array}{rll@{\qquad}rll}
\text{I.} & \sin a & = \tan b \cot \beta, & \text{II.} & \sin a & = \sin \alpha \sin c, \\
\text{III.} & \sin b & = \tan a \cot \alpha, & \text{IV.} & \sin b & = \sin \beta \sin c, \\
\text{V.} & \cos c & = \cot \alpha \cot \beta, & \text{VI.} & \cos c & = \cos a \cos b, \\
\text{VII.} & \cos \alpha & = \tan b \cot c, & \text{VIII.} & \cos \alpha & = \cos a \sin \beta, \\
\text{IX.} & \cos \beta & = \tan a \cot c, & \text{X.} & \cos \beta & = \cos b \sin \alpha.
\end{array}$$

Proofs of these formulas will be given in Art. 337.

336. Napier's Rules. The preceding ten formulas may be summarized in the following two statements which are called Napier's Rules.

In the right spherical triangle ABC, omit the right angle γ and replace the parts α, c, and β, by co-$\alpha = 90° - \alpha$, co-$c = 90° - c$, and co-$\beta = 90° - \beta$, respectively. Arrange the five parts co-α, co-c, co-β, a, and b in a circle, as shown in Figure 208b. Then any one of these five parts has two adjacent parts and two opposite parts. Napier's Rules state that:

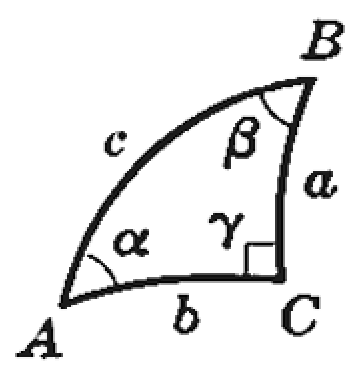


FIG. 208a

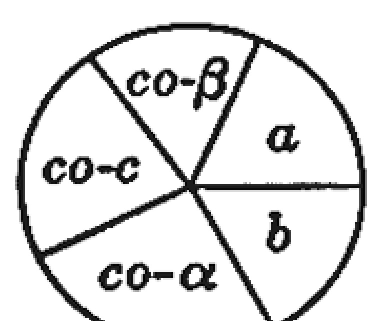


FIG. 208b

I. *The sine of any part equals the product of the tangents of its adjacent parts.*

II. *The sine of any part equals the product of the cosines of its opposite parts.*

The vowel relationships between tangent and adjacent and between cosine and opposite may be helpful in remembering these rules.

The formulas in the first column in Art. 335 are summarized in Rule I and those in the second column in Rule II.

Thus, by Rule I,

$$\begin{array}{ll}
\sin(\text{co-}c) = \tan(\text{co-}\alpha)\tan(\text{co-}\beta), & \text{or} \quad \cos c = \cot \alpha \cot \beta. \\
\sin(\text{co-}\beta) = \tan a \tan(\text{co-}c), & \text{or} \quad \cos \beta = \tan a \cot c.
\end{array}$$

By Rule II,

$$\sin a = \cos(\text{co-}\alpha)\cos(\text{co-}c), \quad \text{or} \quad \sin a = \sin \alpha \sin c.$$
$$\sin(\text{co-}\alpha) = \cos a \cos(\text{co-}\beta), \quad \text{or} \quad \cos \alpha = \cos a \sin \beta.$$

337. Derivation of the Formulas. In the right spherical triangle ABC, let $a < 90°$ and $b < 90°$. Let O be the center of the sphere and draw the radii OA, OB, and OC. Choose any point D on OB and draw through it a plane perpendicular to OA. Let this plane intersect the planes AOB, BOC and AOC in the lines DE, DF, and EF, respectively. Then the angle $FED = \alpha$ (Fig. 209) since both of these angles measure the dihedral angle formed by the planes COA and BOA. Further, by Art. 328,

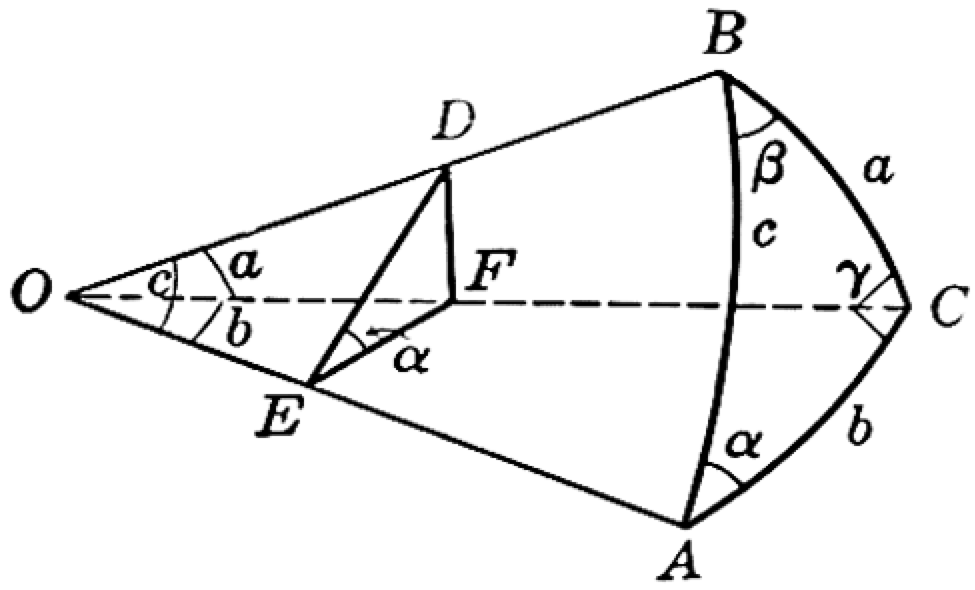


FIG. 209

$$COB = a, \quad COA = b, \quad \text{and} \quad AOB = c.$$

Finally, the triangles OED, OEF, OFD, and EFD are right triangles, the right angle being at the point indicated by the middle letter.

From the figure, we now have the following relations. (The corresponding formulas of Art. 335 are indicated by the Roman numerals.)

$$\sin \alpha = \frac{FD}{ED} = \frac{OD \sin a}{OD \sin c} = \frac{\sin a}{\sin c}. \qquad \text{II}$$

$$\cos \alpha = \frac{EF}{ED} = \frac{OE \tan b}{OE \tan c} = \frac{\tan b}{\tan c}. \qquad \text{VII}$$

$$\tan \alpha = \frac{FD}{EF} = \frac{OF \tan a}{OF \sin b} = \frac{\tan a}{\sin b}. \qquad \text{III}$$

$$\cos c = \frac{OE}{OD} = \frac{OF \cos b}{OF \sec a} = \cos a \cos b. \qquad \text{VI}$$

If, instead of the plane DEF perpendicular to the line OA, we had taken a plane perpendicular to OB, we would have obtained formulas similar to the foregoing but with a and α interchanged with b and β; that is,

$$\sin \beta = \frac{\sin b}{\sin c} \ \text{IV}, \quad \cos \beta = \frac{\tan a}{\tan c} \ \text{IX}, \quad \text{and} \quad \tan \beta = \frac{\tan b}{\sin a} \ \text{I}.$$

The remaining three formulas can be obtained by combining the ones just derived. From III, I, and VI, we have

$$\cot \alpha \cot \beta = \frac{\sin b \sin a}{\tan a \tan b} = \cos b \cos a = \cos c. \qquad \text{V}$$

From VII, IV, and VI, we have

$$\frac{\cos \alpha}{\sin \beta} = \frac{\tan b \sin c}{\tan c \sin b} = \frac{\cos c}{\cos b} = \cos a. \qquad \text{VIII}$$

From IX, II, and VI, we obtain, similarly,

$$\frac{\cos \beta}{\sin \alpha} = \frac{\tan a}{\tan c} \frac{\sin c}{\sin a} = \frac{\cos c}{\cos a} = \cos b. \qquad \text{X}$$

In deriving these formulas, we have supposed that $a < 90°$ and $b < 90°$ but, with slight modifications, the proofs may be extended to the cases where either, or both, of these quantities exceeds $90°$.

338. Rules of Species. Two parts of a spherical triangle are *of the same species* if they are both less, or both greater, than $90°$; otherwise, they are *of opposite species.* The following two rules are sometimes useful in deciding whether there exists a triangle defined by a given set of data and, if one does exist, in determining in what quadrant a specified computed part must lie.

I. *Each side of a right spherical triangle is of the same species as its opposite angle.*

II. *If the hypotenuse is less than 90°, the two sides are of the same species; if the hypotenuse is greater than 90°, the sides are of opposite species.*

To prove Rule I, we use formula VIII.

$$\cos \alpha = \cos a \sin \beta.$$

Since $0 < \beta < 180°$, $\sin \beta$ is positive so that $\cos \alpha$ and $\cos a$ must agree in sign. If $\alpha < 90°$, $\cos \alpha$, and hence $\cos a$, is positive. Hence $a < 90°$. If $\alpha > 90°$, $\cos \alpha$ and $\cos a$ are negative so that $a > 90°$. In a similar way, using formula X, we find that β and b are both less, or both greater, than $90°$.

To prove Rule II, we use formula VI

$$\cos c = \cos a \cos b.$$

If $c < 90°$, $\cos c$ is positive, $\cos a$ and $\cos b$ agree in sign, so that a and b are both less, or both are greater, than $90°$. If $c > 90°$, $\cos c$ is negative, $\cos a$ and $\cos b$ are opposite in sign, so that one of the angles is less, and the other is greater, than $90°$.

339. Solution of Right Spherical Triangles. When any two of the five parts (other than the right angle) of a right spherical triangle are given, we can set up, by Napier's Rules or the equivalent formulas of Art. 335, equations expressing each of the other three parts in terms of the given ones. When we have solved these three equatons, we shall have solved the triangle. As a (partial) check, we shall also solve the equation connecting the three required parts.

There may be no solution; as, for example, when the sine or cosine of a computed part exceeds unity or when a rule of species is violated. There may be just one solution or (in the ambiguous case where a and α or b and β are given) there may be two solutions.

If a required part is found from its sine, we use the rules of species to determine in what quadrant the required angle lies. In the other cases, the sign of the result will determine the required quadrant but the result should be checked by the rules of species.

EXAMPLE 1. Solve the triangle: $a = 51° 35.2'$, $c = 114° 32.6'$.

Given:	Find:
$a = 51° 35.2'$,	$b = 131° 57.3'$,
$c = 114° 32.6'$.	$\alpha = 59° 28.4'$,
	$\beta = 125° 9.6'$.

$$\cos b = \frac{\cos c}{\cos a}, \qquad \sin \alpha = \frac{\sin a}{\sin c},$$

$$\cos \beta = \tan a \cot c. \qquad \cos \beta = \sin \alpha \cos b.$$

$$\begin{aligned} \log \cos c &= 19.61845 - 20 \text{ (n) *} \\ \log \cos a &= \underline{9.79332 - 10} \quad - \\ \log \cos b &= 9.82513 - 10 \text{ (n)} \end{aligned} \qquad \begin{aligned} \log \sin a &= 19.89407 - 20 \\ \log \sin c &= \underline{9.95887 - 10} \ - \\ \log \sin \alpha &= 9.93520 - 10 \end{aligned}$$

$$\begin{aligned} \log \tan a &= 0.10074 \\ \log \cot c &= \underline{9.65957 - 10} \text{ (n)} + \\ \log \cos \beta &= 9.76031 - 10 \text{ (n)} \end{aligned} \qquad \begin{aligned} \log \sin \alpha &= 9.93520 - 10 \text{ (n) †} \\ \log \cos b &= \underline{9.82513 - 10} \quad + \\ \log \cos \beta &= 9.76033 - 10 \text{ (n)} \end{aligned}$$

EXAMPLE 2. Solve the triangle: $\alpha = 129° 41.2'$, $\beta = 27° 58.5'$.

In this case, the triangle does not exist. For, if we compute $\cos c$ from the formula $\cos c = \cot \alpha \cot \beta$, we have

$$\begin{aligned} \log \cot \alpha &= 9.91898 - 10 \text{ (n)} \\ \log \cot \beta &= \underline{0.27478} \qquad + \\ \log \cos c &= 0.19376 \qquad \text{(n)} \end{aligned}$$

This gives $\cos c = -1.5623$ which is impossible since the numerical value of the cosine of an angle cannot exceed unity.

EXAMPLE 3. Solve the triangle: $b = 137° 25.2'$, $\beta = 113° 41.6'$.

This is the ambiguous case. Each required part is computed from its sine. Since $\sin(180° - \theta) = \sin \theta$, we shall find for each required part (provided the logarithm of its sine is negative) two values, one less, and one greater, than 90°. Choose one of the values of c and call it c_1. To find which of the two values of a, and of α, belong with it, use the rules of species. The other values of a and of α must, similarly, go with c_2.

* The notation (n), placed after a logarithm, means that the number whose logarithm is given is a negative number. Observe (a) that the cosine, tangent, and cotangent of an angle in the second quadrant are negative (b) that a product, or quotient, involving an odd number of negative factors is negative and one involving an even number of negative factors is positive.

† This check shows only that the computed logarithms are consistent. A more convincing, but also more laborious, check may be obtained by making the check computation using the natural functions.

Given:	Find:	
$b = 137° 25.2'$,	$c_1 = 47° 38.2'$,	$c_2 = 132° 21.8'$,
$\beta = 113° 41.6'$.	$a_1 = 156° 13.2'$,	$a_2 = 23° 46.8'$,
	$\alpha_1 = 146° 55.5'$.	$\alpha_2 = 33° 4.5'$.

$$\sin c_1 = \frac{\sin b}{\sin \beta}, \qquad \sin a_1 = \tan b \cot \beta,$$

$$c_2 = 180° - c_1, \qquad a_2 = 180° - a_1,$$

$$\sin \alpha_1 = \frac{\cos \beta}{\cos b}, \qquad \sin a_1 = \sin c_1 \sin \alpha_1.$$

$$\alpha_2 = 180° - \alpha_1,$$

$$\begin{aligned} \log \sin b &= 19.83034 - 20 \\ \log \sin \beta &= \underline{9.96176 - 10} - \\ \log \sin c_1 &= 9.86858 - 10 \end{aligned} \qquad \begin{aligned} \log \tan b &= 9.96327 - 10 \text{ (n)} \\ \log \cot \beta &= \underline{9.64229 - 10 \text{ (n)}} + \\ \log \sin a_1 &= 9.60556 - 10 \end{aligned}$$

$$\begin{aligned} \log \cos \beta &= 19.60405 - 20 \text{ (n)} \\ \log \cos b &= \underline{9.86707 - 10 \text{ (n)}} - \\ \log \sin \alpha_1 &= 9.73698 - 10 \end{aligned} \qquad \begin{aligned} \log \sin c_1 &= 9.86858 - 10 \\ \log \sin \alpha_1 &= \underline{9.73698 - 10} + \\ \log \sin a_1 &= 9.60556 - 10 \end{aligned}$$

Denote the computed acute angle c by c_1 and the obtuse angle by c_2. By the laws of species, it follows that a_1 is obtuse and a_2 is acute. It now follows that α_1 is obtuse and α_2 is acute.

Exercises

Solve the following right spherical triangles.

1. $a = 56° 34.0'$, $c = 75° 17.0'$.
2. $b = 143° 52.0'$, $c = 98° 54.0'$.
3. $a = 65° 34.5'$, $\beta = 113° 21.4'$.
4. $b = 127° 49.4'$, $\alpha = 116° 38.5'$.
5. $a = 132° 27.4'$, $b = 78° 19.2'$.
6. $\beta = 118° 20.6'$, $c = 81° 7.6'$.
7. $\alpha = 74° 51.3'$, $c = 61° 48.7'$.
8. $\alpha = 103° 41.3'$, $\beta = 117° 11.4'$.
9. $a = 41° 39.3'$, $\alpha = 66° 13.4'$.
10. $b = 61° 43.4'$, $\beta = 54° 19.8'$.
11. $b = 125° 28.6'$, $c = 104° 16.9'$.
12. $a = 63° 54.1'$, $c = 74° 14.9'$.
13. $b = 43° 28.4'$, $\alpha = 67° 47.2'$.
14. $a = 29° 38.6'$, $\beta = 41° 47.3'$.
15. $\beta = 118° 25.4'$, $c = 143° 19.8'$.
16. $a = 72° 16.8'$, $b = 26° 39.2'$.
17. $\alpha = 64° 43.9'$, $\beta = 55° 29.2'$.
18. $\alpha = 98° 52.8'$, $c = 119° 21.5'$.
19. $b = 118° 54.3'$, $\beta = 107° 24.2'$.
20. $a = 77° 34.2'$, $\alpha = 80° 9.5'$.

21. Find the distance, in nautical miles, and the bearing of Greenwich, Eng. (51° 29′ N, 0°) from New Orleans, La. (29° 57′ N, 90° W).

HINT. Take the third vertex of the triangle at the north pole.

22. Two ships, traveling on great circles, pass at an angle of 90°. If one is going 21 knots (i.e., 21 nautical miles an hour) and the other 17 knots, how far are they apart at the end of 12 hours?

23. A ship is approaching Boston, Mass. (42° 21′ N, 71° 4′ W) along a great circle which is perpendicular to the meridian at Boston. When it crosses

the meridian of 58° W, find its latitude, the bearing of Boston from the ship, and its distance, in nautical miles, from Boston.

24. On a sphere one foot in diameter, the length of the hypotenuse of a right spherical triangle is 11 inches and, of one side, is 7 inches. Find the length of the other side and the area of the triangle.

25. For an observer in latitude 33° 15′ N, the altitude of a star is 17° 30′ and its azimuth is N 90° W. Find its hour angle and declination.

26. The hour angle of a star is 90°, its declination is 57° 16′, and its altitude is 25° 33′. Find its azimuth and the latitude of the observer.

340. Quadrantal Triangles. A spherical triangle in which one side is 90° is a **quadrantal triangle.** From the formulas of Art. 332, it follows that the polar triangle of a quadrantal triangle is a right triangle. Hence, if we are given two parts of a quadrantal triangle, in addition to the 90° side, we can solve its polar triangle and, from these results, we can find the required parts of the given triangle.

EXAMPLE. Solve the quadrantal triangle: $c = 90°$, $a = 67° 24'$, $b = 124° 47'$.

For the polar triangle, we have

Given:

$$\alpha' = 112° 36', \quad \beta' = 55° 13'.$$

Find:

$$a' = 117° 53.9', \quad b' = 51° 50.1', \quad c' = 106° 48.3'.$$

$$\cos c' = \cot \alpha' \cot \beta', \qquad \cos a' = \frac{\cos \alpha'}{\sin \beta'},$$

$$\cos b' = \frac{\cos \beta'}{\sin \alpha'}, \qquad \cos c' = \cos a' \cos b'.$$

$$\begin{array}{lll} \log \cot \alpha' = 9.61936 - 10 \text{ (n)} & & \log \cos \alpha' = 19.58467 - 20 \text{ (n)} \\ \log \cot \beta' = 9.84173 - 10 & + & \log \sin \beta' = 9.91451 - 10 \quad - \\ \log \cos c' = 9.46109 - 10 \text{ (n)} & & \log \cos a' = 9.67016 - 10 \text{ (n)} \end{array}$$

$$\begin{array}{lll} \log \cos \beta' = 19.75624 - 20 & & \log \cos a' = 9.67016 - 10 \text{ (n)} \\ \log \sin \alpha' = 9.96530 - 10 & - & \log \cos b' = 9.79094 - 10 \quad + \\ \log \cos b' = 9.79094 - 10 & & \log \cos c' = 9.46110 - 10 \text{ (n)} \end{array}$$

For the given quadrantal triangle, we now have

$$\alpha = 180° - a' = 62° 6.1', \quad \beta = 180° - b' = 128° 9.9', \quad \gamma = 180° - c' = 73° 11.7'.$$

Exercises

Solve the quadrantal triangle, with $c = 90°$, given:

1. $\alpha = 112° 43'$, $b = 46° 52'$.
2. $a = 117° 31'$, $b = 129° 11'$.
3. $\gamma = 154° 19'$, $b = 70° 44'$.
4. $\alpha = 57° 13'$, $\beta = 68° 51'$.

5. Using a triangle having the north pole as one vertex, find the distance and bearing of Gibraltar (36° 6′ N, 5° 21′ W) from Entebbe, Uganda, (0°, 32° 20′ E).

6. Find the local solar time of sunset at San Francisco, Calif., latitude 37° 47′ N, on June 22, given that the declination of the sun is 23° 27′.

HINT. At sunset, the altitude of the sun is zero.

7. On Dec. 22 (sun's declination − 23° 27′), at a certain place in the northern hemisphere, the sun sets exactly in the southwest. Find the latitude of the place and the local solar time of sunset.

341. Isosceles Spherical Triangles. A spherical triangle is isosceles if two of its sides are equal. Let the triangle ABC (Fig. 210) be isosceles with $a = b$. Then the angles α and β, opposite these sides, are also equal. Further, if a great circle CD is passed through C perpendicular to AB, then this circle bisects the angle γ at C and forms two right spherical triangles ADC and BDC whose parts are respectively equal. By solving one of these right triangles, we can, accordingly, solve the given isosceles triangle.

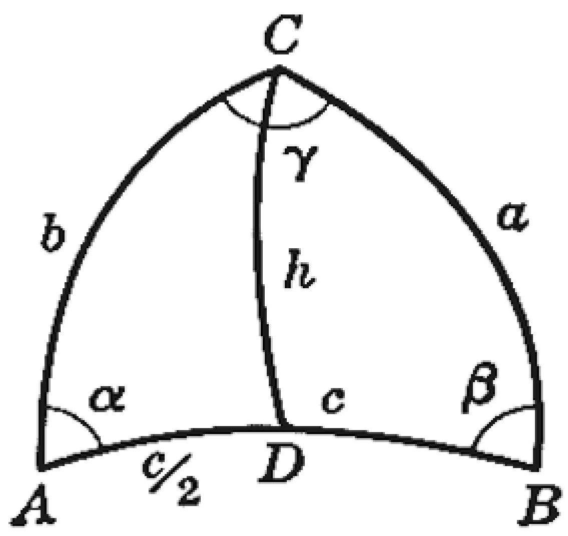


FIG. 210

EXAMPLE. Solve the isosceles spherical triangle, and find its altitude h, given $a = b = 74° 32.6'$, $\gamma = 126° 39'$.

We draw the perpendicular $CD = h$ and solve the right triangle ADC. We have $AD = c/2$ and angle $ACD = \gamma/2$.

Given:	Find:
$b = 74° 32.6'$,	$\alpha = 62° 3.3'$,
$\gamma/2 = 63° 19.5'$.	$c/2 = 59° 27.4'$,
	$h = 58° 22.2'$.

$$\cot \alpha = \cos b \tan \gamma/2, \qquad \tan h = \tan b \cos \gamma/2,$$

$$\sin c/2 = \sin b \sin \gamma/2, \qquad \sin c/2 = \cot \alpha \tan h.$$

$$\begin{array}{rll} \log \cos b = & 9.42571 - 10 & \\ \log \tan \gamma/2 = & \underline{0.29895 \qquad} & + \\ \log \cot \alpha = & 9.72466 - 10 & \end{array} \qquad \begin{array}{rll} \log \tan b = & 0.55829 & \\ \log \cos \gamma/2 = & \underline{9.65218 - 10} & + \\ \log \tan h = & 0.21047 & \end{array}$$

$$\begin{array}{rll} \log \sin b = & 9.98400 - 10 & \\ \log \sin \gamma/2 = & \underline{9.95113 - 10} & + \\ \log \sin c/2 = & 9.93513 - 10 & \end{array} \qquad \begin{array}{rll} \log \cot \alpha = & 9.72466 - 10 & \\ \log \tan h = & \underline{0.21047 \qquad} & + \\ \log \sin c/2 = & 9.93513 - 10 & \end{array}$$

For the given isosceles triangle, we have, accordingly, $\alpha = \beta = 62° 3.3'$; $c = 118° 54.8'$ and $h = 58° 22.2'$.

Exercises

Solve the isosceles triangles.

1. $b = c = 153° 32.0'$, $\alpha = 147° 22.0'$.
2. $a = c = 37° 48.0'$, $\beta = 80° 56.0'$.
3. $\alpha = \beta = 28° 19.0'$, $c = 73° 28.0'$.

4. $\beta = \gamma = 115° 43.0'$, $\alpha = 125° 38.0'$.

5. An airplane was flown along a great circle from Quebec (46° 48′ N, 71° 13′ W) to a point near Olympia, Wash. (46° 48′ N, 122° 52′ W). Find (*a*) the bearing of the destination from Quebec, (*b*) the latitude of the most northerly point on the path (i.e., of the foot of the perpendicular from the pole to the path) and, in statute miles, (*c*) the distance flown and (*d*) how much the distance would have been increased if it had been flown directly west.

6. A star passed through the zenith of an observer in Los Angeles (34° 3′ N) at 7:43 P.M. Find its altitude and azimuth at 11:21 P.M.

Chapter 41

The Oblique Spherical Triangle

342. The Law of Sines. *In a spherical triangle, the sines of the sides are proportional to the sines of the opposite angles;* that is

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}. \tag{1}$$

Through the vertex C, draw an arc of a great circle perpendicular to AB and intersecting AB (produced if necessary) at D. This construction yields two right triangles, ADC and BDC. Denote CD by h.

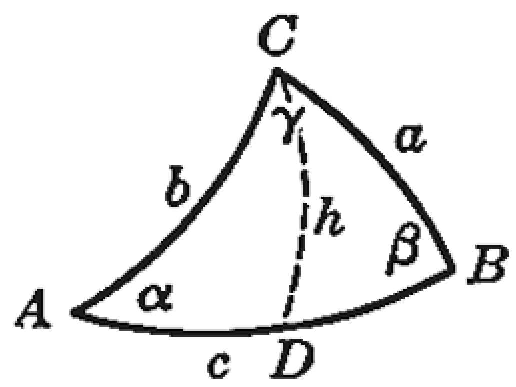


FIG. 211*a*

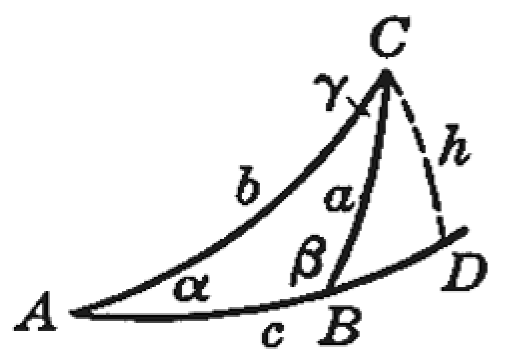


FIG. 211*b*

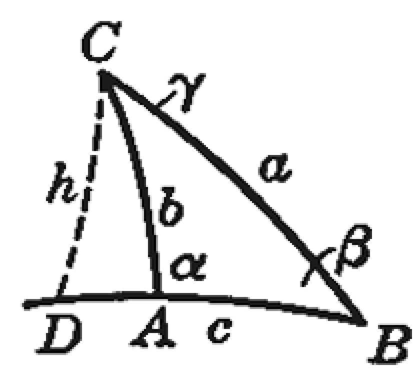


FIG. 211*c*

From Napier's Rules, (Art. 327) we have from the triangles DBC and DAC,

$$\sin h = \sin a \sin DBC, \quad \text{and} \quad \sin h = \sin b \sin DAC.$$

Since $DBC = \beta$ or $180° - \beta$, and $DAC = \alpha$ or $180° - \alpha$, these two equations reduce to

$$\sin h = \sin a \sin \beta, \quad \text{and} \quad \sin h = \sin b \sin \alpha.$$

Equate these two values of $\sin h$ and divide by $\sin \alpha \sin \beta$. We have

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta}.$$

Similarly, if we draw the perpendicular from A to BC, we find that

$$\frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}.$$

These two equations, taken together, give equations (1).

343. The Law of Cosines for Sides. This law states that, *in a spherical triangle,*

$$\begin{aligned} \cos a &= \cos b \cos c + \sin b \sin c \cos \alpha, \\ \cos b &= \cos c \cos a + \sin c \sin a \cos \beta, \\ \cos c &= \cos a \cos b + \sin a \sin b \cos \gamma. \end{aligned} \tag{2}$$

The following proof is for Figure 211*a*. With a few slight changes, it can be modified to apply to either of the other two cases.

Since $DB = c - AD$, we have, on applying Napier's Rules to the right triangles BDC and ADC,

$$\cos a = \cos h \cos(c - AD), \quad \text{and} \quad \cos b = \cos h \cos AD.$$

Hence

$$\frac{\cos a}{\cos b} = \frac{\cos h \cos(c - AD)}{\cos h \cos AD} = \frac{\cos c \cos AD + \sin c \sin AD}{\cos AD}.$$

Multiply by $\cos b$ and simplify. We have

$$\cos a = \cos b \cos c + \sin c \cos b \tan AD. \tag{3}$$

From the triangle ADC, we have further, by Napier's Rules,

$$\cos \alpha = \cot b \tan AD, \quad \text{or} \quad \sin b \cos \alpha = \cos b \tan AD.$$

On substituting this value for $\cos b \tan AD$ in (3), we get

$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha.$$

This is the first of equations (2). The other two may be derived in a similar way.

344. The Law of Cosines for Angles. If we apply the first of equations (2) to $A'B'C'$, the polar triangle of ABC, we have

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos \alpha'.$$

If, in this equation, we replace a', b', c', and α' by their values from Art. 332, apply the trigonometric formulas

$$\cos(180° - \theta) = -\cos \theta, \quad \text{and} \quad \sin(180° - \theta) = \sin \theta,$$

and multiply the resulting equation by -1, we obtain

$$\begin{aligned} \cos \alpha &= -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cos a. \\ \text{Similarly,} \quad \cos \beta &= -\cos \gamma \cos \alpha + \sin \gamma \sin \alpha \cos b. \\ \cos \gamma &= -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cos c. \end{aligned} \tag{4}$$

Equations (4) constitute the Law of Cosines for Angles.

The six laws of cosines, equations (2) and (4), are not adapted to logarithmic computation but they can sometimes be used to advantage for computations with the natural functions.

345. The Half-Side Formulas. If we let

$$s = \tfrac{1}{2}(a + b + c), \tag{5}$$

it follows by a simple algebraic computation that

$$s - a = \tfrac{1}{2}(b + c - a), \quad s - b = \tfrac{1}{2}(a + c - b), \quad \text{and} \quad s - c = \tfrac{1}{2}(a + b - c).$$

Further let

$$r = \sqrt{\frac{\sin(s - a) \sin(s - b) \sin(s - c)}{\sin s}}. \tag{6}$$

We shall show that

$$\tan\frac{\alpha}{2} = \frac{r}{\sin(s-a)}, \quad \tan\frac{\beta}{2} = \frac{r}{\sin(s-b)}, \quad \tan\frac{\gamma}{2} = \frac{r}{\sin(s-c)}. \qquad (7)$$

From the formulas of Chapter XVI and the law of cosines for sides, we have

$$2\sin^2\frac{\alpha}{2} = 1 - \cos\alpha = 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\sin b \sin c + \cos b \cos c - \cos a}{\sin b \sin c}$$

$$= \frac{\cos(b-c) - \cos a}{\sin b \sin c} = \frac{2\sin(s-c)\sin(s-b)}{\sin b \sin c}.$$

Hence, $$\sin^2\frac{\alpha}{2} = \frac{\sin(s-c)\sin(s-b)}{\sin b \sin c}. \qquad (8)$$

In a similar way, we find that

$$2\cos^2\frac{\alpha}{2} = 1 + \cos\alpha = 1 + \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\cos a - \cos(b+c)}{\sin b \sin c} = \frac{2\sin s \sin(s-a)}{\sin b \sin c}.$$

Hence, $$\cos^2\frac{\alpha}{2} = \frac{\sin s \sin(s-a)}{\sin b \sin c}. \qquad (9)$$

If we divide (8) by (9), multiply numerator and denominator by $\sin(s-a)$, and take the square root of both sides, we have

$$\tan\frac{\alpha}{2} = \frac{1}{\sin(s-a)}\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}} = \frac{r}{\sin(s-a)}.$$

The radical is taken as positive because $\alpha/2 < 90°$ and $s - a < 180°$.

The other two formulas (7) are obtained in a similar way.

346. The Half-Angle Formulas. Let

$$S = \tfrac{1}{2}(\alpha+\beta+\gamma), \quad R = \sqrt{\frac{-\cos S}{\cos(S-\alpha)\cos(S-\beta)\cos(S-\gamma)}}. \qquad (10)$$

We shall show that

$$\tan\frac{a}{2} = R\cos(S-\alpha), \; \tan\frac{b}{2} = R\cos(S-\beta), \; \tan\frac{c}{2} = R\cos(S-\gamma) \qquad (11)$$

If we form equations (5), (6), and (7) for the polar triangle, then replace a' by $180° - \alpha$, α' by $180° - a$, and so on, we find that

$$s' = \tfrac{1}{2}(a'+b'+c') = 270° - S, \; s' - a' = 90° - (S-\alpha), \text{ etc.}$$

It follows that

$$r' = \sqrt{\frac{\sin(s' - a') \sin(s' - b') \sin(s' - c')}{\sin s'}}$$

$$= \sqrt{\frac{\cos(S - \alpha) \cos(S - \beta) \cos(S - \gamma)}{- \cos S}} = \frac{1}{R}.$$

It now follows further that

$$\tan \frac{\alpha'}{2} = \frac{r'}{\sin(s' - a')} = \cot \frac{a}{2} = \frac{1}{R \cos(S - \alpha)};$$

that is,

$$\tan \frac{a}{2} = R \cos(S - \alpha),$$

and similarly for the other two formulas.

347. Napier's Analogies. The following four equations, together with eight others which may be obtained from them by interchanging corresponding letters, are called Napier's analogies, the word "analogy" being used in an obsolete sense with the meaning of "proportion." These formulas may be derived from those of the preceding two articles but are given here without proof.

$$\tan \tfrac{1}{2}(\alpha + \beta) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \frac{\gamma}{2} \tag{12}$$

$$\tan \tfrac{1}{2}(\alpha - \beta) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cot \frac{\gamma}{2} \tag{13}$$

$$\tan \tfrac{1}{2}(a + b) = \frac{\cos \frac{1}{2}(\alpha - \beta)}{\cos \frac{1}{2}(\alpha + \beta)} \tan \frac{c}{2} \tag{14}$$

$$\tan \tfrac{1}{2}(a - b) = \frac{\sin \frac{1}{2}(\alpha - \beta)}{\sin \frac{1}{2}(\alpha + \beta)} \tan \frac{c}{2} \tag{15}$$

348. Some Laws of Magnitude and Species. The following theorems are sometimes helpful in determining whether or not a required triangle exists and, if it does exist, in what quadrant a required part must lie.

I. *If two sides of a spherical triangle are unequal, the angles opposite are unequal and the greater angle is opposite the greater side.*

II. *The half-sum of two sides is of the same species as the half-sum of the opposite angles.*

III. *If any side differs from* 90° *more than either of the other sides does, it is of the same species as its opposite angle.*

IV. *If any angle differs from* 90° *more than either of the other angles does, it is of the same species as its opposite side.*

To prove I, we observe, from (13), since $\cot \gamma/2$ and $\sin(a + b)/2$ are both positive, that $\tan(\alpha - \beta)/2$ and $\sin(a - b)/2$ agree in sign. It follows that $\alpha > \beta$ if, and only if, $a > b$.

To prove II, we observe from (12), since $\cot \gamma/2$ and $\cos (a - b)/2$

are both positive, that $\tan(\alpha + \beta)/2$ and $\cos(a + b)/2$ agree in sign. It follows that $(\alpha + \beta)/2$ and $(a + b)/2$ are both less, or both greater, than 90°.

Theorem III follows from equations (2). We may write the first of equations (2) in the following form

$$\cos \alpha = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

If a differs from 90° more than b (or more than c) does, then $\cos a$ is numerically larger than $\cos b$ (or $\cos c$) and, consequently, is larger than the product $\cos b \cos c$. It follows that the sign of the entire second member of the above equation agrees with that of $\cos a$. For, since neither b nor c exceeds 180°, the denominator is positive and, since $\cos a$ is numerically larger than $\cos b \cos c$, the sign of the numerator (and hence of the fraction) agrees with that of $\cos a$. It now follows from the above equation that $\cos a$ and $\cos \alpha$ agree in sign, so that both are less, or both are greater, than 90°. Theorem IV follows in the same way from equations (3).

The laws stated in this article will be found important when we come to the solution of the ambiguous cases (Art. 352). If, for example, a, b, and α are given we first solve for β by the first equation of the law of sines. If there is one solution, β_1, then, equally, $\beta_2 = 180° - \beta_1$ is also a solution but each of these must be checked by the laws of this article to see if the corresponding triangle actually exists.

349. The Solution of Spherical Triangles. If we have given any three parts of a spherical triangle, the other three parts can be found by using the formulas of Arts. 342 and 345 to 347. We shall use the law of sines [equations (1)] throughout, as check formulas.

There are six cases, according as we have given:

I. The three sides.
II. The three angles.
III. Two sides and the included angle.
IV. Two angles and the included side.
V. Two sides and the angle opposite one of them.
VI. Two angles and the side opposite one of them.

We shall group these six cases into three pairs in such a way that the processes of solution of the two cases of each pair are closely similar.

350. Cases I and II. Given Three Sides or Three Angles. If the three sides are given, the three angles may be found by using formulas (5), (6), and (7) of Art. 345. If the three angles are given, we find the sides from formulas (10) and (11) of Art. 346. In either case, the results may be checked by the law of sines. The work may be arranged as shown in the following example.

EXAMPLE. Solve the triangle: $a = 102° 38.3'$, $b = 61° 17.3'$, $c = 74° 31.8'$.

Given:	Find:
$a = 102° 38.3'$,	$\alpha = 114° 14.0'$,
$b = 61° 17.3'$,	$\beta = 55° 3.0'$,
$c = 74° 31.8'$.	$\gamma = 64° 14.8'$.
$2)\overline{238 \quad 27.4}$	
$s = 119° 13.7'$	

$$s = \frac{1}{2}(a + b + c),\ r = \sqrt{\frac{\sin (s - a) \sin (s - b) \sin (s - c)}{\sin s}}$$

$$\tan \frac{\alpha}{2} = \frac{r}{\sin (s - a)},\ \tan \frac{\beta}{2} = \frac{r}{\sin (s - b)},\ \tan \frac{\gamma}{2} = \frac{r}{\sin(s - c)}$$

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}.$$

$$\begin{aligned}
s - a &= 16° 35.4' & \log \sin(s - a) &= 9.45564 - 10\\
s - b &= 57° 56.4' & \log \sin(s - b) &= 9.92813 - 10\\
s - c &= 44° 41.9' & \log \sin(s - c) &= \underline{9.84719 - 10}\,+\\
& & &29.23096 - 30\\
& & \log \sin s &= \underline{9.94085 - 10}\,-\\
& & &2)\overline{19.29011 - 20}\\
& & \log r &= 9.64506 - 10
\end{aligned}$$

$$\begin{aligned}
\alpha/2 &= 57° \ 7.0' & \log \tan \alpha/2 &= 0.18942\\
\beta/2 &= 27° 31.5' & \log \tan \beta/2 &= 9.71693 - 10\\
\gamma/2 &= 32° \ 7.4' & \log \tan \gamma/2 &= 9.79787 - 10
\end{aligned}$$

$$\begin{aligned}
\log \sin a &= 9.98935 - 10 & \log \sin b &= 9.94302 - 10\\
\log \sin \alpha &= \underline{9.95994 - 10}\,- & \log \sin \beta &= \underline{9.91363 - 10}\,-\\
&0.02941 & &0.02939
\end{aligned}$$

$$\begin{aligned}
\log \sin c &= 9.98397 - 10\\
\log \sin \gamma &= \underline{9.95457 - 10}\,-\\
&0.02940
\end{aligned}$$

Exercises

Solve the following spherical triangles.

1. $a = 76° 14.7'$, $b = 85° 21.4'$, $c = 53° 41.7'$.
2. $a = 110° 51.7'$, $b = 73° 14.9'$, $c = 55° 19.2'$.
3. $a = 125° 18.4'$, $b = 117° 15.7'$, $c = 84° 22.7'$.
4. $a = 69° 11.4'$, $b = 76° 39.5'$, $c = 81° 19.6'$.
5. $\alpha = 72° 43.0'$, $\beta = 79° 5.6'$, $\gamma = 67° 33.4'$.
6. $\alpha = 119° 23.4'$, $\beta = 74° 43.8'$, $\gamma = 51° 49.3'$.
7. $\alpha = 127° 44.2'$, $\beta = 118° 4.2'$, $\gamma = 82° 28.4'$.
8. $\alpha = 77° 28.4'$, $\beta = 71° 47.6'$, $\gamma = 123° 51.6'$.
9. Using the formula of Art. 331, find the areas of the triangles of Ex. 5 to 8, assuming that they lie on the surface of the earth. Take $R = 3959$ miles.

10. The bearing of Paris (2° 22′ E) from New York (73° 58′ W) is N 53° 44′ E. The bearing of New York from Paris is N 68° 12′ W. Find the latitude of both places and the distance, in nautical miles, between them.

11. On a sphere of radius one foot, the angles of a triangle are 123° 41′, 81° 14′, and 73° 46′. Find the lengths of the sides in inches.

12. Find the morning solar time of an observer in Annapolis, Md. (38° 59′ N) given that the sun's altitude is 28° 16′ and its declination is − 12° 54′.

351. Cases III and IV. **Given Two Sides and the Included Angle or Two Angles and the Included Side.** If a, b, and γ are given, we can find $(\alpha+\beta)/2$ and $(\alpha-\beta)/2$ from formulas (12) and (13) of Art. 347. The side c can then be found from formula 15. Similarly, if α, β, and c are given, we can find a and b from formulas (14) and (15) and then find γ from (13). The results may be checked by the law of sines.

If a and b are given, with $b > a$, or α and β with $\beta > \alpha$, interchange a and b and also α and β in formulas (12) to (15).

Example. Solve the triangle: $b = 118° 45.2′$, $c = 81° 45.2′$, $\alpha = 94° 36.4′$.

Given:	Find:
$b = 118° 21.4′$,	$\beta = 117° 40.1′$,
$c = 81° 45.2′$,	$\gamma = 84° 52.5′$,
$\alpha = 94° 36.4′$.	$a = 97° 56.2′$.

$$\tan \tfrac{1}{2}(\beta-\gamma) = \frac{\sin \frac{1}{2}(b-c)}{\sin \frac{1}{2}(b+c)} \cot \tfrac{1}{2}\alpha, \qquad \tan \tfrac{1}{2}(\beta+\gamma) = \frac{\cos \frac{1}{2}(b-c)}{\cos \frac{1}{2}(b+c)} \cot \tfrac{1}{2}\alpha,$$

$$\tan \tfrac{1}{2}a = \frac{\sin \frac{1}{2}(\beta+\gamma)}{\sin \frac{1}{2}(\beta-\gamma)} \tan \tfrac{1}{2}(b-c). \qquad \frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}.$$

$(b-c)/2 = 18° 18.1′$, $(b+c)/2 = 100° 3.3′$, $\alpha/2 = 47° 18.2′$.

$$\begin{aligned}
\log \sin(b-c)/2 &= 9.49696 - 10 \\
\log \cot \alpha/2 &= \underline{9.96505 - 10} + \\
& 19.46201 - 20 \\
\log \sin(b+c)/2 &= \underline{9.99327 - 10} - \\
\log \tan(\beta-\gamma)/2 &= 9.46874 - 10 \\
(\beta-\gamma)/2 &= 16° 23.8′.
\end{aligned}
\qquad
\begin{aligned}
\log \cos(b-c)/2 &= 9.97746 - 10 \\
\log \cot \alpha/2 &= \underline{9.96505 - 10} \quad + \\
& 9.94251 - 10 \\
\log \cos(b+c)/2 &= \underline{9.24203 - 10}\ (n) - \\
\log \tan(\beta+\gamma)/2 &= 0.70048 \quad (n) \\
(\beta+\gamma)/2 &= 101° 16.3′.
\end{aligned}$$

$$\begin{aligned}
\log \sin(\beta+\gamma)/2 &= 9.99154 - 10 \\
\log \tan(b-c)/2 &= \underline{9.51950 - 10} + \\
& 9.51104 - 10 \\
\log \sin(\beta-\gamma)/2 &= \underline{9.45069 - 10} - \\
\log \tan a/2 &= 0.06035 \\
a/2 &= 48° 58.1′.
\end{aligned}
\qquad
\begin{aligned}
\log \sin a &= 19.99582 - 10 \\
\log \sin \alpha &= \underline{9.99860 - 10} - \\
& 9.99722 - 10 \\
\log \sin b &= 19.94449 - 10 \\
\log \sin \beta &= \underline{9.94726 - 10} - \\
& 9.99723 - 10 \\
\log \sin c &= 19.99548 - 10 \\
\log \sin \gamma &= \underline{9.99826 - 10} - \\
& 9.99722 - 10
\end{aligned}$$

Exercises

Solve the triangles.

1. $a = 59° 31.0'$,	$b = 23° 17.0'$,	$\gamma = 114° 42.0'$.
2. $a = 97° 16.4'$,	$b = 48° 11.8'$,	$\gamma = 76° 21.8'$.
3. $b = 124° 31.7'$,	$c = 81° 49.5'$,	$\alpha = 111° 52.6'$.
4. $b = 103° 41.5'$,	$c = 129° 16.1'$,	$\alpha = 69° 24.8'$.
5. $a = 121° 50.9'$,	$c = 80° 47.3'$,	$\beta = 64° 37.4'$.
6. $a = 56° 19.1'$,	$c = 108° 21.6'$,	$\beta = 74° 49.5'$.
7. $\alpha = 71° 42.0'$,	$\beta = 54° 36.0'$,	$c = 128° 32.0'$.
8. $\alpha = 76° 41.2'$,	$\beta = 147° 25.6'$,	$c = 72° 54.2'$.
9. $\beta = 116° 37.6'$,	$\gamma = 41° 35.2'$,	$a = 39° 27.6'$.
10. $\beta = 75° 8.9'$,	$\gamma = 57° 54.3'$,	$a = 70° 51.4'$.
11. $\alpha = 131° 23.7'$,	$\gamma = 114° 11.7'$,	$b = 42° 35.8'$.
12. $\alpha = 114° 41.3'$,	$\gamma = 32° 47.5'$,	$b = 126° 54.2'$.

Find the distance, in nautical miles, and the bearing of each of the following cities from the other.

13. San Diego (32° 43′ N, 117° 10′ W) and Colon, Panama, (9° 23′ N, 79° 55′ W).

14. Washington (38° 55′ N, 77° 4′ W) and Moscow (55° 45′ N, 37° 34′ E).

15. Rio de Janeiro (22° 54′ S, 43° 10′ W) and Liverpool (53° 24′ N, 3° 4′ W).

16. Honolulu (21° 18′ N, 157° 52′ W) and Tokyo (35° 39′ N, 139° 45′ E).

17. An observer in latitude 28° 15′ finds the altitude of a star to be 41° 42′ and its azimuth to be N 116° 38′ E. Find the declination and hour angle of the star.

18. An observer in latitude 24° 10′ S finds that the azimuth of a star is 72° 37′ and that its hour angle is N 116° 43′ W. Find the altitude and declination of the star and the angle between the hour circle and the vertical circle of the star.

352. CASES V AND VI. **Given Two Sides and the Angle Opposite One of Them or Two Angles and the Side Opposite One of Them.** These two cases are the **ambiguous cases.** There may be two, one, or no solutions.

If, for example, a, b, and α are given, we first compute β from the law of sines. If log sin β is positive, there is no solution. If log sin β is negative, we find an acute angle β_1 and an obtuse angle β_2. Each of these angles must be tested by the theorems of Art. 348. If solutions exist, the values of c and γ may be found by formulas (13) and (15). The values of c and γ should be checked by the law of sines.

A suitable form for writing down the solution is illustrated by the following example.

EXAMPLE. Solve the triangle: $b = 53° 21.3'$, $c = 102° 47.5'$, $\beta = 47° 36.4'$.

Given:	Find:	
$b = 53° 21.3'$,	$\alpha_1 = 143° 5.0'$,	$\alpha_2 = 64° 11.2'$,
$c = 102° 47.5'$,	$\gamma_1 = 63° 50.7'$,	$\gamma_2 = 116° 9.3'$,
$\beta = 47° 36.4'$.	$a_1 = 139° 15.8'$,	$a_2 = 77° 57.6'$.

$$\sin\gamma = \frac{\sin\beta\sin c}{\sin b},\quad \cot\tfrac{1}{2}\alpha = \frac{\sin\frac{1}{2}(c+b)}{\sin\frac{1}{2}(c-b)}\tan\tfrac{1}{2}(\gamma-\beta),$$

$$\tan\tfrac{1}{2}a = \frac{\sin\frac{1}{2}(\gamma+\beta)}{\sin\frac{1}{2}(\gamma-\beta)}\tan\tfrac{1}{2}(c-b),\quad \frac{\sin\alpha}{\sin a} = \frac{\sin\beta}{\sin b}.$$

$$\begin{aligned}
\log\sin\beta &= 19.86837 - 20\\
\log\sin b &= \underline{\ 9.90437 - 10\ } -\\
&\ \ 9.96400 - 10\\
\log\sin c &= \underline{\ 9.98908 - 10\ } +\\
\log\sin\gamma &= \ 9.95308 - 10
\end{aligned}$$

$(c+b)/2 = 78° 4.4'$, $(\gamma_1+\beta)/2 = 55° 43.55'$, $(\gamma_2+\beta)/2 = 81° 52.85'$,
$(c-b)/2 = 24° 43.1'$, $(\gamma_1-\beta)/2 = 8° 7.15'$, $(\gamma_2-\beta)/2 = 34° 16.45'$.

$$\begin{aligned}
\log\sin(c+b)/2 &= \ 9.99052 - 10\\
\log\tan(\gamma_1-\beta)/2 &= \underline{\ 9.15431 - 10\ } +\\
&\ 19.14483 - 20\\
\log\sin(c-b)/2 &= \underline{\ 9.62134 - 10\ } -\\
\log\cot\alpha_1/2 &= \ 9.52349 - 10
\end{aligned}
\qquad
\begin{aligned}
\log\sin(\gamma_1+\beta)/2 &= \ 9.91716 - 10\\
\log\tan(c-b)/2 &= \underline{\ 9.66307 - 10\ } +\\
&\ \ 9.58023 - 10\\
\log\sin(\gamma_1-\beta)/2 &= \underline{\ 9.14993 - 10\ } -\\
\log\tan a_1/2 &= \ 0.43030
\end{aligned}$$

$\alpha_1/2 = 71° 32.5'$. $\qquad a_1/2 = 69° 37.9'$.

$$\begin{aligned}
\log\sin(c+b)/2 &= \ 9.99052 - 10\\
\log\tan(\gamma_2-\beta)/2 &= \underline{\ 9.83346 - 10\ } +\\
&\ \ 9.82398 - 10\\
\log\sin(c-b)/2 &= \underline{\ 9.62134 - 10\ } -\\
\log\cot\alpha_2/2 &= \ 0.20264
\end{aligned}
\qquad
\begin{aligned}
\log\sin(\gamma_2+\beta)/2 &= \ 9.99563 - 10\\
\log\tan(c-b)/2 &= \underline{\ 9.66307 - 10\ } +\\
&\ 19.65870 - 10\\
\log\sin(\gamma_2-\beta)/2 &= \underline{\ 9.75063 - 10\ } -\\
\log\tan a_2/2 &= \ 9.90807 - 10
\end{aligned}$$

$\alpha_2/2 = 32° 5.6'$. $\qquad a_2/2 = 38° 58.8'$.

$$\begin{aligned}
\log\sin\alpha_1 &= 19.77862 - 20\\
\log\sin a_1 &= \underline{\ 9.81464 - 10\ } -\\
&\ \ 9.96398 - 10
\end{aligned}
\qquad
\begin{aligned}
\log\sin\alpha_2 &= 19.95435 - 20\\
\log\sin a_2 &= \underline{\ 9.99034 - 10\ } -\\
&\ \ 9.96401 - 10
\end{aligned}$$

The last two results should be compared with the value of log sin β − log sin b which was found at the beginning of the computation.

Exercises

Solve the triangles.

1. $a = 81° 36.4'$, $b = 44° 51.8'$, $\alpha = 40° 12.4'$.
2. $a = 98° 43.7'$, $c = 68° 15.4'$, $\alpha = 80° 38.4'$.
3. $b = 71° 49.3'$, $c = 47° 42.5'$, $\gamma = 44° 41.6'$.
4. $a = 82° 21.1'$, $b = 61° 14.5'$, $\beta = 53° 5.7'$.

5. $b = 24° 31.5'$, $c = 76° 53.1'$, $\beta = 72° 21.3'$.
6. $a = 69° 41.8'$, $c = 50° 45.3'$, $\gamma = 45° 57.2'$.
7. $\alpha = 36° 43.9'$, $\beta = 109° 24.3'$, $a = 20° 15.8'$.
8. $\beta = 72° 31.8'$, $\gamma = 117° 14.2'$, $c = 79° 42.6'$.
9. $\alpha = 121° 16.2'$, $\gamma = 43° 24.3'$, $a = 74° 50.8'$.
10. $\alpha = 76° 52.2'$, $\beta = 147° 27.4'$, $b = 148° 31.5'$.
11. $\beta = 138° 27.4'$, $\gamma = 91° 50.3'$, $b = 52° 35.4'$.
12. $\alpha = 162° 19.5'$, $\gamma = 116° 21.8'$, $c = 135° 41.5'$.

13. A ship sails from Honolulu (21° 18′ N, 157° 52′ W) bearing S 40° W. After traveling for 200 hours along a great circle, it crosses the parallel of 16° south latitude. Find the longitude of the point of crossing, the ship's bearing at that point, and its average speed in knots (nautical miles per hour).

14. An airplane, flying on a great circle from New York (40° 49′ N, 73° 58′ W) to Europe, crosses the meridian of 45° W bearing N 77° 30′ E. Find the latitude of the point of crossing, and its distance, in nautical miles, from New York.

15. At 20 minutes past 10 A.M., solar time, an observer in the north temperate zone found the altitude of the sun to be 41° 17′. If the sun's declination on that day was 9° 34′, find the latitude of the observer and the azimuth of the sun.

16. It is known that the declination of a certain star is 31° 46′ and that, on a certain day, it is over the meridian of Greenwich at 8:13 P.M. At 9:48 P.M. Greenwich time, on that day, an observer found the altitude of the star to be 56° 19′ and its azimuth to be N 107° 8′ E. Find the latitude and longitude of the observer.

Tables

Introduction to the Tables

Table I

Table I gives to five decimal places the mantissas of the logarithms of numbers of four significant figures.

1. To Find the Logarithm of a Number. First find the characteristic of the given number by the rules given in Art. 78. The mantissa is then found from Table I, as shown in the following examples.

EXAMPLE 1. Find log 48.35.

The characteristic is 1. To find the mantissa, look, first, in the column headed **N**, in Table I, for the first three significant figures, 483, of the given number. The first two figures, 68, of the required mantissa are immediately to the right of 483. To find the next three numbers, follow the line of 483 to the column headed **5** (the fourth significant figure of the given number). We find 440. The required mantissa is .68440 and the required logarithm is 1.68440.

EXAMPLE 2. Find log 0.06763.

The characteristic is $8 - 10$. Look in the column headed **N** for 676. Directly to the right is 82 but, when we follow the line across to the column headed **3**, we find an asterisk which means that 82 must be increased by unity to 83. The three numbers in the column headed **3** are 014. The required logarithm is $8.83014 - 10$.

2. Interpolation. If the given number contains five significant figures, its logarithm may be found, usually accurately, to five decimal places by the method of interpolation which is illustrated by the following example.

If the given number contains more than five significant figures, it should be rounded off to the nearest fifth figure, as shown in Art. 79.

EXAMPLE. Find log 329.76.

The characteristic is 2. To find its mantissa, we assume (with a small error which we neglect) since 329.76 is 0.6 of the way from 329.70 to 329.80, that log 329.76 is 0.6 of the way from log 329.70 to log 329.80.

The mantissa of the logarithm of 3297 is .51812 and that of 3298 is .51825. The difference between these two mantissas is .00013. To avoid the labor of computing 0.6 of this difference, we look in the column headed Prop. Parts (proportional parts) for the table headed **13** (the significant figures of the difference). In this table, just to the left of the light line, we look down until we come to 6. Just to the right of 6 we find 7.8 which is 0.6 of 13. Round off 7.8 to 8 and add .00008 to the mantissa of 3297, giving .51820. Hence $\log 329.76 = 2.51820$.

Exercises

Verify the following logarithms.

1. $\log 6.7348 = 0.82833$. **2.** $\log 0.0022953 = 7.36084 - 10$.
3. $\log 7458.9 = 3.87267$. **4.** $\log 0.30475 = 9.48394 - 10$.

3. Antilogarithms. To find the number **N** that has a given logarithm, we first look in the body of the table for the mantissa of the given logarithm and thereby determine the sequence of significant digits of N. The position of the decimal point is then fixed by the given characteristic.

EXAMPLE 1. Find **N**, given $\log \mathbf{N} = 8.91882 - 10$.

We first look in the column headed **0** for the first two figures, 91, of the given mantissa. Below, and to the right, we find the next three figures 882. These three numbers are in the row of 829 and in the column headed **5**. The required significant figures are 8295. Since the characteristic is $8 - 10$, $N = 0.08295$.

EXAMPLE 2. Find **N**, given $\log \mathbf{N} = 1.78847$.

This mantissa lies between .78845 and .78852. The smaller of these corresponds to 6144. These are the first four required digits. Subtract .78845 from .78852, giving .00007, and also from the given mantissa .78847, giving .00002. In the table of proportional parts headed **7**, look directly under the **7** for 2. The nearest number is 2.1 which corresponds to 3 (at the left of the light line) which is the required fifth digit. $\mathbf{N} = 61.443$.

Exercises

Verify the following antilogarithms.

1. $9.63889 - 10 = \log 0.4354$. **2.** $3.24879 = \log 1773.3$.
3. $8.83627 - 10 = 0.068592$. **4.** $1.52067 = \log 33.164$.

Table II

4. Logarithms of the Trigonometric Functions. Table II gives, to five decimal places, the logarithms of the sine, cosine, tangent, and cotangent of the angle for every minute from 0° to 90°. The numbers given in the table are, in every case, 10 greater than the required logarithm. If the angle under consideration is less than 45°, the number of degrees will be found at the top of the page, the number of minutes down the left-hand side and the logarithm in the column having the name of the function at the top of the page. If the angle is greater than 45°, the number of degrees and the name of the function are found at the bottom of the page and the number of minutes on the right-hand side.

For problems involving interpolation, the tables of proportional parts are computed for tenths of a minute. The differences between successive numbers in the table are indicated in the columns headed **d** and **cd.** For the sine and the tangent, the interpolation correction should be *added* to the mantissa of the next smaller minute as found in the table; for the cosine and the cotangent, it should be *subtracted* from the mantissa of the next smaller minute.

5. To Find the Logarithm of a Given Function of a Given Angle.

EXAMPLE 1. Find log sin 21° 43′.

From Table II, we find 21° at the top of the page and 43′ down the left-hand side. In the column having **L Sin** at the top of the page, we find, after subtracting 10 from the number given in the table, log sin 21° 43′ = 9.56822 − 10.

EXAMPLE 2. Find log cot 78° 16′.

We find 78° at the bottom of the page and 16′ on the right-hand side. Subtract 10 from the number in the column having **L Cot** at the bottom of the page, giving log cot 78° 16′ = 9.31743 − 10.

EXAMPLE 3. Find log sin 57° 21.5′.

We find log sin 57° 21′ = 9.92530. The difference for 1′, from the column headed **d**, is 8. From the table headed **8** in the column of proportional parts, we find that the correction for 0.5′ is 4. Hence, log sin 57° 21.5′ = 9.92534 − 10.

EXAMPLE 4. Find log cot 41° 52.6′.

We have log cot 41° 52′ = 0.04760. From the column headed **cd,** the difference for 1′ is 26. From the table under proportional parts, the correction for 0.6′ is 16. Since this correction must be subtracted, log cot 41° 52.6′ = 0.04744.

Exercises

Verify the following logarithms.

1. log sin 73° 21.8′ = 9.98143 − 10. **2.** log tan 18° 49.2′ = 9.53252 − 10.
3. log cos 79° 36.8′ = 9.25597 − 10. **4.** log cot 53° 18.4′ = 9.87227 − 10.

6. To Find the Angle, Given the Logarithm of a Given Function.

EXAMPLE 1. Find θ, given log sin θ = 9.74170 − 10.

Look for 9.74170 in a column having L Sin either at the top or bottom of the page. Since we find it in a column having L Sin at the top of the page, we take the number of degrees from the top of the page and the number of minutes at the left. 9.74170 = log sin 33° 29′.

EXAMPLE 2. Find θ, given log tan θ = 9.70132 − 10.

We locate the given logarithm between 9.70121 − 10 = log tan 26° 41′ and the next greater entry in the table. The difference for 1′ is 31. The difference

between .70121 and the given mantissa gives 11. From the table of proportional parts, the nearest tenth of a minute is 0.4. Hence $\theta = 26° 41.4'$.

EXAMPLE 3. Find θ, given $\log \cos \theta = 9.47917 - 10$.

We find $9.47934 - 10 = \log \cos 72° 27'$. The difference for $1'$ is 40. By subtracting the given mantissa from .47934, we get 17. From the table of proportional parts, the corresponding angle is 0.4. Hence $\theta = 72° 27.4'$. Notice that, for the cosine and the cotangent, we work from the mantissa in the table next larger than the given mantissa.

Exercises

Verify the following angles.

1. $9.91091 - 10 = \log \sin 54° 32.4'$. **2.** $9.81624 - 10 = \log \tan 33° 13.5'$.
3. $9.82119 - 10 = \log \cos 48° 30.5'$. **4.** $0.25515 = \log \cot 29° 3.7'$.

Table III

7. Natural Values of the Trigonometric Functions. Table III gives, to four decimal places, the actual values of the sine, cosine, tangent, and cotangent of the angle for every minute from 0° to 90°. The arrangement of the table and the method of using it parallels that of Table II except that, since no tables of proportional parts is given, interpolations, if desired, must be computed out by the student. In this course, however, this table will customarily be used only to the nearest minute.

EXAMPLE 1. Find the value of $\sin 29° 43'$.

We find the number of degrees in the angle at the top of the page and the number of minutes at the left-hand side. From the column having Sin at the top of the page, we find $\sin 29° 43' = 0.4957$.

EXAMPLE 2. Find $\tan 82° 18'$.

We find the number of degrees at the bottom of the page and the minutes at the right-hand side. From the column having Tan at the bottom of the page, we find $\tan 82° 18' = 7.3962$.

EXAMPLE 3. Find θ, given $\cos \theta = 0.2156$.

We find the given number in a column having Cos at the bottom of the page. Hence $\theta = 77° 33'$.

EXAMPLE 4. Find θ to the nearest minute, given $\cot \theta = 0.5382$.

This number lies between 0.5381 and 0.5384. Since the given number is nearer to 0.5381, we have, to the nearest minute, $\theta = 61° 43'$.

Exercises

Verify the following values of the given functions.

1. $\sin 67° 35' = 0.9244$.
2. $\tan 41° 34' = 0.8868$.
3. $\cos 78° 17' = 0.2031$.
4. $\cot 29° 22' = 1.7771$.

Verify the following angles to the nearest minute.

5. $0.7946 = \sin 52° 37'$.
6. $4.1335 = \tan 76° 24'$.
7. $0.3956 = \cos 66° 42'$.
8. $1.7109 = \cot 30° 18'$.

Table IV

8. Squares and Square Roots of Numbers. Table IV gives, to four significant figures, the squares and square roots of numbers from 1.00 to 10.00.

To find the square of a number given in the column headed **N**, look, on the line of the given number, for the number in the column headed $\mathbf{N}^2$. To find the square of 10 times a number in the column headed **N**, look for the square of **N** and multiply this result by 100. To find the square of 100**N**, first find the square of **N**, then multiply this by 10^4, and so on.

To find the square root of a number in the column headed **N**, take the number on the same line in the column headed $\sqrt{\mathbf{N}}$. To find the square root of 10 times a number **N**, take the number on the same line in the column headed $\sqrt{10\mathbf{N}}$. To find the square root of 100**N**, find $\sqrt{\mathbf{N}}$ and multiply it by 10, and so on.

Squares and square roots of numbers having four significant figures may be found by the method of interpolation explained in Art. 2 for Table I and illustrated in the following examples 3, 5, and 6.

Example 1. Find to four significant figures the value of 5.37^2.

Look down the column headed **N** for 5.37. Immediately to the right, in the column headed $\mathbf{N}^2$ is 28.84. Hence $5.37^2 = 28.84$ to four significant figures.

Example 2. Find the value of 273^2.

$273 = 100 \times 2.73$. From the table, $2.73^2 = 7.453$. Hence, to four significant figures, $273^2 = 74530$.

Example 3. Find the value of 71.38^2.

$71.38 = 10 \times 7.138$. From the table, $7.13^2 = 50.84$ and $7.14^2 = 50.98$. The difference between these results is .14 and 0.8 of this difference is .112 which we round off to .11. Add this to $7.13^2 = 50.84$ giving $7.138^2 = 50.95$. Hence $71.38^2 = 5095$.

Example 4. Find $\sqrt{7.58}$, $\sqrt{75.8}$, and $\sqrt{758}$.

Let $\mathbf{N} = 7.58$, then $10\mathbf{N} = 75.8$ and $100\mathbf{N} = 758$. Look for 7.58 in the column headed **N**. From the columns headed $\sqrt{\mathbf{N}}$ and $\sqrt{10\mathbf{N}}$, we find $\sqrt{7.58} = 2.753$ and $\sqrt{75.8} = 8.706$. Further $\sqrt{758} = 10\sqrt{7.58} = 27.53$.

Example 5. Find $\sqrt{5.837}$.

From the table, $\sqrt{5.83} = 2.415$ and $\sqrt{5.84} = 2.417$. The difference is .002 and 0.7 of this difference is .0014 which we round off to .001. Add this to $\sqrt{5.83} = 2.415$, giving $\sqrt{5.837} = 2.416$.

Example 6. Find $\sqrt{193500}$.

$193500 = 10^4 \times 19.35$. Hence $\sqrt{193500} = 100\sqrt{19.35}$. From the table, $\sqrt{19.3} = \sqrt{10 \times 1.93} = 4.393$ and $\sqrt{19.4} = 4.405$. Five tenths of the difference is .006. Add this to $\sqrt{19.3}$, giving $\sqrt{19.35} = 4.399$. Hence $\sqrt{193500} = 439.9$.

Exercises

Verify the following squares and square roots to four significant figures using Table IV.

1. $51.3^2 = 2632$.

2. $327.4^2 = 107200$.

3. $\sqrt{817} = 28.58$.

4. $\sqrt{6340} = 79.62$.

5. $\sqrt{346.3} = 18.61$.

6. $\sqrt{17.48} = 4.181$.

TABLE I

COMMON LOGARITHMS OF NUMBERS

FROM 1 TO 10000

TO FIVE DECIMAL PLACES

1–100

N	Log	N	Log	N	Log	N	Log	N	Log
0	—	**20**	1.30 103	**40**	1.60 206	**60**	1.77 815	**80**	1.90 309
1	0.00 000	21	1.32 222	41	1.61 278	61	1.78 533	81	1.90 849
2	0.30 103	22	1.34 242	42	1.62 325	62	1.79 239	82	1.91 381
3	0.47 712	23	1.36 173	43	1.63 347	63	1.79 934	83	1.91 908
4	0.60 206	24	1.38 021	44	1.64 345	64	1.80 618	84	1.92 428
5	0.69 897	25	1.39 794	45	1.65 321	65	1.81 291	85	1.92 942
6	0.77 815	26	1.41 497	46	1.66 276	66	1.81 954	86	1.93 450
7	0.84 510	27	1.43 136	47	1.67 210	67	1.82 607	87	1.93 952
8	0.90 309	28	1.44 716	48	1.68 124	68	1.83 251	88	1.94 448
9	0.95 424	29	1.46 240	49	1.69 020	69	1.83 885	89	1.94 939
10	1.00 000	**30**	1.47 712	**50**	1.69 897	**70**	1.84 510	**90**	1.95 424
11	1.04 139	31	1.49 136	51	1.70 757	71	1.85 126	91	1.95 904
12	1.07 918	32	1.50 515	52	1.71 600	72	1.85 733	92	1.96 379
13	1.11 394	33	1.51 851	53	1.72 428	73	1.86 332	93	1.96 848
14	1.14 613	34	1.53 148	54	1.73 239	74	1.86 923	94	1.97 313
15	1.17 609	35	1.54 407	55	1.74 036	75	1.87 506	95	1.97 772
16	1.20 412	36	1.55 630	56	1.74 819	76	1.88 081	96	1.98 227
17	1.23 045	37	1.56 820	57	1.75 587	77	1.88 649	97	1.98 677
18	1.25 527	38	1.57 978	58	1.76 343	78	1.89 209	98	1.99 123
19	1.27 875	39	1.59 106	59	1.77 085	79	1.89 763	99	1.99 564
20	1.30 103	**40**	1.60 206	**60**	1.77 815	**80**	1.90 309	**100**	2.00 000

N	0	1	2	3	4	5	6	7	8	9
100	00 000	043	087	130	173	217	260	303	346	389
101	00 432	475	518	561	604	647	689	732	775	817
102	00 860	903	945	988	*030	*072	*115	*157	*199	*242
103	01 284	326	368	410	452	494	536	578	620	662
104	01 703	745	787	828	870	912	953	995	*036	*078
105	02 119	160	202	243	284	325	366	407	449	490
106	02 531	572	612	653	694	735	776	816	857	898
107	02 938	979	*019	*060	*100	*141	*181	*222	*262	*302
108	03 342	383	423	463	503	543	583	623	663	703
109	03 743	782	822	862	902	941	981	*021	*060	*100
110	04 139	179	218	258	297	336	376	415	454	493
111	04 532	571	610	650	689	727	766	805	844	883
112	04 922	961	999	*038	*077	*115	*154	*192	*231	*269
113	05 308	346	385	423	461	500	538	576	614	652
114	05 690	729	767	805	843	881	918	956	994	*032
115	06 070	108	145	183	221	258	296	333	371	408
116	06 446	483	521	558	595	633	670	707	744	781
117	06 819	856	893	930	967	*004	*041	*078	*115	*151
118	07 188	225	262	298	335	372	408	445	482	518
119	07 555	591	628	664	700	737	773	809	846	882
120	07 918	954	990	*027	*063	*099	*135	*171	*207	*243
121	08 279	314	350	386	422	458	493	529	565	600
122	08 636	672	707	743	778	814	849	884	920	955
123	08 991	*026	*061	*096	*132	*167	*202	*237	*272	*307
124	09 342	377	412	447	482	517	552	587	621	656
125	09 691	726	760	795	830	864	899	934	968	*003
126	10 037	072	106	140	175	209	243	278	312	346
127	10 380	415	449	483	517	551	585	619	653	687
128	10 721	755	789	823	857	890	924	958	992	*025
129	11 059	093	126	160	193	227	261	294	327	361
130	11 394	428	461	494	528	561	594	628	661	694
131	11 727	760	793	826	860	893	926	959	992	*024
132	12 057	090	123	156	189	222	254	287	320	352
133	12 385	418	450	483	516	548	581	613	646	678
134	12 710	743	775	808	840	872	905	937	969	*001
135	13 033	066	098	130	162	194	226	258	290	322
136	13 354	386	418	450	481	513	545	577	609	640
137	13 672	704	735	767	799	830	862	893	925	956
138	13 988	*019	*051	*082	*114	*145	*176	*208	*239	*270
139	14 301	333	364	395	426	457	489	520	551	582
140	14 613	644	675	706	737	768	799	829	860	891
141	14 922	953	983	*014	*045	*076	*106	*137	*168	*198
142	15 229	259	290	320	351	381	412	442	473	503
143	15 534	564	594	625	655	685	715	746	776	806
144	15 836	866	897	927	957	987	*017	*047	*077	*107
145	16 137	167	197	227	256	286	316	346	376	406
146	16 435	465	495	524	554	584	613	643	673	702
147	16 732	761	791	820	850	879	909	938	967	997
148	17 026	056	085	114	143	173	202	231	260	289
149	17 319	348	377	406	435	464	493	522	551	580
150	17 609	638	667	696	725	754	782	811	840	869
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	44	43	42
1	4.4	4.3	4.2
2	8.8	8.6	8.4
3	13.2	12.9	12.6
4	17.6	17.2	16.8
5	22.0	21.5	21.0
6	26.4	25.8	25.2
7	30.8	30.1	29.4
8	35.2	34.4	33.6
9	39.6	38.7	37.8

	41	40	39
1	4.1	4.0	3.9
2	8.2	8.0	7.8
3	12.3	12.0	11.7
4	16.4	16.0	15.6
5	20.5	20.0	19.5
6	24.6	24.0	23.4
7	28.7	28.0	27.3
8	32.8	32.0	31.2
9	36.9	36.0	35.1

	38	37	36
1	3.8	3.7	3.6
2	7.6	7.4	7.2
3	11.4	11.1	10.8
4	15.2	14.8	14.4
5	19.0	18.5	18.0
6	22.8	22.2	21.6
7	26.6	25.9	25.2
8	30.4	29.6	28.8
9	34.2	33.3	32.4

	35	34	33
1	3.5	3.4	3.3
2	7.0	6.8	6.6
3	10.5	10.2	9.9
4	14.0	13.6	13.2
5	17.5	17.0	16.5
6	21.0	20.4	19.8
7	24.5	23.8	23.1
8	28.0	27.2	26.4
9	31.5	30.6	29.7

	32	31	30
1	3.2	3.1	3.0
2	6.4	6.2	6.0
3	9.6	9.3	9.0
4	12.8	12.4	12.0
5	16.0	15.5	15.0
6	19.2	18.6	18.0
7	22.4	21.7	21.0
8	25.6	24.8	24.0
9	28.8	27.9	27.0

Prop. Parts

N	0	1	2	3	4	5	6	7	8	9
150	17 609	638	667	696	725	754	782	811	840	869
151	17 898	926	955	984	*013	*041	*070	*099	*127	*156
152	18 184	213	241	270	298	327	355	384	412	441
153	18 469	498	526	554	583	611	639	667	696	724
154	18 752	780	808	837	865	893	921	949	977	*005
155	19 033	061	089	117	145	173	201	229	257	285
156	19 312	340	368	396	424	451	479	507	535	562
157	19 590	618	645	673	700	728	756	783	811	838
158	19 866	893	921	948	976	*003	*030	*058	*085	*112
159	20 140	167	194	222	249	276	303	330	358	385
160	20 412	439	466	493	520	548	575	602	629	656
161	20 683	710	737	763	790	817	844	871	898	925
162	20 952	978	*005	*032	*059	*085	*112	*139	*165	*192
163	21 219	245	272	299	325	352	378	405	431	458
164	21 484	511	537	564	590	617	643	669	696	722
165	21 748	775	801	827	854	880	906	932	958	985
166	22 011	037	063	089	115	141	167	194	220	246
167	22 272	298	324	350	376	401	427	453	479	505
168	22 531	557	583	608	634	660	686	712	737	763
169	22 789	814	840	866	891	917	943	968	994	*019
170	23 045	070	096	121	147	172	198	223	249	274
171	23 300	325	350	376	401	426	452	477	502	528
172	23 553	578	603	629	654	679	704	729	754	779
173	23 805	830	855	880	905	930	955	980	*005	*030
174	24 055	080	105	130	155	180	204	229	254	279
175	24 304	329	353	378	403	428	452	477	502	527
176	24 551	576	601	625	650	674	699	724	748	773
177	24 797	822	846	871	895	920	944	969	993	*018
178	25 042	066	091	115	139	164	188	212	237	261
179	25 285	310	334	358	382	406	431	455	479	503
180	25 527	551	575	600	624	648	672	696	720	744
181	25 768	792	816	840	864	888	912	935	959	983
182	26 007	031	055	079	102	126	150	174	198	221
183	26 245	269	293	316	340	364	387	411	435	458
184	26 482	505	529	553	576	600	623	647	670	694
185	26 717	741	764	788	811	834	858	881	905	928
186	26 951	975	998	*021	*045	*068	*091	*114	*138	*161
187	27 184	207	231	254	277	300	323	346	370	393
188	27 416	439	462	485	508	531	554	577	600	623
189	27 646	669	692	715	738	761	784	807	830	852
190	27 875	898	921	944	967	989	*012	*035	*058	*081
191	28 103	126	149	171	194	217	240	262	285	307
192	28 330	353	375	398	421	443	466	488	511	533
193	28 556	578	601	623	646	668	691	713	735	758
194	28 780	803	825	847	870	892	914	937	959	981
195	29 003	026	048	070	092	115	137	159	181	203
196	29 226	248	270	292	314	336	358	380	403	425
197	29 447	469	491	513	535	557	579	601	623	645
198	29 667	688	710	732	754	776	798	820	842	863
199	29 885	907	929	951	973	994	*016	*038	*060	*081
200	30 103	125	146	168	190	211	233	255	276	298
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	29	28
1	2.9	2.8
2	5.8	5.6
3	8.7	8.4
4	11.6	11.2
5	14.5	14.0
6	17.4	16.8
7	20.3	19.6
8	23.2	22.4
9	26.1	25.2

	27	26
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	25
1	2.5
2	5.0
3	7.5
4	10.0
5	12.5
6	15.0
7	17.5
8	20.0
9	22.5

	24	23
1	2.4	2.3
2	4.8	4.6
3	7.2	6.9
4	9.6	9.2
5	12.0	11.5
6	14.4	13.8
7	16.8	16.1
8	19.2	18.4
9	21.6	20.7

	22	21
1	2.2	2.1
2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

Prop. Parts

N	0	1	2	3	4	5	6	7	8	9
200	30 103	125	146	168	190	211	233	255	276	298
201	30 320	341	363	384	406	428	449	471	492	514
202	30 535	557	578	600	621	643	664	685	707	728
203	30 750	771	792	814	835	856	878	899	920	942
204	30 963	984	*006	*027	*048	*069	*091	*112	*133	*154
205	31 175	197	218	239	260	281	302	323	345	366
206	31 387	408	429	450	471	492	513	534	555	576
207	31 597	618	639	660	681	702	723	744	765	785
208	31 806	827	848	869	890	911	931	952	973	994
209	32 015	035	056	077	098	118	139	160	181	201
210	32 222	243	263	284	305	325	346	366	387	408
211	32 428	449	469	490	510	531	552	572	593	613
212	32 634	654	675	695	715	736	756	777	797	818
213	32 838	858	879	899	919	940	960	980	*001	*021
214	33 041	062	082	102	122	143	163	183	203	224
215	33 244	264	284	304	325	345	365	385	405	425
216	33 445	465	486	506	526	546	566	586	606	626
217	33 646	666	686	706	726	746	766	786	806	826
218	33 846	866	885	905	925	945	965	985	*005	*025
219	34 044	064	084	104	124	143	163	183	203	223
220	34 242	262	282	301	321	341	361	380	400	420
221	34 439	459	479	498	518	537	557	577	596	616
222	34 635	655	674	694	713	733	753	772	792	811
223	34 830	850	869	889	908	928	947	967	986	*005
224	35 025	044	064	083	102	122	141	160	180	199
225	35 218	238	257	276	295	315	334	353	372	392
226	35 411	430	449	468	488	507	526	545	564	583
227	35 603	622	641	660	679	698	717	736	755	774
228	35 793	813	832	851	870	889	908	927	946	965
229	35 984	*003	*021	*040	*059	*078	*097	*116	*135	*154
230	36 173	192	211	229	248	267	286	305	324	342
231	36 361	380	399	418	436	455	474	493	511	530
232	36 549	568	586	605	624	642	661	680	698	717
233	36 736	754	773	791	810	829	847	866	884	903
234	36 922	940	959	977	996	*014	*033	*051	*070	*088
235	37 107	125	144	162	181	199	218	236	254	273
236	37 291	310	328	346	365	383	401	420	438	457
237	37 475	493	511	530	548	566	585	603	621	639
238	37 658	676	694	712	731	749	767	785	803	822
239	37 840	858	876	894	912	931	949	967	985	*003
240	38 021	039	057	075	093	112	130	148	166	184
241	38 202	220	238	256	274	292	310	328	346	364
242	38 382	399	417	435	453	471	489	507	525	543
243	38 561	578	596	614	632	650	668	686	703	721
244	38 739	757	775	792	810	828	846	863	881	899
245	38 917	934	952	970	987	*005	*023	*041	*058	*076
246	39 094	111	129	146	164	182	199	217	235	252
247	39 270	287	305	322	340	358	375	393	410	428
248	39 445	463	480	498	515	533	550	568	585	602
249	39 620	637	655	672	690	707	724	742	759	777
250	39 794	811	829	846	863	881	898	915	933	950
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	22	21
1	2.2	2.1
2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

	20
1	2.0
2	4.0
3	6.0
4	8.0
5	10.0
6	12.0
7	14.0
8	16.0
9	18.0

	19
1	1.9
2	3.8
3	5.7
4	7.6
5	9.5
6	11.4
7	13.3
8	15.2
9	17.1

	18
1	1.8
2	3.6
3	5.4
4	7.2
5	9.0
6	10.8
7	12.6
8	14.4
9	16.2

	17
1	1.7
2	3.4
3	5.1
4	6.8
5	8.5
6	10.2
7	11.9
8	13.6
9	15.3

Prop. Parts

Prop. Parts

	18
1	1.8
2	3.6
3	5.4
4	7.2
5	9.0
6	10.8
7	12.6
8	14.4
9	16.2

	17
1	1.7
2	3.4
3	5.1
4	6.8
5	8.5
6	10.2
7	11.9
8	13.6
9	15.3

	16
1	1.6
2	3.2
3	4.8
4	6.4
5	8.0
6	9.6
7	11.2
8	12.8
9	14.4

	15
1	1.5
2	3.0
3	4.5
4	6.0
5	7.5
6	9.0
7	10.5
8	12.0
9	13.5

	14
1	1.4
2	2.8
3	4.2
4	5.6
5	7.0
6	8.4
7	9.8
8	11.2
9	12.6

N	0	1	2	3	4	5	6	7	8	9
250	39 794	811	829	846	863	881	898	915	933	950
251	39 967	985	*002	*019	*037	*054	*071	*088	*106	*123
252	40 140	157	175	192	209	226	243	261	278	295
253	40 312	329	346	364	381	398	415	432	449	466
254	40 483	500	518	535	552	569	586	603	620	637
255	40 654	671	688	705	722	739	756	773	790	807
256	40 824	841	858	875	892	909	926	943	960	976
257	40 993	*010	*027	*044	*061	*078	*095	*111	*128	*145
258	41 162	179	196	212	229	246	263	280	296	313
259	41 330	347	363	380	397	414	430	447	464	481
260	41 497	514	531	547	564	581	597	614	631	647
261	41 664	681	697	714	731	747	764	780	797	814
262	41 830	847	863	880	896	913	929	946	963	979
263	41 996	*012	*029	*045	*062	*078	*095	*111	*127	*144
264	42 160	177	193	210	226	243	259	275	292	308
265	42 325	341	357	374	390	406	423	439	455	472
266	42 488	504	521	537	553	570	586	602	619	635
267	42 651	667	684	700	716	732	749	765	781	797
268	42 813	830	846	862	878	894	911	927	943	959
269	42 975	991	*008	*024	*040	*056	*072	*088	*104	*120
270	43 136	152	169	185	201	217	233	249	265	281
271	43 297	313	329	345	361	377	393	409	425	441
272	43 457	473	489	505	521	537	553	569	584	600
273	43 616	632	648	664	680	696	712	727	743	759
274	43 775	791	807	823	838	854	870	886	902	917
275	43 933	949	965	981	996	*012	*028	*044	*059	*075
276	44 091	107	122	138	154	170	185	201	217	232
277	44 248	264	279	295	311	326	342	358	373	389
278	44 404	420	436	451	467	483	498	514	529	545
279	44 560	576	592	607	623	638	654	669	685	700
280	44 716	731	747	762	778	793	809	824	840	855
281	44 871	886	902	917	932	948	963	979	994	*010
282	45 025	040	056	071	086	102	117	133	148	163
283	45 179	194	209	225	240	255	271	286	301	317
284	45 332	347	362	378	393	408	423	439	454	469
285	45 484	500	515	530	545	561	576	591	606	621
286	45 637	652	667	682	697	712	728	743	758	773
287	45 788	803	818	834	849	864	879	894	909	924
288	45 939	954	969	984	*000	*015	*030	*045	*060	*075
289	46 090	105	120	135	150	165	180	195	210	225
290	46 240	255	270	285	300	315	330	345	359	374
291	46 389	404	419	434	449	464	479	494	509	523
292	46 538	553	568	583	598	613	627	642	657	672
293	46 687	702	716	731	746	761	776	790	805	820
294	46 835	850	864	879	894	909	923	938	953	967
295	46 982	997	*012	*026	*041	*056	*070	*085	*100	*114
296	47 129	144	159	173	188	202	217	232	246	261
297	47 276	290	305	319	334	349	363	378	392	407
298	47 422	436	451	465	480	494	509	524	538	553
299	47 567	582	596	611	625	640	654	669	683	698
300	47 712	727	741	756	770	784	799	813	828	842
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

Prop. Parts

N	0	1	2	3	4	5	6	7	8	9
300	47 712	727	741	756	770	784	799	813	828	842
301	47 857	871	885	900	914	929	943	958	972	986
302	48 001	015	029	044	058	073	087	101	116	130
303	48 144	159	173	187	202	216	230	244	259	273
304	48 287	302	316	330	344	359	373	387	401	416
305	48 430	444	458	473	487	501	515	530	544	558
306	48 572	586	601	615	629	643	657	671	686	700
307	48 714	728	742	756	770	785	799	813	827	841
308	48 855	869	883	897	911	926	940	954	968	982
309	48 996	*010	*024	*038	*052	*066	*080	*094	*108	*122
310	49 136	150	164	178	192	206	220	234	248	262
311	49 276	290	304	318	332	346	360	374	388	402
312	49 415	429	443	457	471	485	499	513	527	541
313	49 554	568	582	596	610	624	638	651	665	679
314	49 693	707	721	734	748	762	776	790	803	817
315	49 831	845	859	872	886	900	914	927	941	955
316	49 969	982	996	*010	*024	*037	*051	*065	*079	*092
317	50 106	120	133	147	161	174	188	202	215	229
318	50 243	256	270	284	297	311	325	338	352	365
319	50 379	393	406	420	433	447	461	474	488	501
320	50 515	529	542	556	569	583	596	610	623	637
321	50 651	664	678	691	705	718	732	745	759	772
322	50 786	799	813	826	840	853	866	880	893	907
323	50 920	934	947	961	974	987	*001	*014	*028	*041
324	51 055	068	081	095	108	121	135	148	162	175
325	51 188	202	215	228	242	255	268	282	295	308
326	51 322	335	348	362	375	388	402	415	428	441
327	51 455	468	481	495	508	521	534	548	561	574
328	51 587	601	614	627	640	654	667	680	693	706
329	51 720	733	746	759	772	786	799	812	825	838
330	51 851	865	878	891	904	917	930	943	957	970
331	51 983	996	*009	*022	*035	*048	*061	*075	*088	*101
332	52 114	127	140	153	166	179	192	205	218	231
333	52 244	257	270	284	297	310	323	336	349	362
334	52 375	388	401	414	427	440	453	466	479	492
335	52 504	517	530	543	556	569	582	595	608	621
336	52 634	647	660	673	686	699	711	724	737	750
337	52 763	776	789	802	815	827	840	853	866	879
338	52 892	905	917	930	943	956	969	982	994	*007
339	53 020	033	046	058	071	084	097	110	122	135
340	53 148	161	173	186	199	212	224	237	250	263
341	53 275	288	301	314	326	339	352	364	377	390
342	53 403	415	428	441	453	466	479	491	504	517
343	53 529	542	555	567	580	593	605	618	631	643
344	53 656	668	681	694	706	719	732	744	757	769
345	53 782	794	807	820	832	845	857	870	882	895
346	53 908	920	933	945	958	970	983	995	*008	*020
347	54 033	045	058	070	083	095	108	120	133	145
348	54 158	170	183	195	208	220	233	245	258	270
349	54 283	295	307	320	332	345	357	370	382	394
350	54 407	419	432	444	456	469	481	494	506	518
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

Prop. Parts

	15
1	1.5
2	3.0
3	4.5
4	6.0
5	7.5
6	9.0
7	10.5
8	12.0
9	13.5

	14
1	1.4
2	2.8
3	4.2
4	5.6
5	7.0
6	8.4
7	9.8
8	11.2
9	12.6

	13
1	1.3
2	2.6
3	3.9
4	5.2
5	6.5
6	7.8
7	9.1
8	10.4
9	11.7

	12
1	1.2
2	2.4
3	3.6
4	4.8
5	6.0
6	7.2
7	8.4
8	9.6
9	10.8

Prop. Parts

Prop. Parts	N	0	1	2	3	4	5	6	7	8	9
	350	54 407	419	432	444	456	469	481	494	506	518
	351	54 531	543	555	568	580	593	605	617	630	642
	352	54 654	667	679	691	704	716	728	741	753	765
	353	54 777	790	802	814	827	839	851	864	876	888
13	354	54 900	913	925	937	949	962	974	986	998	*011
1 1.3	355	55 023	035	047	060	072	084	096	108	121	133
2 2.6	356	55 145	157	169	182	194	206	218	230	242	255
3 3.9	357	55 267	279	291	303	315	328	340	352	364	376
4 5.2	358	55 388	400	413	425	437	449	461	473	485	497
5 6.5	359	55 509	522	534	546	558	570	582	594	606	618
6 7.8	360	55 630	642	654	666	678	691	703	715	727	739
7 9.1	361	55 751	763	775	787	799	811	823	835	847	859
8 10.4	362	55 871	883	895	907	919	931	943	955	967	979
9 11.7	363	55 991	*003	*015	*027	*038	*050	*062	*074	*086	*098
	364	56 110	122	134	146	158	170	182	194	205	217
12	365	56 229	241	253	265	277	289	301	312	324	336
1 1.2	366	56 348	360	372	384	396	407	419	431	443	455
2 2.4	367	56 467	478	490	502	514	526	538	549	561	573
3 3.6	368	56 585	597	608	620	632	644	656	667	679	691
4 4.8	369	56 703	714	726	738	750	761	773	785	797	808
5 6.0	370	56 820	832	844	855	867	879	891	902	914	926
6 7.2	371	56 937	949	961	972	984	996	*008	*019	*031	*043
7 8.4	372	57 054	066	078	089	101	113	124	136	148	159
8 9.6	373	57 171	183	194	206	217	229	241	252	264	276
9 10.8	374	57 287	299	310	322	334	345	357	368	380	392
	375	57 403	415	426	438	449	461	473	484	496	507
	376	57 519	530	542	553	565	576	588	600	611	623
11	377	57 634	646	657	669	680	692	703	715	726	738
1 1.1	378	57 749	761	772	784	795	807	818	830	841	852
2 2.2	379	57 864	875	887	898	910	921	933	944	955	967
3 3.3	380	57 978	990	*001	*013	*024	*035	*047	*058	*070	*081
4 4.4	381	58 092	104	115	127	138	149	161	172	184	195
5 5.5	382	58 206	218	229	240	252	263	274	286	297	309
6 6.6	383	58 320	331	343	354	365	377	388	399	410	422
7 7.7	384	58 433	444	456	467	478	490	501	512	524	535
8 8.8	385	58 546	557	569	580	591	602	614	625	636	647
9 9.9	386	58 659	670	681	692	704	715	726	737	749	760
	387	58 771	782	794	805	816	827	838	850	861	872
	388	58 883	894	906	917	928	939	950	961	973	984
10	389	58 995	*006	*017	*028	*040	*051	*062	*073	*084	*095
1 1.0	390	59 106	118	129	140	151	162	173	184	195	207
2 2.0	391	59 218	229	240	251	262	273	284	295	306	318
3 3.0	392	59 329	340	351	362	373	384	395	406	417	428
4 4.0	393	59 439	450	461	472	483	494	506	517	528	539
5 5.0	394	59 550	561	572	583	594	605	616	627	638	649
6 6.0	395	59 660	671	682	693	704	715	726	737	748	759
7 7.0	396	59 770	780	791	802	813	824	835	846	857	868
8 8.0	397	59 879	890	901	912	923	934	945	956	966	977
9 9.0	398	59 988	999	*010	*021	*032	*043	*054	*065	*076	*086
	399	60 097	108	119	130	141	152	163	173	184	195
	400	60 206	217	228	239	249	260	271	282	293	304
Prop. Parts	N	0	1	2	3	4	5	6	7	8	9

N	0	1	2	3	4	5	6	7	8	9	Prop. Parts
400	60 206	217	228	239	249	260	271	282	293	304	
401	60 314	325	336	347	358	369	379	390	401	412	
402	60 423	433	444	455	466	477	487	498	509	520	
403	60 531	541	552	563	574	584	595	606	617	627	
404	60 638	649	660	670	681	692	703	713	724	735	
405	60 746	756	767	778	788	799	810	821	831	842	
406	60 853	863	874	885	895	906	917	927	938	949	
407	60 959	970	981	991	*002	*013	*023	*034	*045	*055	11
408	61 066	077	087	098	109	119	130	140	151	162	1 1.1
409	61 172	183	194	204	215	225	236	247	257	268	2 2.2
410	61 278	289	300	310	321	331	342	352	363	374	3 3.3
411	61 384	395	405	416	426	437	448	458	469	479	4 4.4
412	61 490	500	511	521	532	542	553	563	574	584	5 5.5
413	61 595	606	616	627	637	648	658	669	679	690	6 6.6
414	61 700	711	721	731	742	752	763	773	784	794	7 7.7
415	61 805	815	826	836	847	857	868	878	888	899	8 8.8
416	61 909	920	930	941	951	962	972	982	993	*003	9 9.9
417	62 014	024	034	045	055	066	076	086	097	107	
418	62 118	128	138	149	159	170	180	190	201	211	
419	62 221	232	242	252	263	273	284	294	304	315	
420	62 325	335	346	356	366	377	387	397	408	418	
421	62 428	439	449	459	469	480	490	500	511	521	
422	62 531	542	552	562	572	583	593	603	613	624	10
423	62 634	644	655	665	675	685	696	706	716	726	1 1.0
424	62 737	747	757	767	778	788	798	808	818	829	2 2.0
425	62 839	849	859	870	880	890	900	910	921	931	3 3.0
426	62 941	951	961	972	982	992	*002	*012	*022	*033	4 4.0
427	63 043	053	063	073	083	094	104	114	124	134	5 5.0
428	63 144	155	165	175	185	195	205	215	225	236	6 6.0
429	63 246	256	266	276	286	296	306	317	327	337	7 7.0
430	63 347	357	367	377	387	397	407	417	428	438	8 8.0
431	63 448	458	468	478	488	498	508	518	528	538	9 9.0
432	63 548	558	568	579	589	599	609	619	629	639	
433	63 649	659	669	679	689	699	709	719	729	739	
434	63 749	759	769	779	789	799	809	819	829	839	
435	63 849	859	869	879	889	899	909	919	929	939	
436	63 949	959	969	979	988	998	*008	*018	*028	*038	9
437	64 048	058	068	078	088	098	108	118	128	137	1 0.9
438	64 147	157	167	177	187	197	207	217	227	237	2 1.8
439	64 246	256	266	276	286	296	306	316	326	335	3 2.7
440	64 345	355	365	375	385	395	404	414	424	434	4 3.6
441	64 444	454	464	473	483	493	503	513	523	532	5 4.5
442	64 542	552	562	572	582	591	601	611	621	631	6 5.4
443	64 640	650	660	670	680	689	699	709	719	729	7 6.3
444	64 738	748	758	768	777	787	797	807	816	826	8 7.2
445	64 836	846	856	865	875	885	895	904	914	924	9 8.1
446	64 933	943	953	963	972	982	992	*002	*011	*021	
447	65 031	040	050	060	070	079	089	099	108	118	
448	65 128	137	147	157	167	176	186	196	205	215	
449	65 225	234	244	254	263	273	283	292	302	312	
450	65 321	331	341	350	360	369	379	389	398	408	
N	0	1	2	3	4	5	6	7	8	9	Prop. Parts

N	0	1	2	3	4	5	6	7	8	9
450	65 321	331	341	350	360	369	379	389	398	408
451	65 418	427	437	447	456	466	475	485	495	504
452	65 514	523	533	543	552	562	571	581	591	600
453	65 610	619	629	639	648	658	667	677	686	696
454	65 706	715	725	734	744	753	763	772	782	792
455	65 801	811	820	830	839	849	858	868	877	887
456	65 896	906	916	925	935	944	954	963	973	982
457	65 992	*001	*011	*020	*030	*039	*049	*058	*068	*077
458	66 087	096	106	115	124	134	143	153	162	172
459	66 181	191	200	210	219	229	238	247	257	266
460	66 276	285	295	304	314	323	332	342	351	361
461	66 370	380	389	398	408	417	427	436	445	455
462	66 464	474	483	492	502	511	521	530	539	549
463	66 558	567	577	586	596	605	614	624	633	642
464	66 652	661	671	680	689	699	708	717	727	736
465	66 745	755	764	773	783	792	801	811	820	829
466	66 839	848	857	867	876	885	894	904	913	922
467	66 932	941	950	960	969	978	987	997	*006	*015
468	67 025	034	043	052	062	071	080	089	099	108
469	67 117	127	136	145	154	164	173	182	191	201
470	67 210	219	228	237	247	256	265	274	284	293
471	67 302	311	321	330	339	348	357	367	376	385
472	67 394	403	413	422	431	440	449	459	468	477
473	67 486	495	504	514	523	532	541	550	560	569
474	67 578	587	596	605	614	624	633	642	651	660
475	67 669	679	688	697	706	715	724	733	742	752
476	67 761	770	779	788	797	806	815	825	834	843
477	67 852	861	870	879	888	897	906	916	925	934
478	67 943	952	961	970	979	988	997	*006	*015	*024
479	68 034	043	052	061	070	079	088	097	106	115
480	68 124	133	142	151	160	169	178	187	196	205
481	68 215	224	233	242	251	260	269	278	287	296
482	68 305	314	323	332	341	350	359	368	377	386
483	68 395	404	413	422	431	440	449	458	467	476
484	68 485	494	502	511	520	529	538	547	556	565
485	68 574	583	592	601	610	619	628	637	646	655
486	68 664	673	681	690	699	708	717	726	735	744
487	68 753	762	771	780	789	797	806	815	824	833
488	68 842	851	860	869	878	886	895	904	913	922
489	68 931	940	949	958	966	975	984	993	*002	*011
490	69 020	028	037	046	055	064	073	082	090	099
491	69 108	117	126	135	144	152	161	170	179	188
492	69 197	205	214	223	232	241	249	258	267	276
493	69 285	294	302	311	320	329	338	346	355	364
494	69 373	381	390	399	408	417	425	434	443	452
495	69 461	469	478	487	496	504	513	522	531	539
496	69 548	557	566	574	583	592	601	609	618	627
497	69 636	644	653	662	671	679	688	697	705	714
498	69 723	732	740	749	758	767	775	784	793	801
499	69 810	819	827	836	845	854	862	871	880	888
500	69 897	906	914	923	932	940	949	958	966	975
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

Prop. Parts

	10
1	1.0
2	2.0
3	3.0
4	4.0
5	5.0
6	6.0
7	7.0
8	8.0
9	9.0

	9
1	0.9
2	1.8
3	2.7
4	3.6
5	4.5
6	5.4
7	6.3
8	7.2
9	8.1

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

Prop. Parts

N	0	1	2	3	4	5	6	7	8	9
500	69 897	906	914	923	932	940	949	958	966	975
501	69 984	992	*001	*010	*018	*027	*036	*044	*053	*062
502	70 070	079	088	096	105	114	122	131	140	148
503	70 157	165	174	183	191	200	209	217	226	234
504	70 243	252	260	269	278	286	295	303	312	321
505	70 329	338	346	355	364	372	381	389	398	406
506	70 415	424	432	441	449	458	467	475	484	492
507	70 501	509	518	526	535	544	552	561	569	578
508	70 586	595	603	612	621	629	638	646	655	663
509	70 672	680	689	697	706	714	723	731	740	749
510	70 757	766	774	783	791	800	808	817	825	834
511	70 842	851	859	868	876	885	893	902	910	919
512	70 927	935	944	952	961	969	978	986	995	*003
513	71 012	020	029	037	046	054	063	071	079	088
514	71 096	105	113	122	130	139	147	155	164	172
515	71 181	189	198	206	214	223	231	240	248	257
516	71 265	273	282	290	299	307	315	324	332	341
517	71 349	357	366	374	383	391	399	408	416	425
518	71 433	441	450	458	466	475	483	492	500	508
519	71 517	525	533	542	550	559	567	575	584	592
520	71 600	609	617	625	634	642	650	659	667	675
521	71 684	692	700	709	717	725	734	742	750	759
522	71 767	775	784	792	800	809	817	825	834	842
523	71 850	858	867	875	883	892	900	908	917	925
524	71 933	941	950	958	966	975	983	991	999	*008
525	72 016	024	032	041	049	057	066	074	082	090
526	72 099	107	115	123	132	140	148	156	165	173
527	72 181	189	198	206	214	222	230	239	247	255
528	72 263	272	280	288	296	304	313	321	329	337
529	72 346	354	362	370	378	387	395	403	411	419
530	72 428	436	444	452	460	469	477	485	493	501
531	72 509	518	526	534	542	550	558	567	575	583
532	72 591	599	607	616	624	632	640	648	656	665
533	72 673	681	689	697	705	713	722	730	738	746
534	72 754	762	770	779	787	795	803	811	819	827
535	72 835	843	852	860	868	876	884	892	900	908
536	72 916	925	933	941	949	957	965	973	981	989
537	72 997	*006	*014	*022	*030	*038	*046	*054	*062	*070
538	73 078	086	094	102	111	119	127	135	143	151
539	73 159	167	175	183	191	199	207	215	223	231
540	73 239	247	255	263	272	280	288	296	304	312
541	73 320	328	336	344	352	360	368	376	384	392
542	73 400	408	416	424	432	440	448	456	464	472
543	73 480	488	496	504	512	520	528	536	544	552
544	73 560	568	576	584	592	600	608	616	624	632
545	73 640	648	656	664	672	679	687	695	703	711
546	73 719	727	735	743	751	759	767	775	783	791
547	73 799	807	815	823	830	838	846	854	862	870
548	73 878	886	894	902	910	918	926	933	941	949
549	73 957	965	973	981	989	997	*005	*013	*020	*028
550	74 036	044	052	060	068	076	084	092	099	107
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	9	8	7
1	0.9	0.8	0.7
2	1.8	1.6	1.4
3	2.7	2.4	2.1
4	3.6	3.2	2.8
5	4.5	4.0	3.5
6	5.4	4.8	4.2
7	6.3	5.6	4.9
8	7.2	6.4	5.6
9	8.1	7.2	6.3

Prop. Parts

N	0	1	2	3	4	5	6	7	8	9
550	74 036	044	052	060	068	076	084	092	099	107
551	74 115	123	131	139	147	155	162	170	178	186
552	74 194	202	210	218	225	233	241	249	257	265
553	74 273	280	288	296	304	312	320	327	335	343
554	74 351	359	367	374	382	390	398	406	414	421
555	74 429	437	445	453	461	468	476	484	492	500
556	74 507	515	523	531	539	547	554	562	570	578
557	74 586	593	601	609	617	624	632	640	648	656
558	74 663	671	679	687	695	702	710	718	726	733
559	74 741	749	757	764	772	780	788	796	803	811
560	74 819	827	834	842	850	858	865	873	881	889
561	74 896	904	912	920	927	935	943	950	958	966
562	74 974	981	989	997	*005	*012	*020	*028	*035	*043
563	75 051	059	066	074	082	089	097	105	113	120
564	75 128	136	143	151	159	166	174	182	189	197
565	75 205	213	220	228	236	243	251	259	266	274
566	75 282	289	297	305	312	320	328	335	343	351
567	75 358	366	374	381	389	397	404	412	420	427
568	75 435	442	450	458	465	473	481	488	496	504
569	75 511	519	526	534	542	549	557	565	572	580
570	75 587	595	603	610	618	626	633	641	648	656
571	75 664	671	679	686	694	702	709	717	724	732
572	75 740	747	755	762	770	778	785	793	800	808
573	75 815	823	831	838	846	853	861	868	876	884
574	75 891	899	906	914	921	929	937	944	952	959
575	75 967	974	982	989	997	*005	*012	*020	*027	*035
576	76 042	050	057	065	072	080	087	095	103	110
577	76 118	125	133	140	148	155	163	170	178	185
578	76 193	200	208	215	223	230	238	245	253	260
579	76 268	275	283	290	298	305	313	320	328	335
580	76 343	350	358	365	373	380	388	395	403	410
581	76 418	425	433	440	448	455	462	470	477	485
582	76 492	500	507	515	522	530	537	545	552	559
583	76 567	574	582	589	597	604	612	619	626	634
584	76 641	649	656	664	671	678	686	693	701	708
585	76 716	723	730	738	745	753	760	768	775	782
586	76 790	797	805	812	819	827	834	842	849	856
587	76 864	871	879	886	893	901	908	916	923	930
588	76 938	945	953	960	967	975	982	989	997	*004
589	77 012	019	026	034	041	048	056	063	070	078
590	77 085	093	100	107	115	122	129	137	144	151
591	77 159	166	173	181	188	195	203	210	217	225
592	77 232	240	247	254	262	269	276	283	291	298
593	77 305	313	320	327	335	342	349	357	364	371
594	77 379	386	393	401	408	415	422	430	437	444
595	77 452	459	466	474	481	488	495	503	510	517
596	77 525	532	539	546	554	561	568	576	583	590
597	77 597	605	612	619	627	634	641	648	656	663
598	77 670	677	685	692	699	706	714	721	728	735
599	77 743	750	757	764	772	779	786	793	801	808
600	77 815	822	830	837	844	851	859	866	873	880
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

N	0	1	2	3	4	5	6	7	8	9
600	77 815	822	830	837	844	851	859	866	873	880
601	77 887	895	902	909	916	924	931	938	945	952
602	77 960	967	974	981	988	996	*003	*010	*017	*025
603	78 032	039	046	053	061	068	075	082	089	097
604	78 104	111	118	125	132	140	147	154	161	168
605	78 176	183	190	197	204	211	219	226	233	240
606	78 247	254	262	269	276	283	290	297	305	312
607	78 319	326	333	340	347	355	362	369	376	383
608	78 390	398	405	412	419	426	433	440	447	455
609	78 462	469	476	483	490	497	504	512	519	526
610	78 533	540	547	554	561	569	576	583	590	597
611	78 604	611	618	625	633	640	647	654	661	668
612	78 675	682	689	696	704	711	718	725	732	739
613	78 746	753	760	767	774	781	789	796	803	810
614	78 817	824	831	838	845	852	859	866	873	880
615	78 888	895	902	909	916	923	930	937	944	951
616	78 958	965	972	979	986	993	*000	*007	*014	*021
617	79 029	036	043	050	057	064	071	078	085	092
618	79 099	106	113	120	127	134	141	148	155	162
619	79 169	176	183	190	197	204	211	218	225	232
620	79 239	246	253	260	267	274	281	288	295	302
621	79 309	316	323	330	337	344	351	358	365	372
622	79 379	386	393	400	407	414	421	428	435	442
623	79 449	456	463	470	477	484	491	498	505	511
624	79 518	525	532	539	546	553	560	567	574	581
625	79 588	595	602	609	616	623	630	637	644	650
626	79 657	664	671	678	685	692	699	706	713	720
627	79 727	734	741	748	754	761	768	775	782	789
628	79 796	803	810	817	824	831	837	844	851	858
629	79 865	872	879	886	893	900	906	913	920	927
630	79 934	941	948	955	962	969	975	982	989	996
631	80 003	010	017	024	030	037	044	051	058	065
632	80 072	079	085	092	099	106	113	120	127	134
633	80 140	147	154	161	168	175	182	188	195	202
634	80 209	216	223	229	236	243	250	257	264	271
635	80 277	284	291	298	305	312	318	325	332	339
636	80 346	353	359	366	373	380	387	393	400	407
637	80 414	421	428	434	441	448	455	462	468	475
638	80 482	489	496	502	509	516	523	530	536	543
639	80 550	557	564	570	577	584	591	598	604	611
640	80 618	625	632	638	645	652	659	665	672	679
641	80 686	693	699	706	713	720	726	733	740	747
642	80 754	760	767	774	781	787	794	801	808	814
643	80 821	828	835	841	848	855	862	868	875	882
644	80 889	895	902	909	916	922	929	936	943	949
645	80 956	963	969	976	983	990	996	*003	*010	*017
646	81 023	030	037	043	050	057	064	070	077	084
647	81 090	097	104	111	117	124	131	137	144	151
648	81 158	164	171	178	184	191	198	204	211	218
649	81 224	231	238	245	251	258	265	271	278	285
650	81 291	298	305	311	318	325	331	338	345	351
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

Prop. Parts

Prop. Parts

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

N	0	1	2	3	4	5	6	7	8	9
650	81 291	298	305	311	318	325	331	338	345	351
651	81 358	365	371	378	385	391	398	405	411	418
652	81 425	431	438	445	451	458	465	471	478	485
653	81 491	498	505	511	518	525	531	538	544	551
654	81 558	564	571	578	584	591	598	604	611	617
655	81 624	631	637	644	651	657	664	671	677	684
656	81 690	697	704	710	717	723	730	737	743	750
657	81 757	763	770	776	783	790	796	803	809	816
658	81 823	829	836	842	849	856	862	869	875	882
659	81 889	895	902	908	915	921	928	935	941	948
660	81 954	961	968	974	981	987	994	*000	*007	*014
661	82 020	027	033	040	046	053	060	066	073	079
662	82 086	092	099	105	112	119	125	132	138	145
663	82 151	158	164	171	178	184	191	197	204	210
664	82 217	223	230	236	243	249	256	263	269	276
665	82 282	289	295	302	308	315	321	328	334	341
666	82 347	354	360	367	373	380	387	393	400	406
667	82 413	419	426	432	439	445	452	458	465	471
668	82 478	484	491	497	504	510	517	523	530	536
669	82 543	549	556	562	569	575	582	588	595	601
670	82 607	614	620	627	633	640	646	653	659	666
671	82 672	679	685	692	698	705	711	718	724	730
672	82 737	743	750	756	763	769	776	782	789	795
673	82 802	808	814	821	827	834	840	847	853	860
674	82 866	872	879	885	892	898	905	911	918	924
675	82 930	937	943	950	956	963	969	975	982	988
676	82 995	*001	*008	*014	*020	*027	*033	*040	*046	*052
677	83 059	065	072	078	085	091	097	104	110	117
678	83 123	129	136	142	149	155	161	168	174	181
679	83 187	193	200	206	213	219	225	232	238	245
680	83 251	257	264	270	276	283	289	296	302	308
681	83 315	321	327	334	340	347	353	359	366	372
682	83 378	385	391	398	404	410	417	423	429	436
683	83 442	448	455	461	467	474	480	487	493	499
684	83 506	512	518	525	531	537	544	550	556	563
685	83 569	575	582	588	594	601	607	613	620	626
686	83 632	639	645	651	658	664	670	677	683	689
687	83 696	702	708	715	721	727	734	740	746	753
688	83 759	765	771	778	784	790	797	803	809	816
689	83 822	828	835	841	847	853	860	866	872	879
690	83 885	891	897	904	910	916	923	929	935	942
691	83 948	954	960	967	973	979	985	992	998	*004
692	84 011	017	023	029	036	042	048	055	061	067
693	84 073	080	086	092	098	105	111	117	123	130
694	84 136	142	148	155	161	167	173	180	186	192
695	84 198	205	211	217	223	230	236	242	248	255
696	84 261	267	273	280	286	292	298	305	311	317
697	84 323	330	336	342	348	354	361	367	373	379
698	84 386	392	398	404	410	417	423	429	435	442
699	84 448	454	460	466	473	479	485	491	497	504
700	84 510	516	522	528	535	541	547	553	559	566
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

N	0	1	2	3	4	5	6	7	8	9	Prop. Parts
700	84 510	516	522	528	535	541	547	553	559	566	
701	84 572	578	584	590	597	603	609	615	621	628	
702	84 634	640	646	652	658	665	671	677	683	689	
703	84 696	702	708	714	720	726	733	739	745	751	
704	84 757	763	770	776	782	788	794	800	807	813	
705	84 819	825	831	837	844	850	856	862	868	874	
706	84 880	887	893	899	905	911	917	924	930	936	
707	84 942	948	954	960	967	973	979	985	991	997	
708	85 003	009	016	022	028	034	040	046	052	058	
709	85 065	071	077	083	089	095	101	107	114	120	
710	85 126	132	138	144	150	156	163	169	175	181	
711	85 187	193	199	205	211	217	224	230	236	242	
712	85 248	254	260	266	272	278	285	291	297	303	
713	85 309	315	321	327	333	339	345	352	358	364	
714	85 370	376	382	388	394	400	406	412	418	425	
715	85 431	437	443	449	455	461	467	473	479	485	
716	85 491	497	503	509	516	522	528	534	540	546	
717	85 552	558	564	570	576	582	588	594	600	606	
718	85 612	618	625	631	637	643	649	655	661	667	
719	85 673	679	685	691	697	703	709	715	721	727	
720	85 733	739	745	751	757	763	769	775	781	788	
721	85 794	800	806	812	818	824	830	836	842	848	
722	85 854	860	866	872	878	884	890	896	902	908	
723	85 914	920	926	932	938	944	950	956	962	968	
724	85 974	980	986	992	998	*004	*010	*016	*022	*028	
725	86 034	040	046	052	058	064	070	076	082	088	
726	86 094	100	106	112	118	124	130	136	141	147	
727	86 153	159	165	171	177	183	189	195	201	207	
728	86 213	219	225	231	237	243	249	255	261	267	
729	86 273	279	285	291	297	303	308	314	320	326	
730	86 332	338	344	350	356	362	368	374	380	386	
731	86 392	398	404	410	415	421	427	433	439	445	
732	86 451	457	463	469	475	481	487	493	499	504	
733	86 510	516	522	528	534	540	546	552	558	564	
734	86 570	576	581	587	593	599	605	611	617	623	
735	86 629	635	641	646	652	658	664	670	676	682	
736	86 688	694	700	705	711	717	723	729	735	741	
737	86 747	753	759	764	770	776	782	788	794	800	
738	86 806	812	817	823	829	835	841	847	853	859	
739	86 864	870	876	882	888	894	900	906	911	917	
740	86 923	929	935	941	947	953	958	964	970	976	
741	86 982	988	994	999	*005	*011	*017	*023	*029	*035	
742	87 040	046	052	058	064	070	075	081	087	093	
743	87 099	105	111	116	122	128	134	140	146	151	
744	87 157	163	169	175	181	186	192	198	204	210	
745	87 216	221	227	233	239	245	251	256	262	268	
746	87 274	280	286	291	297	303	309	315	320	326	
747	87 332	338	344	349	355	361	367	373	379	384	
748	87 390	396	402	408	413	419	425	431	437	442	
749	87 448	454	460	466	471	477	483	489	495	500	
750	87 506	512	518	523	529	535	541	547	552	558	
N	0	1	2	3	4	5	6	7	8	9	Prop. Parts

Prop. Parts:

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

Prop. Parts	N	0	1	2	3	4	5	6	7	8	9
	750	87 506	512	518	523	529	535	541	547	552	558
	751	87 564	570	576	581	587	593	599	604	610	616
	752	87 622	628	633	639	645	651	656	662	668	674
	753	87 679	685	691	697	703	708	714	720	726	731
	754	87 737	743	749	754	760	766	772	777	783	789
	755	87 795	800	806	812	818	823	829	835	841	846
	756	87 852	858	864	869	875	881	887	892	898	904
	757	87 910	915	921	927	933	938	944	950	955	961
	758	87 967	973	978	984	990	996	*001	*007	*013	*018
	759	88 024	030	036	041	047	053	058	064	070	076
	760	88 081	087	093	098	104	110	116	121	127	133
	761	88 138	144	150	156	161	167	173	178	184	190
6	762	88 195	201	207	213	218	224	230	235	241	247
1 0.6	763	88 252	258	264	270	275	281	287	292	298	304
2 1.2	764	88 309	315	321	326	332	338	343	349	355	360
3 1.8	765	88 366	372	377	383	389	395	400	406	412	417
4 2.4	766	88 423	429	434	440	446	451	457	463	468	474
5 3.0	767	88 480	485	491	497	502	508	513	519	525	530
6 3.6	768	88 536	542	547	553	559	564	570	576	581	587
7 4.2	769	88 593	598	604	610	615	621	627	632	638	643
8 4.8	**770**	88 649	655	660	666	672	677	683	689	694	700
9 5.4	771	88 705	711	717	722	728	734	739	745	750	756
	772	88 762	767	773	779	784	790	795	801	807	812
	773	88 818	824	829	835	840	846	852	857	863	868
	774	88 874	880	885	891	897	902	908	913	919	925
	775	88 930	936	941	947	953	958	964	969	975	981
	776	88 986	992	997	*003	*009	*014	*020	*025	*031	*037
	777	89 042	048	053	059	064	070	076	081	087	092
	778	89 098	104	109	115	120	126	131	137	143	148
	779	89 154	159	165	170	176	182	187	193	198	204
	780	89 209	215	221	226	232	237	243	248	254	260
5	781	89 265	271	276	282	287	293	298	304	310	315
1 0.5	782	89 321	326	332	337	343	348	354	360	365	371
2 1.0	783	89 376	382	387	393	398	404	409	415	421	426
3 1.5	784	89 432	437	443	448	454	459	465	470	476	481
4 2.0	785	89 487	492	498	504	509	515	520	526	531	537
5 2.5	786	89 542	548	553	559	564	570	575	581	586	592
6 3.0	787	89 597	603	609	614	620	625	631	636	642	647
7 3.5	788	89 653	658	664	669	675	680	686	691	697	702
8 4.0	789	89 708	713	719	724	730	735	741	746	752	757
9 4.5	**790**	89 763	768	774	779	785	790	796	801	807	812
	791	89 818	823	829	834	840	845	851	856	862	867
	792	89 873	878	883	889	894	900	905	911	916	922
	793	89 927	933	938	944	949	955	960	966	971	977
	794	89 982	988	993	998	*004	*009	*015	*020	*026	*031
	795	90 037	042	048	053	059	064	069	075	080	086
	796	90 091	097	102	108	113	119	124	129	135	140
	797	90 146	151	157	162	168	173	179	184	189	195
	798	90 200	206	211	217	222	227	233	238	244	249
	799	90 255	260	266	271	276	282	287	293	298	304
	800	90 309	314	320	325	331	336	342	347	352	358
Prop. Parts	**N**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

N	0	1	2	3	4	5	6	7	8	9
800	90 309	314	320	325	331	336	342	347	352	358
801	90 363	369	374	380	385	390	396	401	407	412
802	90 417	423	428	434	439	445	450	455	461	466
803	90 472	477	482	488	493	499	504	509	515	520
804	90 526	531	536	542	547	553	558	563	569	574
805	90 580	585	590	596	601	607	612	617	623	628
806	90 634	639	644	650	655	660	666	671	677	682
807	90 687	693	698	703	709	714	720	725	730	736
808	90 741	747	752	757	763	768	773	779	784	789
809	90 795	800	806	811	816	822	827	832	838	843
810	90 849	854	859	865	870	875	881	886	891	897
811	90 902	907	913	918	924	929	934	940	945	950
812	90 956	961	966	972	977	982	988	993	998	*004
813	91 009	014	020	025	030	036	041	046	052	057
814	91 062	068	073	078	084	089	094	100	105	110
815	91 116	121	126	132	137	142	148	153	158	164
816	91 169	174	180	185	190	196	201	206	212	217
817	91 222	228	233	238	243	249	254	259	265	270
818	91 275	281	286	291	297	302	307	312	318	323
819	91 328	334	339	344	350	355	360	365	371	376
820	91 381	387	392	397	403	408	413	418	424	429
821	91 434	440	445	450	455	461	466	471	477	482
822	91 487	492	498	503	508	514	519	524	529	535
823	91 540	545	551	556	561	566	572	577	582	587
824	91 593	598	603	609	614	619	624	630	635	640
825	91 645	651	656	661	666	672	677	682	687	693
826	91 698	703	709	714	719	724	730	735	740	745
827	91 751	756	761	766	772	777	782	787	793	798
828	91 803	808	814	819	824	829	834	840	845	850
829	91 855	861	866	871	876	882	887	892	897	903
830	91 908	913	918	924	929	934	939	944	950	955
831	91 960	965	971	976	981	986	991	997	*002	*007
832	92 012	018	023	028	033	038	044	049	054	059
833	92 065	070	075	080	085	091	096	101	106	111
834	92 117	122	127	132	137	143	148	153	158	163
835	92 169	174	179	184	189	195	200	205	210	215
836	92 221	226	231	236	241	247	252	257	262	267
837	92 273	278	283	288	293	298	304	309	314	319
838	92 324	330	335	340	345	350	355	361	366	371
839	92 376	381	387	392	397	402	407	412	418	423
840	92 428	433	438	443	449	454	459	464	469	474
841	92 480	485	490	495	500	505	511	516	521	526
842	92 531	536	542	547	552	557	562	567	572	578
843	92 583	588	593	598	603	609	614	619	624	629
844	92 634	639	645	650	655	660	665	670	675	681
845	92 686	691	696	701	706	711	716	722	727	732
846	92 737	742	747	752	758	763	768	773	778	783
847	92 788	793	799	804	809	814	819	824	829	834
848	92 840	845	850	855	860	865	870	875	881	886
849	92 891	896	901	906	911	916	921	927	932	937
850	92 942	947	952	957	962	967	973	978	983	988
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

Prop. Parts

Prop. Parts

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

N	0	1	2	3	4	5	6	7	8	9
850	92 942	947	952	957	962	967	973	978	983	988
851	92 993	998	*003	*008	*013	*018	*024	*029	*034	*039
852	93 044	049	054	059	064	069	075	080	085	090
853	93 095	100	105	110	115	120	125	131	136	141
854	93 146	151	156	161	166	171	176	181	186	192
855	93 197	202	207	212	217	222	227	232	237	242
856	93 247	252	258	263	268	273	278	283	288	293
857	93 298	303	308	313	318	323	328	334	339	344
858	93 349	354	359	364	369	374	379	384	389	394
859	93 399	404	409	414	420	425	430	435	440	445
860	93 450	455	460	465	470	475	480	485	490	495
861	93 500	505	510	515	520	526	531	536	541	546
862	93 551	556	561	566	571	576	581	586	591	596
863	93 601	606	611	616	621	626	631	636	641	646
864	93 651	656	661	666	671	676	682	687	692	697
865	93 702	707	712	717	722	727	732	737	742	747
866	93 752	757	762	767	772	777	782	787	792	797
867	93 802	807	812	817	822	827	832	837	842	847
868	93 852	857	862	867	872	877	882	887	892	897
869	93 902	907	912	917	922	927	932	937	942	947
870	93 952	957	962	967	972	977	982	987	992	997
871	94 002	007	012	017	022	027	032	037	042	047
872	94 052	057	062	067	072	077	082	086	091	096
873	94 101	106	111	116	121	126	131	136	141	146
874	94 151	156	161	166	171	176	181	186	191	196
875	94 201	206	211	216	221	226	231	236	240	245
876	94 250	255	260	265	270	275	280	285	290	295
877	94 300	305	310	315	320	325	330	335	340	345
878	94 349	354	359	364	369	374	379	384	389	394
879	94 399	404	409	414	419	424	429	433	438	443
880	94 448	453	458	463	468	473	478	483	488	493
881	94 498	503	507	512	517	522	527	532	537	542
882	94 547	552	557	562	567	571	576	581	586	591
883	94 596	601	606	611	616	621	626	630	635	640
884	94 645	650	655	660	665	670	675	680	685	689
885	94 694	699	704	709	714	719	724	729	734	738
886	94 743	748	753	758	763	768	773	778	783	787
887	94 792	797	802	807	812	817	822	827	832	836
888	94 841	846	851	856	861	866	871	876	880	885
889	94 890	895	900	905	910	915	919	924	929	934
890	94 939	944	949	954	959	963	968	973	978	983
891	94 988	993	998	*002	*007	*012	*017	*022	*027	*032
892	95 036	041	046	051	056	061	066	071	075	080
893	95 085	090	095	100	105	109	114	119	124	129
894	95 134	139	143	148	153	158	163	168	173	177
895	95 182	187	192	197	202	207	211	216	221	226
896	95 231	236	240	245	250	255	260	265	270	274
897	95 279	284	289	294	299	303	308	313	318	323
898	95 328	332	337	342	347	352	357	361	366	371
899	95 376	381	386	390	395	400	405	410	415	419
900	95 424	429	434	439	444	448	453	458	463	468
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

N	0	1	2	3	4	5	6	7	8	9
900	95 424	429	434	439	444	448	453	458	463	468
901	95 472	477	482	487	492	497	501	506	511	516
902	95 521	525	530	535	540	545	550	554	559	564
903	95 569	574	578	583	588	593	598	602	607	612
904	95 617	622	626	631	636	641	646	650	655	660
905	95 665	670	674	679	684	689	694	698	703	708
906	95 713	718	722	727	732	737	742	746	751	756
907	95 761	766	770	775	780	785	789	794	799	804
908	95 809	813	818	823	828	832	837	842	847	852
909	95 856	861	866	871	875	880	885	890	895	899
910	95 904	909	914	018	923	928	933	938	942	947
911	95 952	957	961	966	971	976	980	985	990	995
912	95 999	*004	*009	*014	*019	*023	*028	*033	*038	*042
913	96 047	052	057	061	066	071	076	080	085	090
914	96 095	099	104	109	114	118	123	128	133	137
915	96 142	147	152	156	161	166	171	175	180	185
916	96 190	194	199	204	209	213	218	223	227	232
917	96 237	242	246	251	256	261	265	270	275	280
918	96 284	289	294	298	303	308	313	317	322	327
919	96 332	336	341	346	350	355	360	365	369	374
920	96 379	384	388	393	398	402	407	412	417	421
921	96 426	431	435	440	445	450	454	459	464	468
922	96 473	478	483	487	492	497	501	506	511	515
923	96 520	525	530	534	539	544	548	553	558	562
924	96 567	572	577	581	586	591	595	600	605	609
925	96 614	619	624	628	633	638	642	647	652	656
926	96 661	666	670	675	680	685	689	694	699	703
927	96 708	713	717	722	727	731	736	741	745	750
928	96 755	759	764	769	774	778	783	788	792	797
929	96 802	806	811	816	820	825	830	834	839	844
930	96 848	853	858	862	867	872	876	881	886	890
931	96 895	900	904	909	914	918	923	928	932	937
932	96 942	946	951	956	960	965	970	974	979	984
933	96 988	993	997	*002	*007	*011	*016	*021	*025	*030
934	97 035	039	044	049	053	058	063	067	072	077
935	97 081	086	090	095	100	104	109	114	118	123
936	97 128	132	137	142	146	151	155	160	165	169
937	97 174	179	183	188	192	197	202	206	211	216
938	97 220	225	230	234	239	243	248	253	257	262
939	97 267	271	276	280	285	290	294	299	304	308
940	97 313	317	322	327	331	336	340	345	350	354
941	97 359	364	368	373	377	382	387	391	396	400
942	97 405	410	414	419	424	428	433	437	442	447
943	97 451	456	460	465	470	474	479	483	488	493
944	97 497	502	506	511	516	520	525	529	534	539
945	97 543	548	552	557	562	566	571	575	580	585
946	97 589	594	598	603	607	612	617	621	626	630
947	97 635	640	644	649	653	658	663	667	672	676
948	97 681	685	690	695	699	704	708	713	717	722
949	97 727	731	736	740	745	749	754	759	763	768
950	97 772	777	782	786	791	795	800	804	809	813
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

Prop. Parts

N	0	1	2	3	4	5	6	7	8	9
950	97 772	777	782	786	791	795	800	804	809	813
951	97 818	823	827	832	836	841	845	850	855	859
952	97 864	868	873	877	882	886	891	896	900	905
953	97 909	914	918	923	928	932	937	941	946	950
954	97 955	959	964	968	973	978	982	987	991	996
955	98 000	005	009	014	019	023	028	032	037	041
956	98 046	050	055	059	064	068	073	078	082	087
957	98 091	096	100	105	109	114	118	123	127	132
958	98 137	141	146	150	155	159	164	168	173	177
959	98 182	186	191	195	200	204	209	214	218	223
960	98 227	232	236	241	245	250	254	259	263	268
961	98 272	277	281	286	290	295	299	304	308	313
962	98 318	322	327	331	336	340	345	349	354	358
963	98 363	367	372	376	381	385	390	394	399	403
964	98 408	412	417	421	426	430	435	439	444	448
965	98 453	457	462	466	471	475	480	484	489	493
966	98 498	502	507	511	516	520	525	529	534	538
967	98 543	547	552	556	561	565	570	574	579	583
968	98 588	592	597	601	605	610	614	619	623	628
969	98 632	637	641	646	650	655	659	664	668	673
970	98 677	682	686	691	695	700	704	709	713	717
971	98 722	726	731	735	740	744	749	753	758	762
972	98 767	771	776	780	784	789	793	798	802	807
973	98 811	816	820	825	829	834	838	843	847	851
974	98 856	860	865	869	874	878	883	887	892	896
975	98 900	905	909	914	918	923	927	932	936	941
976	98 945	949	954	958	963	967	972	976	981	985
977	98 989	994	998	*003	*007	*012	*016	*021	*025	*029
978	99 034	038	043	047	052	056	061	065	069	074
979	99 078	083	087	092	096	100	105	109	114	118
980	99 123	127	131	136	140	145	149	154	158	162
981	99 167	171	176	180	185	189	193	198	202	207
982	99 211	216	220	224	229	233	238	242	247	251
983	99 255	260	264	269	273	277	282	286	291	295
984	99 300	304	308	313	317	322	326	330	335	339
985	99 344	348	352	357	361	366	370	374	379	383
986	99 388	392	396	401	405	410	414	419	423	427
987	99 432	436	441	445	449	454	458	463	467	471
988	99 476	480	484	489	493	498	502	506	511	515
989	99 520	524	528	533	537	542	546	550	555	559
990	99 564	568	572	577	581	585	590	594	599	603
991	99 607	612	616	621	625	629	634	638	642	647
992	99 651	656	660	664	669	673	677	682	686	691
993	99 695	699	704	708	712	717	721	726	730	734
994	99 739	743	747	752	756	760	765	769	774	778
995	99 782	787	791	795	800	804	808	813	817	822
996	99 826	830	835	839	843	848	852	856	861	865
997	99 870	874	878	883	887	891	896	900	904	909
998	99 913	917	922	926	930	935	939	944	948	952
999	99 957	961	965	970	974	978	983	987	991	996
1000	00 000	004	009	013	017	022	026	030	035	039
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

Prop. Parts

IIa. TABLES OF S AND T FOR ANGLES NEAR 0° AND 90°

Interpolation by the ordinary method leads to inaccurate results if the differences are large.

For angles less than 3°, the values of log sin θ, log tan θ, and log cot θ should be found by the following formulas.

$$\begin{aligned} \log \sin \theta &= S + \log \theta' - 10, \\ \log \tan \theta &= T + \log \theta' - 10 = -\log \cot \theta, \end{aligned}$$

where θ' is the number of minutes in the angle θ and S and T are found from the tables given below.

Similary, if $87° < \theta < 90°$, the values of log cos θ, log cot θ, and log tan θ are found from the following formulas.

$$\begin{aligned} \log \cos \theta &= S + \log\,(90° - \theta)' - 10, \\ \log \cot \theta &= T + \log\,(90° - \theta)' - 10 = -\log \tan \theta, \end{aligned}$$

where $(90° - \theta)'$ is the number of minutes in the angle $90° - \theta$.

θ' or $(90° - \theta)'$	S
0′– 13′	6.46 373
14′– 42′	72
43′– 58′	71
59′– 71′	6.46 370
72′– 81′	69
82′– 91′	68
92′– 99′	6.46 367
100′–107′	66
108′–115′	65
116′–121′	6.46 364
122′–128′	63
129′–134′	62
135′–140′	6.46 361
141′–146′	60
147′–151′	59
152′–157′	6.46 358
158′–162′	57
163′–167′	56
168′–171′	6.46 355
172′–176′	54
177′–181′	53

θ' or $(90° - \theta)'$	T	θ' or $(90° - \theta)'$	T
0′– 26′	6.46 373	131′–133′	6.46 394
27′– 39′	74	134′–136′	95
40′– 48′	75	137′–139′	96
49′– 56′	6.46 376	140′–142′	6.46 397
57′– 63′	77	143′–145′	98
64′– 69′	78	146′–148′	99
70′– 74′	6.46 379	149′–150′	6.46 400
75′– 80′	80	151′–153′	01
81′– 85′	81	154′–156′	02
86′– 89′	6.46 382	157′–158′	6.46 403
90′– 94′	83	159′–161′	04
95′– 98′	84	162′–163′	05
99′–102′	6.46 385	164′–166′	6.46 406
103′–106′	86	167′–168′	07
107′–110′	87	169′–171′	08
111′–113′	6.46 388	172′–173′	6.46 409
114′–117′	89	174′–175′	10
118′–120′	90	176′–178′	11
121′–124′	6.46 391	179′–180′	6.46 412
125′–127′	92	181′–182′	13
128′–130′	93	183′–184′	14

To avoid interpolation use Table IIa.

′	L Sin	L Tan	L Cot	L Cos	
0	—	—	—	10.00 000	60
1	6.46 373	6.46 373	13.53 627	10.00 000	59
2	6.76 476	6.76 476	13.23 524	10.00 000	58
3	6.94 085	6.94 085	13.05 915	10.00 000	57
4	7.06 579	7.06 579	12.93 421	10.00 000	56
5	7.16 270	7.16 270	12.83 730	10.00 000	55
6	7.24 188	7.24 188	12.75 812	10.00 000	54
7	7.30 882	7.30 882	12.69 118	10.00 000	53
8	7.36 682	7.36 682	12.63 318	10.00 000	52
9	7.41 797	7.41 797	12.58 203	10.00 000	51
10	7.46 373	7.46 373	12.53 627	10.00 000	50
11	7.50 512	7.50 512	12.49 488	10.00 000	49
12	7.54 291	7.54 291	12.45 709	10.00 000	48
13	7.57 767	7.57 767	12.42 233	10.00 000	47
14	7.60 985	7.60 986	12.39 014	10.00 000	46
15	7.63 982	7.63 982	12.36 018	10.00 000	45
16	7.66 784	7.66 785	12.33 215	10.00 000	44
17	7.69 417	7.69 418	12.30 582	9.99 999	43
18	7.71 900	7.71 900	12.28 100	9.99 999	42
19	7.74 248	7.74 248	12.25 752	9.99 999	41
20	7.76 475	7.76 476	12.23 524	9.99 999	40
21	7.78 594	7.78 595	12.21 405	9.99 999	39
22	7.80 615	7.80 615	12.19 385	9.99 999	38
23	7.82 545	7.82 546	12.17 454	9.99 999	37
24	7.84 393	7.84 394	12.15 606	9.99 999	36
25	7.86 166	7.86 167	12.13 833	9.99 999	35
26	7.87 870	7.87 871	12.12 129	9.99 999	34
27	7.89 509	7.89 510	12.10 490	9.99 999	33
28	7.91 088	7.91 089	12.08 911	9.99 999	32
29	7.92 612	7.92 613	12.07 387	9.99 998	31
30	7.94 084	7.94 086	12.05 914	9.99 998	30
31	7.95 508	7.95 510	12.04 490	9.99 998	29
32	7.96 887	7.96 889	12.03 111	9.99 998	28
33	7.98 223	7.98 225	12.01 775	9.99 998	27
34	7.99 520	7.99 522	12.00 478	9.99 998	26
35	8.00 779	8.00 781	11.99 219	9.99 998	25
36	8.02 002	8.02 004	11.97 996	9.99 998	24
37	8.03 192	8.03 194	11.96 806	9.99 997	23
38	8.04 350	8.04 353	11.95 647	9.99 997	22
39	8.05 478	8.05 481	11.94 519	9.99 997	21
40	8.06 578	8.06 581	11.93 419	9.99 997	20
41	8.07 650	8.07 653	11.92 347	9.99 997	19
42	8.08 696	8.08 700	11.91 300	9.99 997	18
43	8.09 718	8.09 722	11.90 278	9.99 997	17
44	8.10 717	8.10 720	11.89 280	9.99 996	16
45	8.11 693	8.11 696	11.88 304	9.99 996	15
46	8.12 647	8.12 651	11.87 349	9.99 996	14
47	8.13 581	8.13 585	11.86 415	9.99 996	13
48	8.14 495	8.14 500	11.85 500	9.99 996	12
49	8.15 391	8.15 395	11.84 605	9.99 996	11
50	8.16 268	8.16 273	11.83 727	9.99 995	10
51	8.17 128	8.17 133	11.82 867	9.99 995	9
52	8.17 971	8.17 976	11.82 024	9.99 995	8
53	8.18 798	8.18 804	11.81 196	9.99 995	7
54	8.19 610	8.19 616	11.80 384	9.99 995	6
55	8.20 407	8.20 413	11.79 587	9.99 994	5
56	8.21 189	8.21 195	11.78 805	9.99 994	4
57	8.21 958	8.21 964	11.78 036	9.99 994	3
58	8.22 713	8.22 720	11.77 280	9.99 994	2
59	8.23 456	8.23 462	11.76 538	9.99 994	1
60	8.24 186	8.24 192	11.75 808	9.99 993	0
	L Cos	L Cot	L Tan	L Sin	′

Prop. Pts. | Prop. Pts.

89°

′	L Sin	L Tan	L Cot	L Cos		Prop. Pts.
0	8.24 186	8.24 192	11.75 808	9.99 993	60	
1	8.24 903	8.24 910	11.75 090	9.99 993	59	
2	8.25 609	8.25 616	11.74 384	9.99 993	58	
3	8.26 304	8.26 312	11.73 688	9.99 993	57	
4	8.26 988	8.26 996	11.73 004	9.99 992	56	
5	8.27 661	8.27 669	11.72 331	9.99 992	55	
6	8.28 324	8.28 332	11.71 668	9.99 992	54	
7	8.28 977	8.28 986	11.71 014	9.99 992	53	
8	8.29 621	8.29 629	11.70 371	9.99 992	52	
9	8.30 255	8.30 263	11.69 737	9.99 991	51	
10	8.30 879	8.30 888	11.69 112	9.99 991	50	
11	8.31 495	8.31 505	11.68 495	9.99 991	49	
12	8.32 103	8.32 112	11.67 888	9.99 990	48	
13	8.32 702	8.32 711	11.67 289	9.99 990	47	
14	8.33 292	8.33 302	11.66 698	9.99 990	46	
15	8.33 875	8 33 886	11.66 114	9.99 990	45	
16	8.34 450	8.34 461	11.65 539	9.99 989	44	
17	8.35 018	8.35 029	11.64 971	9.99 989	43	
18	8.35 578	8.35 590	11.64 410	9.99 989	42	
19	8.36 131	8.36 143	11.63 857	9.99 989	41	
20	8.36 678	8.36 689	11 63 311	9.99 988	40	
21	8.37 217	8.37 229	11.62 771	9.99 988	39	
22	8.37 750	8.37 762	11.62 238	9.99 988	38	
23	8.38 276	8.38 289	11.61 711	9.99 987	37	
24	8.38 796	8.38 809	11.61 191	9.99 987	36	
25	8.39 310	8.39 323	11.60 677	9.99 987	35	
26	8.39 818	8.39 832	11.60 168	9.99 986	34	
27	8.40 320	8.40 334	11.59 666	9.99 986	33	
28	8.40 816	8.40 830	11.59 170	9.99 986	32	
29	8.41 307	8.41 321	11.58 679	9.99 985	31	
30	8.41 792	8.41 807	11.58 193	9.99 985	30	
31	8.42 272	8.42 287	11.57 713	9.99 985	29	
32	8.42 746	8.42 762	11.57 238	9.99 984	28	
33	8.43 216	8.43 232	11.56 768	9.99 984	27	
34	8.43 680	8.43 696	11.56 304	9.99 984	26	
35	8.44 139	8.44 156	11.55 844	9.99 983	25	
36	8.44 594	8.44 611	11.55 389	9.99 983	24	
37	8.45 044	8.45 061	11.54 939	9.99 983	23	
38	8.45 489	8.45 507	11.54 493	9.99 982	22	
39	8.45 930	8.45 948	11.54 052	9.99 982	21	
40	8.46 366	8.46 385	11.53 615	9.99 982	20	
41	8.46 799	8.46 817	11.53 183	9.99 981	19	
42	8.47 226	8.47 245	11.52 755	9.99 981	18	
43	8.47 650	8.47 669	11.52 331	9.99 981	17	
44	8.48 069	8.48 089	11.51 911	9.99 980	16	
45	8.48 485	8.48 505	11.51 495	9.99 980	15	
46	8.48 896	8.48 917	11.51 083	9.99 979	14	
47	8.49 304	8.49 325	11.50 675	9.99 979	13	
48	8.49 708	8.49 729	11.50 271	9.99 979	12	
49	8.50 108	8.50 130	11.49 870	9.99 978	11	
50	8.50 504	8.50 527	11.49 473	9.99 978	10	
51	8.50 897	8.50 920	11.49 080	9.99 977	9	
52	8.51 287	8.51 310	11.48 690	9.99 977	8	
53	8.51 673	8.51 696	11.48 304	9.99 977	7	
54	8.52 055	8.52 079	11.47 921	9.99 976	6	
55	8.52 434	8.52 459	11.47 541	9.99 976	5	
56	8.52 810	8.52 835	11.47 165	9.99 975	4	
57	8.53 183	8.53 208	11.46 792	9.99 975	3	
58	8.53 552	8.53 578	11.46 422	9.99 974	2	
59	8.53 919	8.53 945	11.46 055	9.99 974	1	
60	8.54 282	8.54 308	11.45 692	9.99 974	0	
	L Cos	L Cot	L Tan	L Sin	′	Prop. Pts.

To avoid interpolation use Table IIa.

To avoid interpolation use Table IIa.

Prop. Pts.	′	L Sin	L Tan	L Cot	L Cos	
	0	8.54 282	8.54 308	11.45 692	9.99 974	60
	1	8.54 642	8.54 669	11.45 331	9.99 973	59
	2	8.54 999	8.55 027	11.44 973	9.99 973	58
	3	8.55 354	8.55 382	11.44 618	9.99 972	57
	4	8.55 705	8.55 734	11.44 266	9.99 972	56
	5	8.56 054	8.56 083	11.43 917	9.99 971	55
	6	8.56 400	8.56 429	11.43 571	9.99 971	54
	7	8.56 743	8.56 773	11.43 227	9.99 970	53
	8	8.57 084	8.57 114	11.42 886	9.99 970	52
	9	8.57 421	8.57 452	11.42 548	9.99 969	51
	10	8.57 757	8.57 788	11.42 212	9.99 969	50
	11	8.58 089	8.58 121	11.41 879	9.99 968	49
	12	8.58 419	8.58 451	11.41 549	9.99 968	48
	13	8.58 747	8.58 779	11.41 221	9.99 967	47
	14	8.59 072	8.59 105	11.40 895	9.99 967	46
	15	8.59 395	8.59 428	11.40 572	9.99 967	45
	16	8.59 715	8.59 749	11.40 251	9.99 966	44
	17	8.60 033	8.60 068	11.39 932	9.99 966	43
	18	8.60 349	8.60 384	11.39 616	9.99 965	42
	19	8.60 662	8.60 698	11.39 302	9.99 964	41
	20	8.60 973	8.61 009	11.38 991	9.99 964	40
	21	8.61 282	8.61 319	11.38 681	9.99 963	39
	22	8.61 589	8.61 626	11.38 374	9.99 963	38
	23	8.61 894	8.61 931	11.38 069	9.99 962	37
	24	8.62 196	8.62 234	11.37 766	9.99 962	36
	25	8.62 497	8.62 535	11.37 465	9.99 961	35
	26	8.62 795	8.62 834	11.37 166	9.99 961	34
	27	8.63 091	8.63 131	11.36 869	9.99 960	33
	28	8.63 385	8.63 426	11.36 574	9.99 960	32
	29	8.63 678	8.63 718	11.36 282	9.99 959	31
	30	8.63 968	8.64 009	11.35 991	9.99 959	30
	31	8.64 256	8.64 298	11.35 702	9.99 958	29
	32	8.64 543	8.64 585	11.35 415	9.99 958	28
	33	8.64 827	8.64 870	11.35 130	9.99 957	27
	34	8.65 110	8.65 154	11.34 846	9.99 956	26
	35	8.65 391	8.65 435	11.34 565	9.99 956	25
	36	8.65 670	8.65 715	11.34 285	9.99 955	24
	37	8.65 947	8.65 993	11.34 007	9.99 955	23
	38	8.66 223	8.66 269	11.33 731	9.99 954	22
	39	8.66 497	8.66 543	11.33 457	9.99 954	21
	40	8.66 769	8.66 816	11.33 184	9.99 953	20
	41	8.67 039	8.67 087	11.32 913	9.99 952	19
	42	8.67 308	8.67 356	11.32 644	9.99 952	18
	43	8.67 575	8.67 624	11.32 376	9.99 951	17
	44	8.67 841	8.67 890	11.32 110	9.99 951	16
	45	8.68 104	8.68 154	11.31 846	9.99 950	15
	46	8.68 367	8.68 417	11.31 583	9.99 949	14
	47	8.68 627	8.68 678	11.31 322	9.99 949	13
	48	8.68 886	8.68 938	11.31 062	9.99 948	12
	49	8.69 144	8.69 196	11.30 804	9.99 948	11
	50	8.69 400	8.69 453	11.30 547	9.99 947	10
	51	8.69 654	8.69 708	11.30 292	9.99 946	9
	52	8.69 907	8.69 962	11.30 038	9.99 946	8
	53	8.70 159	8.70 214	11.29 786	9.99 945	7
	54	8.70 409	8.70 465	11.29 535	9.99 944	6
	55	8.70 658	8.70 714	11.29 286	9.99 944	5
	56	8.70 905	8.70 962	11.29 038	9.99 943	4
	57	8.71 151	8.71 208	11.28 792	9.99 942	3
	58	8.71 395	8.71 453	11.28 547	9.99 942	2
	59	8.71 638	8.71 697	11.28 303	9.99 941	1
	60	8.71 880	8.71 940	11.28 060	9.99 940	0
Prop. Pts.		L Cos	L Cot	L Tan	L Sin	′

′	L Sin	d	L Tan	c d	L Cot	L Cos	′
0	8.71 880	240	8.71 940	241	11.28 060	9.99 940	60
1	8.72 120	239	8.72 181	239	11.27 819	9.99 940	59
2	8.72 359	238	8.72 420	239	11.27 580	9.99 939	58
3	8.72 597	237	8.72 659	237	11.27 341	9.99 938	57
4	8.72 834	235	8.72 896	236	11.27 104	9.99 938	56
5	8.73 069	234	8.73 132	234	11.26 868	9.99 937	55
6	8.73 303	232	8.73 366	234	11.26 634	9.99 936	54
7	8.73 535	232	8.73 600	232	11.26 400	9.99 936	53
8	8.73 767	230	8.73 832	231	11.26 168	9.99 935	52
9	8.73 997	229	8.74 063	229	11.25 937	9.99 934	51
10	8.74 226	228	8.74 292	229	11.25 708	9.99 934	50
11	8.74 454	226	8.74 521	227	11.25 479	9.99 933	49
12	8.74 680	226	8.74 748	226	11.25 252	9.99 932	48
13	8.74 906	224	8.74 974	225	11.25 026	9.99 932	47
14	8.75 130	223	8.75 199	224	11.24 801	9.99 931	46
15	8.75 353	222	8.75 423	222	11.24 577	9.99 930	45
16	8.75 575	220	8.75 645	222	11.24 355	9.99 929	44
17	8.75 795	220	8.75 867	220	11.24 133	9.99 929	43
18	8.76 015	219	8.76 087	219	11.23 913	9.99 928	42
19	8.76 234	217	8.76 306	219	11.23 694	9.99 927	41
20	8.76 451	216	8.76 525	217	11.23 475	9.99 926	40
21	8.76 667	216	8.76 742	216	11.23 258	9.99 926	39
22	8.76 883	214	8.76 958	215	11.23 042	9.99 925	38
23	8.77 097	213	8.77 173	214	11.22 827	9.99 924	37
24	8.77 310	212	8.77 387	213	11.22 613	9.99 923	36
25	8.77 522	211	8.77 600	211	11.22 400	9.99 923	35
26	8.77 733	210	8.77 811	211	11.22 189	9.99 922	34
27	8.77 943	209	8.78 022	210	11.21 978	9.99 921	33
28	8.78 152	208	8.78 232	209	11.21 768	9.99 920	32
29	8.78 360	208	8.78 441	208	11.21 559	9.99 920	31
30	8.78 568	206	8.78 649	206	11.21 351	9.99 919	30
31	8.78 774	205	8.78 855	206	11.21 145	9.99 918	29
32	8.78 979	204	8.79 061	205	11.20 939	9.99 917	28
33	8.79 183	203	8.79 266	204	11.20 734	9.99 917	27
34	8.79 386	202	8.79 470	203	11.20 530	9.99 916	26
35	8.79 588	201	8.79 673	202	11.20 327	9.99 915	25
36	8.79 789	201	8.79 875	201	11.20 125	9.99 914	24
37	8.79 990	199	8.80 076	201	11.19 924	9.99 913	23
38	8.80 189	199	8.80 277	199	11.19 723	9.99 913	22
39	8.80 388	197	8.80 476	198	11.19 524	9.99 912	21
40	8.80 585	197	8.80 674	198	11.19 326	9.99 911	20
41	8.80 782	196	8.80 872	196	11.19 128	9.99 910	19
42	8.80 978	195	8.81 068	196	11.18 932	9.99 909	18
43	8.81 173	194	8.81 264	195	11.18 736	9.99 909	17
44	8.81 367	193	8.81 459	194	11.18 541	9.99 908	16
45	8.81 560	192	8.81 653	193	11.18 347	9.99 907	15
46	8.81 752	192	8.81 846	192	11.18 154	9.99 906	14
47	8.81 944	190	8.82 038	192	11.17 962	9.99 905	13
48	8.82 134	190	8.82 230	190	11.17 770	9.99 904	12
49	8.82 324	189	8.82 420	190	11.17 580	9.99 904	11
50	8.82 513	188	8.82 610	189	11.17 390	9.99 903	10
51	8.82 701	187	8.82 799	188	11.17 201	9.99 902	9
52	8.82 888	187	8.82 987	188	11.17 013	9.99 901	8
53	8.83 075	186	8.83 175	186	11.16 825	9.99 900	7
54	8.83 261	185	8.83 361	186	11.16 639	9.99 899	6
55	8.83 446	184	8.83 547	185	11.16 453	9.99 898	5
56	8.83 630	183	8.83 732	184	11.16 268	9.99 898	4
57	8.83 813	183	8.83 916	184	11.16 084	9.99 897	3
58	8.83 996	181	8.84 100	182	11.15 900	9.99 896	2
59	8.84 177	181	8.84 282	182	11.15 718	9.99 895	1
60	8.84 358		8.84 464		11.15 536	9.99 894	0
	L Cos	d	L Cot	c d	L Tan	L Sin	′

Prop. Pts.

	241	239	237	235
.1	24.1	23.9	23.7	23.5
.2	48.2	47.8	47.4	47.0
.3	72.3	71.7	71.1	70.5
.4	96.4	95.6	94.8	94.0
.5	120.5	119.5	118.5	117.5
.6	144.6	143.4	142.2	141.0
.7	168.7	167.3	165.9	164.5
.8	192.8	191.2	189.6	188.0
.9	216.9	215.1	213.3	211.5

	234	232	230	228
.1	23.4	23.2	23.0	22.8
.2	46.8	46.4	46.0	45.6
.3	70.2	69.6	69.0	68.4
.4	93.6	92.8	92.0	91.2
.5	117.0	116.0	115.0	114.0
.6	140.4	139.2	138.0	136.8
.7	163.8	162.4	161.0	159.6
.8	187.2	185.6	184.0	182.4
.9	210.6	208.8	207.0	205.2

	227	225	223	221
.1	22.7	22.5	22.3	22.1
.2	45.4	45.0	44.6	44.2
.3	68.1	67.5	66.9	66.3
.4	90.8	90.0	89.2	88.4
.5	113.5	112.5	111.5	110.5
.6	136.2	135.0	133.8	132.6
.7	158.9	157.5	156.1	154.7
.8	181.6	180.0	178.4	176.8
.9	204.3	202.5	200.7	198.9

	220	218	216	214
.1	22.0	21.8	21.6	21.4
.2	44.0	43.6	43.2	42.8
.3	66.0	65.4	64.8	64.2
.4	88.0	87.2	86.4	85.6
.5	110.0	109.0	108.0	107.0
.6	132.0	130.8	129.6	128.4
.7	154.0	152.6	151.2	149.8
.8	176.0	174.4	172.8	171.2
.9	198.0	196.2	194.4	192.6

	213	211	209	207
.1	21.3	21.1	20.9	20.7
.2	42.6	42.2	41.8	41.4
.3	63.9	63.3	62.7	62.1
.4	85.2	84.4	83.6	82.8
.5	106.5	105.5	104.5	103.5
.6	127.8	126.6	125.4	124.2
.7	149.1	147.7	146.3	144.9
.8	170.4	168.8	167.2	165.6
.9	191.7	189.9	188.1	186.3

	206	204	202	200
.1	20.6	20.4	20.2	20.0
.2	41.2	40.8	40.4	40.0
.3	61.8	61.2	60.6	60.0
.4	82.4	81.6	80.8	80.0
.5	103.0	102.0	101.0	100.0
.6	123.6	122.4	121.2	120.0
.7	144.2	142.8	141.4	140.0
.8	164.8	163.2	161.6	160.0
.9	185.4	183.6	181.8	180.0

	199	197	195	193
.1	19.9	19.7	19.5	19.3
.2	39.8	39.4	39.0	38.6
.3	59.7	59.1	58.5	57.9
.4	79.6	78.8	78.0	77.2
.5	99.5	98.5	97.5	96.5
.6	119.4	118.2	117.0	115.8
.7	139.3	137.9	136.5	135.1
.8	159.2	157.6	156.0	154.4
.9	179.1	177.3	175.5	173.7

	192	190	188	186
.1	19.2	19.0	18.8	18.6
.2	38.4	38.0	37.6	37.2
.3	57.6	57.0	56.4	55.8
.4	76.8	76.0	75.2	74.4
.5	96.0	95.0	94.0	93.0
.6	115.2	114.0	112.8	111.6
.7	134.4	133.0	131.6	130.2
.8	153.6	152.0	150.4	148.8
.9	172.8	171.0	169.2	167.4

Prop. Pts.

Prop. Pts.				
	185	**183**	**181**	**179**
.1	18.5	18.3	18.1	17.9
.2	37.0	36.6	36.2	35.8
.3	55.5	54.9	54.3	53.7
.4	74.0	73.2	72.4	71.6
.5	92.5	91.5	90.5	89.5
.6	111.0	109.8	108.6	107.4
.7	129.5	128.1	126.7	125.3
.8	148.0	146.4	144.8	143.2
.9	166.5	164.7	162.9	161.1
	178	**176**	**174**	**172**
.1	17.8	17.6	17.4	17.2
.2	35.6	35.2	34.8	34.4
.3	53.4	52.8	52.2	51.6
.4	71.2	70.4	69.6	68.8
.5	89.0	88.0	87.0	86.0
.6	106.8	105.6	104.4	103.2
.7	124.6	123.2	121.8	120.4
.8	142.4	140.8	139.2	137.6
.9	160.2	158.4	156.6	154.8
	171	**169**	**167**	**165**
.1	17.1	16.9	16.7	16.5
.2	34.2	33.8	33.4	33.0
.3	51.3	50.7	50.1	49.5
.4	68.4	67.6	66.8	66.0
.5	85.5	84.5	83.5	82.5
.6	102.6	101.4	100.2	99.0
.7	119.7	118.3	116.9	115.5
.8	136.8	135.2	133.6	132.0
.9	153.9	152.1	150.3	148.5
	164	**163**	**162**	**161**
.1	16.4	16.3	16.2	16.1
.2	32.8	32.6	32.4	32.2
.3	49.2	48.9	48.6	48.3
.4	65.6	65.2	64.8	64.4
.5	82.0	81.5	81.0	80.5
.6	98.4	97.8	97.2	96.6
.7	114.8	114.1	113.4	112.7
.8	131.2	130.4	129.6	128.8
.9	147.6	146.7	145.8	144.9
	160	**159**	**158**	**157**
.1	16.0	15.9	15.8	15.7
.2	32.0	31.8	31.6	31.4
.3	48.0	47.7	47.4	47.1
.4	64.0	63.6	63.2	62.8
.5	80.0	79.5	79.0	78.5
.6	96.0	95.4	94.8	94.2
.7	112.0	111.3	110.6	109.9
.8	128.0	127.2	126.4	125.6
.9	144.0	143.1	142.2	141.3
	156	**155**	**154**	**153**
.1	15.6	15.5	15.4	15.3
.2	31.2	31.0	30.8	30.6
.3	46.8	46.5	46.2	45.9
.4	62.4	62.0	61.6	61.2
.5	78.0	77.5	77.0	76.5
.6	93.6	93.0	92.4	91.8
.7	109.2	108.5	107.8	107.1
.8	124.8	124.0	123.2	122.4
.9	140.4	139.5	138.6	137.7
	152	**151**	**150**	**149**
.1	15.2	15.1	15.0	14.9
.2	30.4	30.2	30.0	29.8
.3	45.6	45.3	35.0	44.7
.4	60.8	60.4	60.0	59.6
.5	76.0	75.5	75.0	74.5
.6	91.2	90.6	90.0	89.4
.7	106.4	105.7	105.0	104.3
.8	121.6	120.8	120.0	119.2
.9	136.8	135.9	135.0	134.1
	148	**147**	**146**	**145**
.1	14.8	14.7	14.6	14.5
.2	29.6	29.4	29.2	29.0
.3	44.4	44.1	43.8	43.5
.4	59.2	58.8	58.4	58.0
.5	74.0	73.5	73.0	72.5
.6	88.8	88.2	87.6	87.0
.7	103.6	102.9	102.2	101.5
.8	118.4	117.6	116.8	116.0
.9	133.2	132.3	131.4	130.5
Prop. Pts.				

′	L Sin	d	L Tan	c d	L Cot	L Cos	
0	8.84 358	181	8.84 464	182	11.15 536	9.99 894	60
1	8.84 530	179	8.84 646	180	11.15 354	9.99 893	59
2	8.84 718	179	8.84 826	180	11.15 174	9.99 892	58
3	8.84 897	178	8.85 006	179	11.14 994	9.99 891	57
4	8.85 075	177	8.85 185	178	11.14 815	9.99 891	56
5	8.85 252	177	8.85 363	177	11.14 637	9.99 890	55
6	8.85 429	176	8.85 540	177	11.14 460	9.99 889	54
7	8.85 605	175	8.85 717	176	11.14 283	9.99 888	53
8	8.85 780	175	8.85 893	176	11.14 107	9.99 887	52
9	8.85 955	173	8.86 069	174	11.13 931	9.99 886	51
10	8.86 128	173	8.86 243	174	11.13 757	9.99 885	50
11	8.86 301	173	8.86 417	174	11.13 583	9.99 884	49
12	8.86 474	171	8.86 591	172	11.13 409	9.99 883	48
13	8.86 645	171	8.86 763	172	11.13 237	9.99 882	47
14	8.86 816	171	8.86 935	171	11.13 065	9.99 881	46
15	8.86 987	169	8.87 106	171	11.12 894	9.99 880	45
16	8.87 156	169	8.87 277	170	11.12 723	9.99 879	44
17	8.87 325	169	8.87 447	169	11.12 553	9.99 879	43
18	8.87 494	167	8.87 616	169	11.12 384	9.99 878	42
19	8.87 661	168	8.87 785	168	11.12 215	9.99 877	41
20	8.87 829	166	8.87 953	167	11.12 047	9.99 876	40
21	8.87 995	166	8.88 120	167	11.11 880	9.99 875	39
22	8.88 161	165	8.88 287	166	11.11 713	9.99 874	38
23	8.88 326	164	8.88 453	165	11.11 547	9.99 873	37
24	8.88 490	164	8.88 618	165	11.11 382	9.99 872	36
25	8.88 654	163	8.88 783	165	11.11 217	9.99 871	35
26	8.88 817	163	8.88 948	163	11.11 052	9.99 870	34
27	8.88 980	162	8.89 111	163	11.10 889	9.99 869	33
28	8.89 142	162	8.89 274	163	11.10 726	9.99 868	32
29	8.89 304	160	8.89 437	161	11.10 563	9.99 867	31
30	8.89 464	161	8.89 598	162	11.10 402	9.99 866	30
31	8.89 625	159	8.89 760	160	11.10 240	9.99 865	29
32	8.89 784	159	8.89 920	160	11.10 080	9.99 864	28
33	8.89 943	159	8.90 080	160	11.09 920	9.99 863	27
34	8.90 102	158	8.90 240	159	11.09 760	9.99 862	26
35	8.90 260	157	8.90 399	158	11.09 601	9.99 861	25
36	8.90 417	157	8.90 557	158	11.09 443	9.99 860	24
37	8.90 574	156	8.90 715	157	11.09 285	9.99 859	23
38	8.90 730	155	8.90 872	157	11.09 128	9.99 858	22
39	8.90 885	155	8.91 029	156	11.08 971	9.99 857	21
40	8.91 040	155	8.91 185	155	11.08 815	9.99 856	20
41	8.91 195	154	8.91 340	155	11.08 660	9.99 855	19
42	8.91 349	153	8.91 495	155	11.08 505	9.99 854	18
43	8.91 502	153	8.91 650	153	11.08 350	9.99 853	17
44	8.91 655	152	8.91 803	154	11.08 197	9.99 852	16
45	8.91 807	152	8.91 957	153	11.08 043	9.99 851	15
46	8.91 959	151	8.92 110	152	11.07 890	9.99 850	14
47	8.92 110	151	8.92 262	152	11.07 738	9.99 848	13
48	8.92 261	150	8.92 414	151	11.07 586	9.99 847	12
49	8.92 411	150	8.92 565	151	11.07 435	9.99 846	11
50	8.92 561	149	8.92 716	150	11.07 284	9.99 845	10
51	8.92 710	149	8.92 866	150	11.07 134	9.99 844	9
52	8.92 859	148	8.93 016	149	11.06 984	9.99 843	8
53	8.93 007	147	8.93 165	148	11.06 835	9.99 842	7
54	8.93 154	147	8.93 313	149	11.06 687	9.99 841	6
55	8.93 301	147	8.93 462	147	11.06 538	9.99 840	5
56	8.93 448	146	8.93 609	147	11.06 391	9.99 839	4
57	8.93 594	146	8.93 756	147	11.06 244	9.99 838	3
58	8.93 740	145	8.93 903	146	11.06 097	9.99 837	2
59	8.93 885	145	8.94 049	146	11.05 951	9.99 836	1
60	8.94 030		8.94 195		11.05 805	9.99 834	0
	L Cos	d	L Cot	c d	L Tan	L Sin	′

85°

′	L Sin	d	L Tan	c d	L Cot	L Cos	
0	8.94 030	144	8.94 195	145	11.05 805	9.99 834	60
1	8.94 174	143	8.94 340	145	11.05 660	9.99 833	59
2	8.94 317	144	8.94 485	145	11.05 515	9.99 832	58
3	8.94 461	142	8.94 630	143	11.05 370	9.99 831	57
4	8.94 603	143	8.94 773	144	11.05 227	9.99 830	56
5	8.94 746	141	8.94 917	143	11.05 083	9.99 829	55
6	8.94 887	142	8.95 060	142	11.04 940	9.99 828	54
7	8.95 029	141	8.95 202	142	11.04 798	9.99 827	53
8	8.95 170	140	8.95 344	142	11.04 656	9.99 825	52
9	8.95 310	140	8.95 486	141	11.04 514	9.99 824	51
10	8.95 450	139	8.95 627	140	11.04 373	9.99 823	50
11	8.95 589	139	8.95 767	141	11.04 233	9.99 822	49
12	8.95 728	139	8.95 908	139	11.04 092	9.99 821	48
13	8.95 867	138	8.96 047	140	11.03 953	9.99 820	47
14	8.96 005	138	8.96 187	138	11.03 813	9.99 819	46
15	8.96 143	137	8.96 325	139	11.03 675	9.99 817	45
16	8.96 280	137	8.96 464	138	11.03 536	9.99 816	44
17	8.96 417	136	8.96 602	137	11.03 398	9.99 815	43
18	8.96 553	136	8.96 739	138	11.03 261	9.99 814	42
19	8.96 689	136	8.96 877	136	11.03 123	9.99 813	41
20	8.96 825	135	8.97 013	137	11.02 987	9.99 812	40
21	8.96 960	135	8.97 150	135	11.02 850	9.99 810	39
22	8.97 095	134	8.97 285	136	11.02 715	9.99 809	38
23	8.97 229	134	8.97 421	135	11.02 579	9.99 808	37
24	8.97 363	133	8.97 556	135	11.02 444	9.99 807	36
25	8.97 496	133	8.97 691	134	11.02 309	9.99 806	35
26	8.97 629	133	8.97 825	134	11.02 175	9.99 804	34
27	8.97 762	132	8.97 959	133	11.02 041	9.99 803	33
28	8.97 894	132	8.98 092	133	11.01 908	9.99 802	32
29	8.98 026	131	8.98 225	133	11.01 775	9.99 801	31
30	8.98 157	131	8.98 358	132	11.01 642	9.99 800	30
31	8.98 288	131	8.98 490	132	11.01 510	9.99 798	29
32	8.98 419	130	8.98 622	131	11.01 378	9.99 797	28
33	8.98 549	130	8.98 753	131	11.01 247	9.99 796	27
34	8.98 679	129	8.98 884	131	11.01 116	9.99 795	26
35	8.98 808	129	8.99 015	130	11.00 985	9.99 793	25
36	8.98 937	129	8.99 145	130	11.00 855	9.99 792	24
37	8.99 066	128	8.99 275	130	11.00 725	9.99 791	23
38	8.99 194	128	8.99 405	129	11.00 595	9.99 790	22
39	8.99 322	128	8.99 534	128	11.00 466	9.99 788	21
40	8.99 450	127	8.99 662	129	11.00 338	9.99 787	20
41	8.99 577	127	8.99 791	128	11.00 209	9.99 786	19
42	8.99 704	126	8.99 919	127	11.00 081	9.99 785	18
43	8.99 830	126	9.00 046	128	10.99 954	9.99 783	17
44	8.99 956	126	9.00 174	127	10.99 826	9.99 782	16
45	9.00 082	125	9.00 301	126	10.99 699	9.99 781	15
46	9.00 207	125	9.00 427	126	10.99 573	9.99 780	14
47	9.00 332	124	9.00 553	126	10.99 447	9.99 778	13
48	9.00 456	125	9.00 679	126	10.99 321	9.99 777	12
49	9.00 581	123	9.00 805	125	10.99 195	9.99 776	11
50	9.00 704	124	9.00 930	125	10.99 070	9.99 775	10
51	9.00 828	123	9.01 055	124	10.98 945	9.99 773	9
52	9.00 951	123	9.01 179	124	10.98 821	9.99 772	8
53	9.01 074	122	9.01 303	124	10.98 697	9.99 771	7
54	9.01 196	122	9.01 427	123	10.98 573	9.99 769	6
55	9.01 318	122	9.01 550	123	10.98 450	9.99 768	5
56	9.01 440	121	9.01 673	123	10.98 327	9.99 767	4
57	9.01 561	121	9.01 796	122	10.98 204	9.99 765	3
58	9.01 682	121	9.01 918	122	10.98 082	9.99 764	2
59	9.01 803	120	9.02 040	122	10.97 960	9.99 763	1
60	9.01 923		9.02 162		10.97 838	9.99 761	0
	L Cos	d	L Cot	c d	L Tan	L Sin	′

Prop. Pts.

	145	143	141
.1	14.5	14.3	14.1
.2	29.0	28.6	28.2
.3	43.5	42.9	42.3
.4	58.0	57.2	56.4
.5	72.5	71.5	70.5
.6	87.0	85.8	84.6
.7	101.5	100.1	98.7
.8	116.0	114.4	112.8
.9	130.5	128.7	126.9

	140	138	136
.1	14.0	13.8	13.6
.2	28.0	27.6	27.2
.3	42.0	41.4	40.8
.4	56.0	55.2	54.4
.5	70.0	69.0	68.0
.6	84.0	82.8	81.6
.7	98.0	96.6	95.2
.8	112.0	110.4	108.8
.9	126.0	124.2	122.4

	135	133	131
.1	13.5	13.3	13.1
.2	27.0	26.6	26.2
.3	40.5	39.9	39.3
.4	54.0	53.2	52.4
.5	67.5	66.5	65.5
.6	81.0	79.8	78.6
.7	94.5	93.1	91.7
.8	108.0	106.4	104.8
.9	121.5	119.7	117.9

	130	128	126
.1	13.0	12.8	12.6
.2	26.0	25.6	25.2
.3	39.0	38.4	37.8
.4	52.0	51.2	50.4
.5	65.0	64.0	63.0
.6	78.0	76.8	75.6
.7	91.0	89.6	88.2
.8	104.0	102.4	100.8
.9	117.0	115.2	113.4

	125	124	123
.1	12.5	12.4	12.3
.2	25.0	24.8	24.6
.3	37.5	37.2	36.9
.4	50.0	49.6	49.2
.5	62.5	62.0	61.5
.6	75.0	74.4	73.8
.7	87.5	86.8	86.1
.8	100.0	99.2	98.4
.9	112.5	111.6	110.7

	122	121	120
.1	12.2	12.1	12.0
.2	24.4	24.2	24.0
.3	36.6	36.3	36.0
.4	48.8	48.4	48.0
.5	61.0	60.5	60.0
.6	73.2	72.6	72.0
.7	85.4	84.7	84.0
.8	97.6	96.8	96.0
.9	109.8	108.9	108.0

Prop. Pts.

Prop. Pts.

	120	119	118
.1	12.0	11.9	11.8
.2	24.0	23.8	23.6
.3	36.0	35.7	35.4
.4	48.0	47.6	47.2
.5	60.0	59.5	59.0
.6	72.0	71.4	70.8
.7	84.0	83.3	82.6
.8	96.0	95.2	94.4
.9	108.0	107.1	106.2

	117	116	115
.1	11.7	11.6	11.5
.2	23.4	23.2	23.0
.3	35.1	34.8	34.5
.4	46.8	46.4	46.0
.5	58.5	58.0	57.5
.6	70.2	69.6	69.0
.7	81.9	81.2	80.5
.8	93.6	92.8	92.0
.9	105.3	104.4	103.5

	114	113	112
.1	11.4	11.3	11.2
.2	22.8	22.6	22.4
.3	34.2	33.9	33.6
.4	45.6	45.2	44.8
.5	57.0	56.5	56.0
.6	68.4	67.8	67.2
.7	79.8	79.1	78.4
.8	91.2	90.4	89.6
.9	102.6	101.7	100.8

	111	110	109
.1	11.1	11.0	10.9
.2	22.2	22.0	21.8
.3	33.3	33.0	32.7
.4	44.4	44.0	43.6
.5	55.5	55.0	54.5
.6	66.6	66.0	65.4
.7	77.7	77.0	76.3
.8	88.8	88.0	87.2
.9	99.9	99.0	98.1

	108	107	106
.1	10.8	10.7	10.6
.2	21.6	21.4	21.2
.3	32.4	32.1	31.8
.4	43.2	42.8	42.4
.5	54.0	53.5	53.0
.6	64.8	64.2	63.6
.7	75.6	74.9	74.2
.8	86.4	85.6	84.8
.9	97.2	96.3	95.4

	105	104	103
.1	10.5	10.4	10.3
.2	21.0	20.8	20.6
.3	31.5	31.2	30.9
.4	42.0	41.6	41.2
.5	52.5	52.0	51.5
.6	63.0	62.4	61.8
.7	73.5	72.8	72.1
.8	84.0	83.2	82.4
.9	94.5	93.6	92.7

Prop. Pts.

′	L Sin	d	L Tan	c d	L Cot	L Cos	
0	9.01 923	120	9.02 162	121	10.97 838	9.99 761	60
1	9.02 043	120	9.02 283	121	10.97 717	9.99 760	59
2	9.02 163	120	9.02 404	121	10.97 596	9.99 759	58
3	9.02 283	119	9.02 525	120	10.97 475	9.99 757	57
4	9.02 402	118	9.02 645	121	10.97 355	9.99 756	56
5	9.02 520	119	9.02 766	119	10.97 234	9.99 755	55
6	9.02 639	118	9.02 885	120	10.97 115	9.99 753	54
7	9.02 757	117	9.03 005	119	10.96 995	9.99 752	53
8	9.02 874	118	9.03 124	118	10.96 876	9.99 751	52
9	9.02 992	117	9.03 242	119	10.96 758	9.99 749	51
10	9.03 109	117	9.03 361	118	10.96 639	9.99 748	50
11	9.03 226	116	9.03 479	118	10.96 521	9.99 747	49
12	9.03 342	116	9.03 597	117	10.96 403	9.99 745	48
13	9.03 458	116	9.03 714	118	10.96 286	9.99 744	47
14	9.03 574	116	9.03 832	116	10.96 168	9.99 742	46
15	9.03 690	115	9.03 948	117	10.96 052	9.99 741	45
16	9.03 805	115	9.04 065	116	10.95 935	9.99 740	44
17	9.03 920	114	9.04 181	116	10.95 819	9.99 738	43
18	9.04 034	115	9.04 297	116	10.95 703	9.99 737	42
19	9.04 149	113	9.04 413	115	10.95 587	9.99 736	41
20	9.04 262	114	9.04 528	115	10.95 472	9.99 734	40
21	9.04 376	114	9.04 643	115	10.95 357	9.99 733	39
22	9.04 490	113	9.04 758	115	10.95 242	9.99 731	38
23	9.04 603	112	9.04 873	114	10.95 127	9.99 730	37
24	9.04 715	113	9.04 987	114	10.95 013	9.99 728	36
25	9.04 828	112	9.05 101	113	10.94 899	9.99 727	35
26	9.04 940	112	9.05 214	114	10.94 786	9.99 726	34
27	9.05 052	112	9.05 328	113	10.94 672	9.99 724	33
28	9.05 164	111	9.05 441	112	10.94 559	9.99 723	32
29	9.05 275	111	9.05 553	113	10.94 447	9.99 721	31
30	9.05 386	111	9.05 666	112	10.94 334	9.99 720	30
31	9.05 497	110	9.05 778	112	10.94 222	9.99 718	29
32	9.05 607	110	9.05 890	112	10.94 110	9.99 717	28
33	9.05 717	110	9.06 002	111	10.93 998	9.99 716	27
34	9.05 827	110	9.06 113	111	10.93 887	9.99 714	26
35	9.05 937	109	9.06 224	111	10.93 776	9.99 713	25
36	9.06 046	109	9.06 335	110	10.93 665	9.99 711	24
37	9.06 155	109	9.06 445	111	10.93 555	9.99 710	23
38	9.06 264	108	9.06 556	110	10.93 444	9.99 708	22
39	9.06 372	109	9.06 666	109	10.93 334	9.99 707	21
40	9.06 481	108	9.06 775	110	10.93 225	9.99 705	20
41	9.06 589	107	9.06 885	109	10.93 115	9.99 704	19
42	9.06 696	108	9.06 994	109	10.93 006	9.99 702	18
43	9.06 804	107	9.07 103	108	10.92 897	9.99 701	17
44	9.06 911	107	9.07 211	109	10.92 789	9.99 699	16
45	9.07 018	106	9.07 320	108	10.92 680	9.99 698	15
46	9.07 124	107	9.07 428	108	10.92 572	9.99 696	14
47	9.07 231	106	9.07 536	107	10.92 464	9.99 695	13
48	9.07 337	105	9.07 643	108	10.92 357	9.99 693	12
49	9.07 442	106	9.07 751	107	10.92 249	9.99 692	11
50	9.07 548	105	9.07 858	106	10.92 142	9.99 690	10
51	9.07 653	105	9.07 964	107	10.92 036	9.99 689	9
52	9.07 758	105	9.08 071	106	10.91 929	9.99 687	8
53	9.07 863	105	9.08 177	106	10.91 823	9.99 686	7
54	9.07 968	104	9.08 283	106	10.91 717	9.99 684	6
55	9.08 072	104	9.08 389	106	10.91 611	9.99 683	5
56	9.08 176	104	9.08 495	105	10.91 505	9.99 681	4
57	9.08 280	103	9.08 600	105	10.91 400	9.99 680	3
58	9.08 383	103	9.08 705	105	10.91 295	9.99 678	2
59	9.08 486	103	9.08 810	104	10.91 190	9.99 677	1
60	9.08 589		9.08 914		10.91 086	9.99 675	0
	L Cos	d	L Cot	c d	L Tan	L Sin	′

83°

′	L Sin	d	L Tan	c d	L Cot	L Cos	
0	9.08 589	103	9.08 914	105	10.91 086	9.99 675	60
1	9.08 692	103	9.09 019	104	10.90 981	9.99 674	59
2	9.08 795	102	9.09 123	104	10.90 877	9.99 672	58
3	9.08 897	102	9.09 227	103	10.90 773	9.99 670	57
4	9.08 999	102	9.09 330	104	10.90 670	9.99 669	56
5	9.09 101	101	9.09 434	103	10.90 566	9.99 667	55
6	9.09 202	102	9.09 537	103	10.90 463	9.99 666	54
7	9.09 304	101	9.09 640	102	10.90 360	9.99 664	53
8	9.09 405	101	9.09 742	103	10.90 258	9.99 663	52
9	9.09 506	100	9.09 845	102	10.90 155	9.99 661	51
10	9.09 606	101	9.09 947	102	10.90 053	9.99 659	50
11	9.09 707	100	9.10 049	101	10.89 951	9.99 658	49
12	9.09 807	100	9.10 150	102	10.89 850	9.99 656	48
13	9.09 907	99	9.10 252	101	10.89 748	9.99 655	47
14	9.10 006	100	9.10 353	101	10.89 647	9.99 653	46
15	9.10 106	99	9.10 454	101	10.89 546	9.99 651	45
16	9.10 205	99	9.10 555	101	10.89 445	9.99 650	44
17	9.10 304	98	9.10 656	100	10.89 344	9.99 648	43
18	9.10 402	99	9.10 756	100	10.89 244	9.99 647	42
19	9.10 501	98	9.10 856	100	10.89 144	9.99 645	41
20	9.10 599	98	9.10 956	100	10.89 044	9.99 643	40
21	9.10 697	98	9.11 056	99	10.88 944	9.99 642	39
22	9.10 795	98	9.11 155	99	10.88 845	9.99 640	38
23	9.10 893	97	9.11 254	99	10.88 746	9.99 638	37
24	9.10 990	97	9.11 353	99	10.88 647	9.99 637	36
25	9.11 087	97	9.11 452	99	10.88 548	9.99 635	35
26	9.11 184	97	9.11 551	98	10.88 449	9.99 633	34
27	9.11 281	96	9.11 649	98	10.88 351	9.99 632	33
28	9.11 377	97	9.11 747	98	10.88 253	9.99 630	32
29	9.11 474	96	9.11 845	98	10.88 155	9.99 629	31
30	9.11 570	96	9.11 943	97	10.88 057	9.99 627	30
31	9.11 666	95	9.12 040	98	10.87 960	9.99 625	29
32	9.11 761	96	9.12 138	97	10.87 862	9.99 624	28
33	9.11 857	95	9.12 235	97	10.87 765	9.99 622	27
34	9.11 952	95	9.12 332	96	10.87 668	9.99 620	26
35	9.12 047	95	9.12 428	97	10.87 572	9.99 618	25
36	9.12 142	94	9.12 525	96	10.87 475	9.99 617	24
37	9.12 236	95	9.12 621	96	10.87 379	9.99 615	23
38	9.12 331	94	9.12 717	96	10.87 283	9.99 613	22
39	9.12 425	94	9.12 813	96	10.87 187	9.99 612	21
40	9.12 519	93	9.12 909	95	10.87 091	9.99 610	20
41	9.12 612	94	9.13 004	95	10.86 996	9.99 608	19
42	9.12 706	93	9.13 099	95	10.86 901	9.99 607	18
43	9.12 799	93	9.13 194	95	10.86 806	9.99 605	17
44	9.12 892	93	9.13 289	95	10.86 711	9.99 603	16
45	9.12 985	93	9.13 384	94	10.86 616	9.99 601	15
46	9.13 078	93	9.13 478	95	10.86 522	9.99 600	14
47	9.13 171	92	9.13 573	94	10.86 427	9.99 598	13
48	9.13 263	92	9.13 667	94	10.86 333	9.99 596	12
49	9.13 355	92	9.13 761	93	10.86 239	9.99 595	11
50	9.13 447	92	9.13 854	94	10.86 146	9.99 593	10
51	9.13 539	91	9.13 948	93	10.86 052	9.99 591	9
52	9.13 630	92	9.14 041	93	10.85 959	9.99 589	8
53	9.13 722	91	9.14 134	93	10.85 866	9.99 588	7
54	9.13 813	91	9.14 227	93	10.85 773	9.99 586	6
55	9.13 904	90	9.14 320	92	10.85 680	9.99 584	5
56	9.13 994	91	9.14 412	92	10.85 588	9.99 582	4
57	9.14 085	90	9.14 504	93	10.85 496	9.99 581	3
58	9.14 175	91	9.14 597	91	10.85 403	9.99 579	2
59	9.14 266	90	9.14 688	92	10.85 312	9.99 577	1
60	9.14 356		9.14 780		10.85 220	9.99 575	0
	L Cos	d	L Cot	c d	L Tan	L Sin	′

Prop. Pts.

	105	104	103
.1	10.5	10.4	10.3
.2	21.0	20.8	20.6
.3	31.5	31.2	30.9
.4	42.0	41.6	41.2
.5	52.5	52.0	51.5
.6	63.0	62.4	61.8
.7	73.5	72.8	72.1
.8	84.0	83.2	82.4
.9	94.5	93.6	92.7

	102	101	100
.1	10.2	10.1	10.0
.2	20.4	20.2	20.0
.3	30.6	30.3	30.0
.4	40.8	40.4	40.0
.5	51.0	50.5	50.0
.6	61.2	60.6	60.0
.7	71.4	70.7	70.0
.8	81.6	80.8	80.0
.9	91.8	90.9	90.0

	99	98	97
.1	9.9	9.8	9.7
.2	19.8	19.6	19.4
.3	29.7	29.4	29.1
.4	39.6	39.2	38.8
.5	49.5	49.0	48.5
.6	59.4	58.8	58.2
.7	69.3	68.6	67.9
.8	79.2	78.4	77.6
.9	89.1	88.2	87.3

	96	95	94
.1	9.6	9.5	9.4
.2	19.2	19.0	18.8
.3	28.8	28.5	28.2
.4	38.4	38.0	37.6
.5	48.0	47.5	47.0
.6	57.6	57.0	56.4
.7	67.2	66.5	65.8
.8	76.8	76.0	75.2
.9	86.4	85.5	84.6

	93	92	91
.1	9.3	9.2	9.1
.2	18.6	18.4	18.2
.3	27.9	27.6	27.3
.4	37.2	36.8	36.4
.5	46.5	46.0	45.5
.6	55.8	55.2	54.6
.7	65.1	64.4	63.7
.8	74.4	73.6	72.8
.9	83.7	82.8	81.9

	90	2	1
.1	9.0	0.2	0.1
.2	18.0	0.4	0.2
.3	27.0	0.6	0.3
.4	36.0	0.8	0.4
.5	45.0	1.0	0.5
.6	54.0	1.2	0.6
.7	63.0	1.4	0.7
.8	72.0	1.6	0.8
.9	81.0	1.8	0.9

Prop. Pts.

'	L Sin	d	L Tan	c d	L Cot	L Cos	
0	9.14 356	89	9.14 780	92	10.85 220	9.99 575	60
1	9.14 445	90	9.14 872	91	10.85 128	9.99 574	59
2	9.14 535	89	9.14 963	91	10.85 037	9.99 572	58
3	9.14 624	90	9.15 054	91	10.84 946	9.99 570	57
4	9.14 714	89	9.15 145	91	10.84 855	9.99 568	56
5	9.14 803	88	9.15 236	91	10.84 764	9.99 566	55
6	9.14 891	89	9.15 327	90	10.84 673	9.99 565	54
7	9.14 980	89	9.15 417	91	10.84 583	9.99 563	53
8	9.15 069	88	9.15 508	90	10.84 492	9.99 561	52
9	9.15 157	88	9.15 598	90	10.84 402	9.99 559	51
10	9.15 245	88	9.15 688	89	10.84 312	9.99 557	50
11	9.15 333	88	9.15 777	90	10.84 223	9.99 556	49
12	9.15 421	87	9.15 867	89	10.84 133	9.99 554	48
13	9.15 508	88	9.15 956	90	10.84 044	9.99 552	47
14	9.15 596	87	9.16 046	89	10.83 954	9.99 550	46
15	9.15 683	87	9.16 135	89	10.83 865	9.99 548	45
16	9.15 770	87	9.16 224	88	10.83 776	9.99 546	44
17	9.15 857	87	9.16 312	89	10.83 688	9.99 545	43
18	9.15 944	86	9.16 401	88	10.83 599	9.99 543	42
19	9.16 030	86	9.16 489	88	10.83 511	9.99 541	41
20	9.16 116	87	9.16 577	88	10.83 423	9.99 539	40
21	9.16 203	86	9.16 665	88	10.83 335	9.99 537	39
22	9.16 289	85	9.16 753	88	10.83 247	9.99 535	38
23	9.16 374	86	9.16 841	87	10.83 159	9.99 533	37
24	9.16 460	85	9.16 928	88	10.83 072	9.99 532	36
25	9.16 545	86	9.17 016	87	10.82 984	9.99 530	35
26	9.16 631	85	9.17 103	87	10.82 897	9.99 528	34
27	9.16 716	85	9.17 190	87	10.82 810	9.99 526	33
28	9.16 801	85	9.17 277	86	10.82 723	9.99 524	32
29	9.16 886	84	9.17 363	87	10.82 637	9.99 522	31
30	9.16 970	85	9.17 450	86	10.82 550	9.99 520	30
31	9.17 055	84	9.17 536	86	10.82 464	9.99 518	29
32	9.17 139	84	9.17 622	86	10.82 378	9.99 517	28
33	9.17 223	84	9.17 708	86	10.82 292	9.99 515	27
34	9.17 307	84	9.17 794	86	10.82 206	9.99 513	26
35	9.17 391	83	9.17 880	85	10.82 120	9.99 511	25
36	9.17 474	84	9.17 965	86	10.82 035	9.99 509	24
37	9.17 558	83	9.18 051	85	10.81 949	9.99 507	23
38	9.17 641	83	9.18 136	85	10.81 864	9.99 505	22
39	9.17 724	83	9.18 221	85	10.81 779	9.99 503	21
40	9.17 807	83	9.18 306	85	10.81 694	9.99 501	20
41	9.17 890	83	9.18 391	84	10.81 609	9.99 499	19
42	9.17 973	82	9.18 475	85	10.81 525	9.99 497	18
43	9.18 055	82	9.18 560	84	10.81 440	9.99 495	17
44	9.18 137	83	9.18 644	84	10.81 356	9.99 494	16
45	9.18 220	82	9.18 728	84	10.81 272	9.99 492	15
46	9.18 302	81	9.18 812	84	10.81 188	9.99 490	14
47	9.18 383	82	9.18 896	83	10.81 104	9.99 488	13
48	9.18 465	82	9.18 979	84	10.81 021	9.99 486	12
49	9.18 547	81	9.19 063	83	10.80 937	9.99 484	11
50	9.18 628	81	9.19 146	83	10.80 854	9.99 482	10
51	9.18 709	81	9.19 229	83	10.80 771	9.99 480	9
52	9.18 790	81	9.19 312	83	10.80 688	9.99 478	8
53	9.18 871	81	9.19 395	83	10.80 605	9.99 476	7
54	9.18 952	81	9.19 478	83	10.80 522	9.99 474	6
55	9.19 033	80	9.19 561	82	10.80 439	9.99 472	5
56	9.19 113	80	9.19 643	82	10.80 357	9.99 470	4
57	9.19 193	80	9.19 725	82	10.80 275	9.99 468	3
58	9.19 273	80	9.19 807	82	10.80 193	9.99 466	2
59	9.19 353	80	9.19 889	82	10.80 111	9.99 464	1
60	9.19 433		9.19 971		10.80 029	9.99 462	0
	L Cos	d	L Cot	c d	L Tan	L Sin	'

Prop. Pts.

	92	91	90
.1	9.2	9.1	9.0
.2	18.4	18.2	18.0
.3	27.6	27.3	27.0
.4	36.8	36.4	36.0
.5	46.0	45.5	45.0
.6	55.2	54.6	54.0
.7	64.4	63.7	63.0
.8	73.6	72.8	72.0
.9	82.8	81.9	81.0

	89	88	87
.1	8.9	8.8	8.7
.2	17.8	17.6	17.4
.3	26.7	26.4	26.1
.4	35.6	35.2	34.8
.5	44.5	44.0	43.5
.6	53.4	52.8	52.2
.7	62.3	61.6	60.9
.8	71.2	70.4	69.6
.9	80.1	79.2	78.3

	86	85	84
.1	8.6	8.5	8.4
.2	17.2	17.0	16.8
.3	25.8	25.5	25.2
.4	34.4	34.0	33.6
.5	43.0	42.5	42.0
.6	51.6	51.0	50.4
.7	60.2	59.5	58.8
.8	68.8	68.0	67.2
.9	77.4	76.5	75.6

	83	82	81
.1	8.3	8.2	8.1
.2	16.6	16.4	16.2
.3	24.9	24.6	24.3
.4	33.2	32.8	32.4
.5	41.5	41.0	40.5
.6	49.8	49.2	48.6
.7	58.1	57.4	56.7
.8	66.4	65.6	64.8
.9	74.7	73.8	72.9

	80	2	1
.1	8.0	0.2	0.1
.2	16.0	0.4	0.2
.3	24.0	0.6	0.3
.4	32.0	0.8	0.4
.5	40.0	1.0	0.5
.6	48.0	1.2	0.6
.7	56.0	1.4	0.7
.8	64.0	1.6	0.8
.9	72.0	1.8	0.9

Prop. Pts.

′	L Sin	d	L Tan	c d	L Cot	L Cos	′
0	9.19 433	80	9.19 971	82	10.80 029	9.99 462	60
1	9.19 513	79	9.20 053	81	10.79 947	9.99 460	59
2	9.19 592	80	9.20 134	82	10.79 866	9.99 458	58
3	9.19 672	79	9.20 216	81	10.79 784	9.99 456	57
4	9.19 751	79	9.20 297	81	10.79 703	9.99 454	56
5	9.19 830	79	9.20 378	81	10.79 622	9.99 452	55
6	9.19 909	79	9.20 459	81	10.79 541	9.99 450	54
7	9.19 988	79	9.20 540	81	10.79 460	9.99 448	53
8	9.20 067	78	9.20 621	80	10.79 379	9.99 446	52
9	9.20 145	78	9.20 701	81	10.79 299	9.99 444	51
10	9.20 223	79	9.20 782	80	10.79 218	9.99 442	50
11	9.20 302	78	9.20 862	80	10.79 138	9.99 440	49
12	9.20 380	78	9.20 942	80	10.79 058	9.99 438	48
13	9.20 458	77	9.21 022	80	10.78 978	9.99 436	47
14	9.20 535	78	9.21 102	80	10.78 898	9.99 434	46
15	9.20 613	78	9.21 182	79	10.78 818	9.99 432	45
16	9.20 691	77	9.21 261	80	10.78 739	9.99 429	44
17	9.20 768	77	9.21 341	79	10.78 659	9.99 427	43
18	9.20 845	77	9.21 420	79	10.78 580	9.99 425	42
19	9.20 922	77	9.21 499	79	10.78 501	9.99 423	41
20	9.20 999	77	9.21 578	79	10.78 422	9.99 421	40
21	9.21 076	77	9.21 657	79	10.78 343	9.99 419	39
22	9.21 153	76	9.21 736	78	10.78 264	9.99 417	38
23	9.21 229	77	9.21 814	79	10.78 186	9.99 415	37
24	9.21 306	76	9.21 893	78	10.78 107	9.99 413	36
25	9.21 382	76	9.21 971	78	10.78 029	9.99 411	35
26	9.21 458	76	9.22 049	78	10.77 951	9.99 409	34
27	9.21 534	76	9.22 127	78	10.77 873	9.99 407	33
28	9.21 610	75	9.22 205	78	10.77 795	9.99 404	32
29	9.21 685	76	9.22 283	78	10.77 717	9.99 402	31
30	9.21 761	75	9.22 361	77	10.77 639	9.99 400	30
31	9.21 836	76	9.22 438	78	10.77 562	9.99 398	29
32	9.21 912	75	9.22 516	77	10.77 484	9.99 396	28
33	9.21 987	75	9.22 593	77	10.77 407	9.99 394	27
34	9.22 062	75	9.22 670	77	10.77 330	9.99 392	26
35	9.22 137	74	9.22 747	77	10.77 253	9.99 390	25
36	9.22 211	75	9.22 824	77	10.77 176	9.99 388	24
37	9.22 286	75	9.22 901	76	10.77 099	9.99 385	23
38	9.22 361	74	9.22 977	77	10.77 023	9.99 383	22
39	9.22 435	74	9.23 054	76	10.76 946	9.99 381	21
40	9.22 509	74	9.23 130	76	10.76 870	9.99 379	20
41	9.22 583	74	9.23 206	77	10.76 794	9.99 377	19
42	9.22 657	74	9.23 283	76	10.76 717	9.99 375	18
43	9.22 731	74	9.23 359	76	10.76 641	9.99 372	17
44	9.22 805	73	9.23 435	75	10.76 565	9.99 370	16
45	9.22 878	74	9.23 510	76	10.76 490	9.99 368	15
46	9.22 952	73	9.23 586	75	10.76 414	9.99 366	14
47	9.23 025	73	9.23 661	76	10.76 339	9.99 364	13
48	9.23 098	73	9.23 737	75	10.76 263	9.99 362	12
49	9.23 171	73	9.23 812	75	10.76 188	9.99 359	11
50	9.23 244	73	9.23 887	75	10.76 113	9.99 357	10
51	9.23 317	73	9.23 962	75	10.76 038	9.99 355	9
52	9.23 390	72	9.24 037	75	10.75 963	9.99 353	8
53	9.23 462	73	9.24 112	74	10.75 888	9.99 351	7
54	9.23 535	72	9.24 186	75	10.75 814	9.99 348	6
55	9.23 607	72	9.24 261	74	10.75 739	9.99 346	5
56	9.23 679	73	9.24 335	75	10.75 665	9.99 344	4
57	9.23 752	71	9.24 410	74	10.75 590	9.99 342	3
58	9.23 823	72	9.24 484	74	10.75 516	9.99 340	2
59	9.23 895	72	9.24 558	74	10.75 442	9.99 337	1
60	9.23 967		9.24 632		10.75 368	9.99 335	0
	L Cos	d	L Cot	c d	L Tan	L Sin	′

Prop. Pts.

82	81
8.2	8.1
16.4	16.2
24.6	24.3
32.8	32.4
41.0	40.5
49.2	48.6
57.4	56.7
65.6	64.8
73.8	72.9

	80	79	78
.1	8.0	7.9	7.8
.2	16.0	15.8	15.6
.3	24.0	23.7	13.4
.4	32.0	31.6	31.2
.5	40.0	39.5	39.0
.6	48.0	47.4	46.8
.7	56.0	55.3	54.6
.8	64.0	63.2	62.4
.9	72.0	71.1	70.2

	77	76	75
.1	7.7	7.6	7.5
.2	15.4	15.2	15.0
.3	23.1	22.8	22.5
.4	30.8	30.4	30.0
.5	38.5	38.0	37.5
.6	46.2	45.6	45.0
.7	53.9	53.2	52.5
.8	61.6	60.8	60.0
.9	69.3	68.4	67.5

	74	73	72
.1	7.4	7.3	7.2
.2	14.8	14.6	14.4
.3	22.2	21.9	21.6
.4	29.6	29.2	28.8
.5	37.0	36.5	36.0
.6	44.4	43.8	43.2
.7	51.8	51.1	50.4
.8	59.2	58.4	57.6
.9	66.6	65.7	64.8

	71	3	2
.1	7.1	0.3	0.2
.2	14.2	0.6	0.4
.3	21.3	0.9	0.6
.4	28.4	1.2	0.8
.5	35.5	1.5	1.0
.6	42.6	1.8	1.2
.7	49.7	2.1	1.4
.8	56.8	2.4	1.6
.9	63.9	2.7	1.8

Prop. Pts.

′	L Sin	d	L Tan	c d	L Cot	L Cos	′
0	9.23 967	72	9.24 632	74	10.75 368	9.99 335	60
1	9.24 039	71	9.24 706	73	10.75 294	9.99 333	59
2	9.24 110	71	9.24 779	74	10.75 221	9.99 331	58
3	9.24 181	72	9.24 853	73	10.75 147	9.99 328	57
4	9.24 253	71	9.24 926	74	10.75 074	9.99 326	56
5	9.24 324	71	9.25 000	73	10.75 000	9.99 324	55
6	9.24 395	71	9.25 073	73	10.74 927	9.99 322	54
7	9.24 466	70	9.25 146	73	10.74 854	9.99 319	53
8	9.24 536	71	9.25 219	73	10.74 781	9.99 317	52
9	9.24 607	70	9.25 292	73	10.74 708	9.99 315	51
10	9.24 677	71	9.25 365	72	10.74 635	9.99 313	50
11	9.24 748	70	9.25 437	73	10.74 563	9.99 310	49
12	9.24 818	70	9.25 510	72	10.74 490	9.99 308	48
13	9.24 888	70	9.25 582	73	10.74 418	9.99 306	47
14	9.24 958	70	9.25 655	72	10.74 345	9.99 304	46
15	9.25 028	70	9.25 727	72	10.74 273	9.99 301	45
16	9.25 098	79	9.25 799	72	10.74 201	9.99 299	44
17	9.25 168	60	9.25 871	72	10.74 129	9.99 297	43
18	9.25 237	70	9.25 943	72	10.74 057	9.99 294	42
19	9.25 307	69	9.26 015	71	10.73 985	9.99 292	41
20	9.25 376	69	9.26 086	72	10.73 914	9.99 290	40
21	9.25 445	69	9.26 158	71	10.73 842	9.99 288	39
22	9.25 514	69	9.26 229	72	10.73 771	9.99 285	38
23	9.25 583	69	9.26 301	71	10.73 699	9.99 283	37
24	9.25 652	69	9.26 372	71	10.73 628	9.99 281	36
25	9.25 721	69	9.26 443	71	10.73 557	9.99 278	35
26	9.25 790	68	9.26 514	71	10.73 486	9.99 276	34
27	9.25 858	69	9.26 585	70	10.73 415	9.99 274	33
28	9.25 927	68	9.26 655	71	10.73 345	9.99 271	32
29	9.25 995	68	9.26 726	71	10.73 274	9.99 269	31
30	9.26 063	68	9.26 797	70	10.73 203	9.99 267	30
31	9.26 131	68	9.26 867	70	10.73 133	9.99 264	29
32	9.26 199	68	9.26 937	71	10.73 063	9.99 262	28
33	9.26 267	68	9.27 008	70	10.72 992	9.99 260	27
34	9.26 335	68	9.27 078	70	10.72 922	9.99 257	26
35	9.26 403	67	9.27 148	70	10.72 852	9.99 255	25
36	9.26 470	68	9.27 218	70	10.72 782	9.99 252	24
37	9.26 538	67	9.27 288	69	10.72 712	9.99 250	23
38	9.26 605	67	9.27 357	70	10.72 643	9.99 248	22
39	9.26 672	67	9.27 427	69	10.72 573	9.99 245	21
40	9.26 739	67	9.27 496	70	10.72 504	9.99 243	20
41	9.26 806	67	9.27 566	69	10.72 434	9.99 241	19
42	9.26 873	67	9.27 635	69	10.72 365	9.99 238	18
43	9.26 940	67	9.27 704	69	10.72 296	9.99 236	17
44	9.27 007	66	9.27 773	69	10.72 227	9.99 233	16
45	9.27 073	67	9.27 842	69	10.72 158	9.99 231	15
46	9.27 140	66	9.27 911	69	10.72 089	9.99 229	14
47	9.27 206	67	9.27 980	69	10.72 020	9.99 226	13
48	9.27 273	66	9.28 049	68	10.71 951	9.99 224	12
49	9.27 339	66	9.28 117	69	10.71 883	9.99 221	11
50	9.27 405	66	9.28 186	68	10.71 814	9.99 219	10
51	9.27 471	66	9.28 254	69	10.71 746	9.99 217	9
52	9.27 537	65	9.28 323	68	10.71 677	9.99 214	8
53	9.27 602	66	9.28 391	68	10.71 609	9.99 212	7
54	9.27 668	66	9.28 459	68	10.71 541	9.99 209	6
55	9.27 734	65	9.28 527	68	10.71 473	9.99 207	5
56	9.27 799	65	9.28 595	67	10.71 405	9.99 204	4
57	9.27 864	66	9.28 662	68	10.71 338	9.99 202	3
58	9.27 930	65	9.28 730	68	10.71 270	9.99 200	2
59	9.27 995	65	9.28 798	67	10.71 202	9.99 197	1
60	9.28 060		9.28 865		10.71 135	9.99 195	0
	L Cos	d	L Cot	c d	L Tan	L Sin	′

Prop. Pts.

	74	73	72
.1	7.4	7.3	7.2
.2	14.8	14.6	14.4
.3	22.2	21.9	21.6
.4	29.6	29.2	28.8
.5	37.0	36.5	36.0
.6	44.4	43.8	43.2
.7	51.8	51.1	50.4
.8	59.2	58.4	57.6
.9	66.6	65.7	64.8

	71	70	69
.1	7.1	7.0	6.9
.2	14.2	14.0	13.8
.3	21.3	21.0	20.7
.4	28.4	28.0	27.6
.5	35.5	35.0	34.5
.6	42.6	42.0	41.4
.7	49.7	49.0	48.3
.8	56.8	56.0	55.2
.9	63.9	63.0	62.1

	68	67	66
.1	6.8	6.7	6.6
.2	13.6	13.4	13.2
.3	20.4	20.1	19.8
.4	27.2	26.8	26.4
.5	34.0	33.5	33.0
.6	40.8	40.2	39.6
.7	47.6	46.9	46.2
.8	54.4	53.6	52.8
.9	61.2	60.3	59.4

	65	3	2
.1	6.5	0.3	0.2
.2	13.0	0.6	0.4
.3	19.5	0.9	0.6
.4	26.0	1.2	0.8
.5	32.5	1.5	1.0
.6	39.0	1.8	1.2
.7	45.5	2.1	1.4
.8	52.0	2.4	1.6
.9	58.5	2.7	1.8

Prop. Pts.

′	L Sin	d	L Tan	c d	L Cot	L Cos	
0	9.28 060	65	9.28 865	68	10.71 135	9.99 195	**60**
1	9.28 125	65	9.28 933	67	10.71 067	9.99 192	59
2	9.28 190	64	9.29 000	67	10.71 000	9.99 190	58
3	9.28 254	65	9.29 067	67	10.70 933	9.99 187	57
4	9.28 319	65	9.29 134	67	10.70 866	9.99 185	56
5	9.28 384	64	9.29 201	67	10.70 799	9.99 182	55
6	9.28 448	64	9.29 268	67	10.70 732	9.99 180	54
7	9.28 512	65	9.29 335	67	10.70 665	9.99 177	53
8	9.28 577	64	9.29 402	66	10.70 598	9.99 175	52
9	9.28 641	64	9.29 468	67	10.70 532	9.99 172	51
10	9.28 705	64	9.29 535	66	10.70 465	9.99 170	**50**
11	9.28 769	64	9.29 601	67	10.70 399	9.99 167	49
12	9.28 833	63	9.29 668	66	10.70 332	9.99 165	48
13	9.28 896	64	9.29 734	66	10.70 266	9.99 162	47
14	9.28 960	64	9.29 800	66	10.70 200	9.99 160	46
15	9.29 024	63	9.29 866	66	10.70 134	9.99 157	45
16	9.29 087	63	9.29 932	66	10.70 068	9.99 155	44
17	9.29 150	64	9.29 998	66	10.70 002	9.99 152	43
18	9.29 214	63	9.30 064	66	10.69 936	9.99 150	42
19	9.29 277	63	9.30 130	65	10.69 870	9.99 147	41
20	9.29 340	63	9.30 195	66	10.69 805	9.99 145	**40**
21	9.29 403	63	9.30 261	65	10.69 739	9.99 142	39
22	9.29 466	63	9.30 326	65	10.69 674	9.99 140	38
23	9.29 529	62	9.30 391	66	10.69 609	9.99 137	37
24	9.29 591	63	9.30 457	65	10.69 543	9.99 135	36
25	9.29 654	62	9.30 522	65	10.69 478	9.99 132	35
26	9.29 716	63	9.30 587	65	10.69 413	9.99 130	34
27	9.29 779	62	9.30 652	65	10.69 348	9.99 127	33
28	9.29 841	62	9.30 717	65	10.69 283	9.99 124	32
29	9.29 903	63	9.30 782	64	10.69 218	9.99 122	31
30	9.29 966	62	9.30 846	65	10.69 154	9.99 119	**30**
31	9.30 028	62	9.30 911	64	10.69 089	9.99 117	29
32	9.30 090	61	9.30 975	65	10.69 025	9.99 114	28
33	9.30 151	62	9.31 040	64	10.68 960	9.99 112	27
34	9.30 213	62	9.31 104	64	10.68 896	9.99 109	26
35	9.30 275	61	9.31 168	65	10.68 832	9.99 106	25
36	9.30 336	62	9.31 233	64	10.68 767	9.99 104	24
37	9.30 398	61	9.31 297	64	10.68 703	9.99 101	23
38	9.30 459	62	9.31 361	64	10.68 639	9.99 099	22
39	9.30 521	61	9.31 425	64	10.68 575	9.99 096	21
40	9.30 582	61	9.31 489	63	10.68 511	9.99 093	**20**
41	9.30 643	61	9.31 552	64	10.68 448	9.99 091	19
42	9.30 704	61	9.31 616	63	10.68 384	9.99 088	18
43	9.30 765	61	9.31 679	64	10.68 321	9.99 086	17
44	9.30 826	61	9.31 743	63	10.68 257	9.99 083	16
45	9.30 887	60	9.31 806	64	10.68 194	9.99 080	15
46	9.30 947	61	9.31 870	63	10.68 130	9.99 078	14
47	9.31 008	60	9.31 933	63	10.68 067	9.99 075	13
48	9.31 068	61	9.31 996	63	10.68 004	9.99 072	12
49	9.31 129	60	9.32 059	63	10.67 941	9.99 070	11
50	9.31 189	61	9.32 122	63	10.67 878	9.99 067	**10**
51	9.31 250	60	9.32 185	63	10.67 815	9.99 064	9
52	9.31 310	60	9.32 248	63	10.67 752	9.99 062	8
53	9.31 370	60	9.32 311	62	10.67 689	9.99 059	7
54	9.31 430	60	9.32 373	63	10.67 627	9.99 056	6
55	9.31 490	59	9.32 436	62	10.67 564	9.99 054	5
56	9.31 549	60	9.32 498	63	10.67 502	9.99 051	4
57	9.31 609	60	9.32 561	62	10.67 439	9.99 048	3
58	9.31 669	59	9.32 623	62	10.67 377	9.99 046	2
59	9.31 728	60	9.32 685	62	10.67 315	9.99 043	1
60	9.31 788		9.32 747		10.67 253	9.99 040	**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	′

Prop. Pts.

	68	67	66
.1	6.8	6.7	6.6
.2	13.6	13.4	13.2
.3	20.4	20.1	19.8
.4	27.2	26.8	26.4
.5	34.0	33.5	33.0
.6	40.8	40.2	39.6
.7	47.6	46.9	46.2
.8	54.4	53.6	52.8
.9	61.2	60.3	59.4

	65	64	63
.1	6.5	6.4	6.3
.2	13.0	12.8	12.6
.3	19.5	19.2	18.9
.4	26.0	25.6	25.2
.5	32.5	32.0	31.5
.6	39.0	38.4	37.8
.7	45.5	44.8	44.1
.8	52.0	51.2	50.4
.9	58.5	57.6	56.7

	62	61	60
.1	6.2	6.1	6.0
.2	12.4	12.2	12.0
.3	18.6	18.3	18.0
.4	24.8	24.4	24.0
.5	31.0	30.5	30.0
.6	37.2	36.6	36.0
.7	43.4	42.7	42.0
.8	49.6	48.8	48.0
.9	55.8	54.9	54.0

	59	3	2
.1	5.9	0.3	0.2
.2	11.8	0.6	0.4
.3	17.7	0.9	0.6
.4	23.6	1.2	0.8
.5	29.5	1.5	1.0
.6	35.4	1.8	1.2
.7	41.3	2.1	1.4
.8	47.2	2.4	1.6
.9	53.1	2.7	1.8

Prop. Pts.

′	L Sin	d	L Tan	c d	L Cot	L Cos	
0	9.31 788	59	9.32 747	63	10.67 253	9.99 040	60
1	9.31 847	60	9.32 810	62	10.67 190	9.99 038	59
2	9.31 907	59	9.32 872	61	10.67 128	9.99 035	58
3	9.31 966	59	9.32 933	62	10.67 067	9.99 032	57
4	9.32 025	59	9.32 995	62	10.67 005	9.99 030	56
5	9.32 084	59	9.33 057	62	10.66 943	9.99 027	55
6	9.32 143	59	9.33 119	61	10.66 881	9.99 024	54
7	9.32 202	59	9.33 180	62	10.66 820	9.99 022	53
8	9.32 261	58	9.33 242	61	10.66 758	9.99 019	52
9	9.32 319	59	9.33 303	62	10.66 697	9.99 016	51
10	9.32 378	59	9.33 365	61	10.66 635	9.99 013	50
11	9.32 437	58	9.33 426	61	10.66 574	9.99 011	49
12	9.32 495	58	9.33 487	61	10.66 513	9.99 008	48
13	9.32 553	59	9.33 548	61	10.66 452	9.99 005	47
14	9.32 612	58	9.33 609	61	10.66 391	9.99 002	46
15	9.32 670	58	9.33 670	61	10.66 330	9.99 000	45
16	9.32 728	58	9.33 731	61	10.66 269	9.98 997	44
17	9.32 786	58	9.33 792	61	10.66 208	9.98 994	43
18	9.32 844	58	9.33 853	60	10.66 147	9.98 991	42
19	9.32 902	58	9.33 913	61	10.66 087	9.98 989	41
20	9.32 960	58	9.33 974	60	10.66 026	9.98 986	40
21	9.33 018	57	9.34 034	61	10.65 966	9.98 983	39
22	9.33 075	58	9.34 095	60	10.65 905	9.98 980	38
23	9.33 133	57	9.34 155	60	10.65 845	9.98 978	37
24	9.33 190	58	9.34 215	61	10.65 785	9.98 975	36
25	9.33 248	57	9.34 276	60	10.65 724	9.98 972	35
26	9.33 305	57	9.34 336	60	10.65 664	9.98 969	34
27	9.33 362	58	9.34 396	60	10.65 604	9.98 967	33
28	9.33 420	57	9.34 456	60	10.65 544	9.98 964	32
29	9.33 477	57	9.34 516	60	10.65 484	9.98 961	31
30	9.33 534	57	9.34 576	59	10.65 424	9.98 958	30
31	9.33 591	56	9.34 635	60	10.65 365	9.98 955	29
32	9.33 647	57	9.34 695	60	10.65 305	9.98 953	28
33	9.33 704	57	9.34 755	59	10.65 245	9.98 950	27
34	9.33 761	57	9.34 814	60	10.65 186	9.98 947	26
35	9.33 818	56	9.34 874	59	10.65 126	9.98 944	25
36	9.33 874	57	9.34 933	59	10.65 067	9.98 941	24
37	9.33 931	56	9.34 992	59	10.65 008	9.98 938	23
38	9.33 987	56	9.35 051	60	10.64 949	9.98 936	22
39	9.34 043	57	9.35 111	59	10.64 889	9.98 933	21
40	9.34 100	56	9.35 170	59	10.64 830	9.98 930	20
41	9.34 156	56	9.35 229	59	10.64 771	9.98 927	19
42	9.34 212	56	9.35 288	59	10.64 712	9.98 924	18
43	9.34 268	56	9.35 347	58	10.64 653	9.98 921	17
44	9.34 324	56	9.35 405	59	10.64 595	9.98 919	16
45	9.34 380	56	9.35 464	59	10.64 536	9.98 916	15
46	9.34 436	55	9.35 523	58	10.64 477	9.98 913	14
47	9.34 491	56	9.35 581	59	10.64 419	9.98 910	13
48	9.34 547	55	9.35 640	58	10.64 360	9.98 907	12
49	9.34 602	56	9.35 698	59	10.64 302	9.98 904	11
50	9.34 658	55	9.35 757	58	10.64 243	9.98 901	10
51	9.34 713	56	9.35 815	58	10.64 185	9.98 898	9
52	9.34 769	55	9.35 873	58	10.64 127	9.98 896	8
53	9.34 824	55	9.35 931	58	10.64 069	9.98 893	7
54	9.34 879	55	9.35 989	58	10.64 011	9.98 890	6
55	9.34 934	55	9.36 047	58	10.63 953	9.98 887	5
56	9.34 989	55	9.36 105	58	10.63 895	9.98 884	4
57	9.35 044	55	9.36 163	58	10.63 837	9.98 881	3
58	9.35 099	55	9.36 221	58	10.63 779	9.98 878	2
59	9.35 154	55	9.36 279	57	10.63 721	9.98 875	1
60	9.35 209		9.36 336		10.63 664	9.98 872	0
	L Cos	d	L Cot	c d	L Tan	L Sin	′

Prop. Pts.

	63	62	61
.1	6.3	6.2	6.1
.2	12.6	12.4	12.2
.3	18.9	18.6	18.3
.4	25.2	24.8	24.4
.5	31.5	31.0	30.5
.6	37.8	37.2	36.6
.7	44.1	43.4	42.7
.8	50.4	49.6	48.8
.9	56.7	55.8	54.9

	60	59	58
.1	6.0	5.9	5.8
.2	12.0	11.8	11.6
.3	18.0	17.7	17.4
.4	24.0	23.6	23.2
.5	30.0	29.5	29.0
.6	36.0	35.4	34.8
.7	42.0	41.3	40.6
.8	48.0	47.2	46.4
.9	54.0	53.1	52.2

	57	56	55
.1	5.7	5.6	5.5
.2	11.4	11.2	11.0
.3	17.1	16.8	16.5
.4	22.8	22.4	22.0
.5	28.5	28.0	27.5
.6	34.2	33.6	33.0
.7	39.9	39.2	38.5
.8	45.6	44.8	44.0
.9	51.3	50.4	49.5

	3	2
.1	0.3	0.2
.2	0.6	0.4
.3	0.9	0.6
.4	1.2	0.8
.5	1.5	1.0
.6	1.8	1.2
.7	2.1	1.4
.8	2.4	1.6
.9	2.7	1.8

Prop. Pts.

77°

′	L Sin	d	L Tan	c d	L Cot	L Cos	
0	9.35 209	54	9.36 336	58	10.63 664	9.98 872	60
1	9.35 263	55	9.36 394	58	10.63 606	9.98 869	59
2	9.35 318	55	9.36 452	57	10.63 548	9.98 867	58
3	9.35 373	54	9.36 509	57	10.63 491	9.98 864	57
4	9.35 427	54	9.36 566	58	10.63 434	9.98 861	56
5	9.35 481	55	9.36 624	57	10.63 376	9.98 858	55
6	9.35 536	54	9.36 681	57	10.63 319	9.98 855	54
7	9.35 590	54	9.36 738	57	10.63 262	9.98 852	53
8	9.35 644	54	9.36 795	57	10.63 205	9.98 849	52
9	9.35 698	54	9.36 852	57	10.63 148	9.98 846	51
10	9.35 752	54	9.36 909	57	10.63 091	9.98 843	50
11	9.35 806	54	9.36 966	57	10.63 034	9.98 840	49
12	9.35 860	54	9.37 023	57	10.62 977	9.98 837	48
13	9.35 914	54	9.37 080	57	10.62 920	9.98 834	47
14	9.35 968	54	9.37 137	56	10.62 863	9.98 831	46
15	9.36 022	53	9.37 193	57	10.62 807	9.98 828	45
16	9.36 075	54	9.37 250	56	10.62 750	9.98 825	44
17	9.36 129	53	9.37 306	57	10.62 694	9.98 822	43
18	9.36 182	54	9.37 363	56	10.62 637	9.98 819	42
19	9.36 236	53	9.37 419	57	10.62 581	9.98 816	41
20	9.36 289	53	9.37 476	56	10.62 524	9.98 813	40
21	9.36 342	53	9.37 532	56	10.62 468	9.98 810	39
22	9.36 395	54	9.37 588	56	10.62 412	9.98 807	38
23	9.36 449	53	9.37 644	56	10.62 356	9.98 804	37
24	9.36 502	53	9.37 700	56	10.62 300	9.98 801	36
25	9.36 555	53	9.37 756	56	10.62 244	9.98 798	35
26	9.36 608	52	9.37 812	56	10.62 188	9.98 795	34
27	9.36 660	53	9.37 868	56	10.62 132	9.98 792	33
28	9.36 713	53	9.37 924	56	10.62 076	9.98 789	32
29	9.36 766	53	9.37 980	55	10.62 020	9.98 786	31
30	9.36 819	52	9.38 035	56	10.61 965	9.98 783	30
31	9.36 871	53	9.38 091	56	10.61 909	9.98 780	29
32	9.36 924	52	9.38 147	55	10.61 853	9.98 777	28
33	9.36 976	52	9.38 202	55	10.61 798	9.98 774	27
34	9.37 028	53	9.38 257	56	10.61 743	9.98 771	26
35	9.37 081	52	9.38 313	55	10.61 687	9.98 768	25
36	9.37 133	52	9.38 368	55	10.61 632	9.98 765	24
37	9.37 185	52	9.38 423	56	10.61 577	9.98 762	23
38	9.37 237	52	9.38 479	55	10.61 521	9.98 759	22
39	9.37 289	52	9.38 534	55	10.61 466	9.98 756	21
40	9.37 341	52	9.38 589	55	10.61 411	9.98 753	20
41	9.37 393	52	9.38 644	55	10.61 356	9.98 750	19
42	9.37 445	52	9.38 699	55	10.61 301	9.98 746	18
43	9.37 497	52	9.38 754	54	10.61 246	9.98 743	17
44	9.37 549	51	9.38 808	55	10.61 192	9.98 740	16
45	9.37 600	52	9.38 863	55	10.61 137	9.98 737	15
46	9.37 652	51	9.38 918	54	10.61 082	9.98 734	14
47	9.37 703	52	9.38 972	55	10.61 028	9.98 731	13
48	9.37 755	51	9.39 027	55	10.60 973	9.98 728	12
49	9.37 806	52	9.39 082	54	10.60 918	9.98 725	11
50	9.37 858	51	9.39 136	54	10.60 864	9.98 722	10
51	9.37 909	51	9.39 190	55	10.60 810	9.98 719	9
52	9.37 960	51	9.39 245	54	10.60 755	9.98 715	8
53	9.38 011	51	9.39 299	54	10.60 701	9.98 712	7
54	9.38 062	51	9.39 353	54	10.60 647	9.98 709	6
55	9.38 113	51	9.39 407	54	10.60 593	9.98 706	5
56	9.38 164	51	9.39 461	54	10.60 539	9.98 703	4
57	9.38 215	51	9.39 515	54	10.60 485	9.98 700	3
58	9.38 266	51	9.39 569	54	10.60 431	9.98 697	2
59	9.38 317	51	9.39 623	54	10.60 377	9.98 694	1
60	9.38 368		9.39 677		10.60 323	9.98 690	0
	L Cos	d	L Cot	c d	L Tan	L Sin	′

Prop. Pts.

	58	57	56
.1	5.8	5.7	5.6
.2	11.6	11.4	11.2
.3	17.4	17.1	16.8
.4	23.2	22.8	22.4
.5	29.0	28.5	28.0
.6	34.8	34.2	33.6
.7	40.6	39.9	39.2
.8	46.4	45.6	44.8
.9	52.2	51.3	50.4

	55	54	53
.1	5.5	5.4	5.3
.2	11.0	10.8	10.6
.3	16.5	16.2	15.9
.4	22.0	21.6	21.2
.5	27.5	27.0	26.5
.6	33.0	32.4	31.8
.7	38.5	37.8	37.1
.8	44.0	43.2	42.4
.9	49.5	48.6	47.7

	52	51	4
.1	5.2	5.1	0.4
.2	10.4	10.2	0.8
.3	15.6	15.3	1.2
.4	20.8	20.4	1.6
.5	26.0	25.5	2.0
.6	31.2	30.6	2.4
.7	36.4	35.7	2.8
.8	41.6	40.8	3.2
.9	46.8	45.9	3.6

	3	2
.1	0.3	0.2
.2	0.6	0.4
.3	0.9	0.6
.4	1.2	0.8
.5	1.5	1.0
.6	1.8	1.2
.7	2.1	1.4
.8	2.4	1.6
.9	2.7	1.8

Prop. Pts.

′	L Sin	d	L Tan	c d	L Cot	L Cos	
0	9.38 368	50	9.39 677	54	10.60 323	9.98 690	60
1	9.38 418	51	9.39 731	54	10.60 269	9.98 687	59
2	9.38 469	50	9.39 785	53	10.60 215	9.98 684	58
3	9.38 519	51	9.39 838	54	10.60 162	9.98 681	57
4	9.38 570	50	9.39 892	53	10.60 108	9.98 678	56
5	9.38 620	50	9.39 945	54	10.60 055	9.98 675	55
6	9.38 670	51	9.39 999	53	10.60 001	9.98 671	54
7	9.38 721	50	9.40 052	54	10.59 948	9.98 668	53
8	9.38 771	50	9.40 106	53	10.59 894	9.98 665	52
9	9.38 821	50	9.40 159	53	10.59 841	9.98 662	51
10	9.38 871	50	9.40 212	54	10.59 788	9.98 659	50
11	9.38 921	50	9.40 266	53	10.59 734	9.98 656	49
12	9.38 971	50	9.40 319	53	10.59 681	9.98 652	48
13	9.39 021	50	9.40 372	53	10.59 628	9.98 649	47
14	9.39 071	50	9.40 425	53	10.59 575	9.98 646	46
15	9.39 121	49	9.40 478	53	10.59 522	9.98 643	45
16	9.39 170	50	9.40 531	53	10.59 469	9.98 640	44
17	9.39 220	50	9.40 584	52	10.59 416	9.98 636	43
18	9.39 270	49	9.40 636	53	10.59 364	9.98 633	42
19	9.39 319	50	9.40 689	53	10.59 311	9.98 630	41
20	9.39 369	49	9.40 742	53	10.59 258	9.98 627	40
21	9.39 418	49	9.40 795	52	10.59 205	9.98 623	39
22	9.39 467	50	9.40 847	53	10.59 153	9.98 620	38
23	9.39 517	49	9.40 900	52	10.59 100	9.98 617	37
24	9.39 566	49	9.40 952	53	10.59 048	9.98 614	36
25	9.39 615	49	9.41 005	52	10.58 995	9.98 610	35
26	9.39 664	49	9.41 057	52	10.58 943	9.98 607	34
27	9.39 713	49	9.41 109	52	10.58 891	9.98 604	33
28	9.39 762	49	9.41 161	53	10.58 839	9.98 601	32
29	9.39 811	49	9.41 214	52	10.58 786	9.98 597	31
30	9.39 860	49	9.41 266	52	10.58 734	9.98 594	30
31	9.39 909	49	9.41 318	52	10.58 682	9.98 591	29
32	9.39 958	48	9.41 370	52	10.58 630	9.98 588	28
33	9.40 006	49	9.41 422	52	10.58 578	9.98 584	27
34	9.40 055	48	9.41 474	52	10.58 526	9.98 581	26
35	9.40 103	49	9.41 526	52	10.58 474	9.98 578	25
36	9.40 152	48	9.41 578	51	10.58 422	9.98 574	24
37	9.40 200	49	9.41 629	52	10.58 371	9.98 571	23
38	9.40 249	48	9.41 681	52	10.58 319	9.98 568	22
39	9.40 297	49	9.41 733	51	10.58 267	9.98 565	21
40	9.40 346	48	9.41 784	52	10.58 216	9.98 561	20
41	9.40 394	48	9.41 836	51	10.58 164	9.98 558	19
42	9.40 442	48	9.41 887	52	10.58 113	9.98 555	18
43	9.40 490	48	9.41 939	51	10.58 061	9.98 551	17
44	9.40 538	48	9.41 990	51	10.58 010	9.98 548	16
45	9.40 586	48	9.42 041	52	10.57 959	9.98 545	15
46	9.40 634	48	9.42 093	51	10.57 907	9.98 541	14
47	9.40 682	48	9.42 144	51	10.57 856	9.98 538	13
48	9.40 730	48	9.42 195	51	10.57 805	9.98 535	12
49	9.40 778	47	9.42 246	51	10.57 754	9.98 531	11
50	9.40 825	48	9.42 297	51	10.57 703	9.98 528	10
51	9.40 873	48	9.42 348	51	10.57 652	9.98 525	9
52	9.40 921	47	9.42 399	51	10.57 601	9.98 521	8
53	9.40 968	48	9.42 450	51	10.57 550	9.98 518	7
54	9.41 016	47	9.42 501	51	10.57 499	9.98 515	6
55	9.41 063	48	9.42 552	51	10.57 448	9.98 511	5
56	9.41 111	47	9.42 603	50	10.57 397	9.98 508	4
57	9.41 158	47	9.42 653	51	10.57 347	9.98 505	3
58	9.41 205	47	9.42 704	51	10.57 296	9.98 501	2
59	9.41 252	48	9.42 755	50	10.57 245	9.98 498	1
60	9.41 300		9.42 805		10.57 195	9.98 494	0
	L Cos	d	L Cot	c d	L Tan	L Sin	′

Prop. Pts.

	54	53	52
.1	5.4	5.3	5.2
.2	10.8	10.6	10.4
.3	16.2	15.9	15.6
.4	21.6	21.2	20.8
.5	27.0	26.5	26.0
.6	32.4	31.8	31.2
.7	37.8	37.1	36.4
.8	43.2	42.4	41.6
.9	48.6	47.7	46.8

	51	50	49
.1	5.1	5.0	4.9
.2	10.2	10.0	9.8
.3	15.3	15.0	14.7
.4	20.4	20.0	19.6
.5	25.5	25.0	24.5
.6	30.6	30.0	29.4
.7	35.7	35.0	34.3
.8	40.8	40.0	39.2
.9	45.9	45.0	44.1

	48	47
.1	4.8	4.7
.2	9.6	9.4
.3	14.4	14.1
.4	19.2	18.8
.5	24.0	23.5
.6	28.8	28.2
.7	33.6	32.9
.8	38.4	37.6
.9	43.2	42.3

	4	3
.1	0.4	0.3
.2	0.8	0.6
.3	1.2	0.9
.4	1.6	1.2
.5	2.0	1.5
.6	2.4	1.8
.7	2.8	2.1
.8	3.2	2.4
.9	3.6	2.7

Prop. Pts.

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.41 300	47	9.42 805	51	10.57 195	9.98 494	3	60
1	9.41 347	47	9.42 856	50	10.57 144	9.98 491	3	59
2	9.41 394	47	9.42 906	51	10.57 094	9.98 488	4	58
3	9.41 441	47	9.42 957	50	10.57 043	9.98 484	3	57
4	9.41 488	47	9.43 007	50	10.56 993	9.98 481	4	56
5	9.41 535	47	9.43 057	51	10.56 943	9.98 477	3	55
6	9.41 582	46	9.43 108	50	10.56 892	9.98 474	3	54
7	9.41 628	47	9.43 158	50	10.56 842	9.98 471	4	53
8	9.41 675	47	9.43 208	50	10.56 792	9.98 467	3	52
9	9.41 722	46	9.43 258	50	10.56 742	9.98 464	4	51
10	9.41 768	47	9.43 308	50	10.56 692	9.98 460	3	50
11	9.41 815	46	9.43 358	50	10.56 642	9.98 457	4	49
12	9.41 861	47	9.43 408	50	10.56 592	9.98 453	3	48
13	9.41 908	46	9.43 458	50	10.56 542	9.98 450	3	47
14	9.41 954	47	9.43 508	50	10.56 492	9.98 447	4	46
15	9.42 001	46	9.43 558	49	10.56 442	9.98 443	3	45
16	9.42 047	46	9.43 607	50	10.56 393	9.98 440	4	44
17	9.42 093	47	9.43 657	50	10.56 343	9.98 436	3	43
18	9.42 140	46	9.43 707	49	10.56 293	9.98 433	4	42
19	9.42 186	46	9.43 756	50	10.56 244	9.98 429	3	41
20	9.42 232	46	9.43 806	49	10.56 194	9.98 426	4	40
21	9.42 278	46	9.43 855	50	10.56 145	9.98 422	3	39
22	9.42 324	46	9.43 905	49	10.56 095	9.98 419	4	38
23	9.42 370	46	9.43 954	50	10.56 046	9.98 415	3	37
24	9.42 416	45	9.44 004	49	10.55 996	9.98 412	3	36
25	9.42 461	46	9.44 053	49	10.55 947	9.98 409	4	35
26	9.42 507	46	9.44 102	49	10.55 898	9.98 405	3	34
27	9.42 553	46	9.44 151	50	10.55 849	9.98 402	4	33
28	9.42 599	45	9.44 201	49	10.55 799	9.98 398	3	32
29	9.42 644	46	9.44 250	49	10.55 750	9.98 395	4	31
30	9.42 690	45	9.44 299	49	10.55 701	9.98 391	3	30
31	9.42 735	46	9.44 348	49	10.55 652	9.98 388	4	29
32	9.42 781	45	9.44 397	49	10.55 603	9.98 384	3	28
33	9.42 826	46	9.44 446	49	10.55 554	9.98 381	4	27
34	9.42 872	45	9.44 495	49	10.55 505	9.98 377	4	26
35	9.42 917	45	9.44 544	48	10.55 456	9.98 373	3	25
36	9.42 962	46	9.44 592	49	10.55 408	9.98 370	4	24
37	9.43 008	45	9.44 641	49	10.55 359	9.98 366	3	23
38	9.43 053	45	9.44 690	48	10.55 310	9.98 363	4	22
39	9.43 098	45	9.44 738	49	10.55 262	9.98 359	3	21
40	9.43 143	45	9.44 787	49	10.55 213	9.98 356	4	20
41	9.43 188	45	9.44 836	48	10.55 164	9.98 352	3	19
42	9.43 233	45	9.44 884	49	10.55 116	9.98 349	4	18
43	9.43 278	45	9.44 933	48	10.55 067	9.98 345	3	17
44	9.43 323	44	9.44 981	48	10.55 019	9.98 342	4	16
45	9.43 367	45	9.45 029	49	10.54 971	9.98 338	4	15
46	9.43 412	45	9.45 078	48	10.54 922	9.98 334	3	14
47	9.43 457	45	9.45 126	48	10.54 874	9.98 331	4	13
48	9.43 502	44	9.45 174	48	10.54 826	9.98 327	3	12
49	9.43 546	45	9.45 222	49	10.54 778	9.98 324	4	11
50	9.43 591	44	9.45 271	48	10.54 729	9.98 320	3	10
51	9.43 635	45	9.45 319	48	10.54 681	9.98 317	4	9
52	9.43 680	44	9.45 367	48	10.54 633	9.98 313	4	8
53	9.43 724	45	9.45 415	48	10.54 585	9.98 309	3	7
54	9.43 769	44	9.45 463	48	10.54 537	9.98 306	4	6
55	9.43 813	44	9.45 511	48	10.54 489	9.98 302	3	5
56	9.43 857	44	9.45 559	47	10.54 441	9.98 299	4	4
57	9.43 901	45	9.45 606	48	10.54 394	9.98 295	4	3
58	9.43 946	44	9.45 654	48	10.54 346	9.98 291	3	2
59	9.43 990	44	9.45 702	48	10.54 298	9.98 288	4	1
60	9.44 034		9.45 750		10.54 250	9.98 284		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

	51	50	49
.1	5.1	5.0	4.9
.2	10.2	10.0	9.8
.3	15.3	15.0	14.7
.4	20.4	20.0	19.6
.5	25.5	25.0	24.5
.6	30.6	30.0	29.4
.7	35.7	35.0	34.3
.8	40.8	40.0	39.2
.9	45.9	45.0	44.1

	48	47	46
.1	4.8	4.7	4.6
.2	9.6	9.4	9.2
.3	14.4	14.1	13.8
.4	19.2	18.8	18.4
.5	24.0	23.5	23.0
.6	28.8	28.2	27.6
.7	33.6	32.9	32.2
.8	38.4	37.6	36.8
.9	43.2	42.3	41.4

	45	44
.1	4.5	4.4
.2	9.0	8.8
.3	13.5	13.2
.4	18.0	17.6
.5	22.5	22.0
.6	27.0	26.4
.7	31.5	30.8
.8	36.0	35.2
.9	40.5	39.6

	4	3
.1	0.4	0.3
.2	0.8	0.6
.3	1.2	0.9
.4	1.6	1.2
.5	2.0	1.5
.6	2.4	1.8
.7	2.8	2.1
.8	3.2	2.4
.9	3.6	2.7

Prop. Pts.

74°

'	L Sin	d	L Tan	c d	L Cot	L Cos	d	'
0	9.44 034	44	9.45 750	47	10.54 250	9.98 284	3	60
1	9.44 078	44	9.45 797	48	10.54 203	9.98 281	4	59
2	9.44 122	44	9.45 845	47	10.54 155	9.98 277	4	58
3	9.44 166	44	9.45 892	48	10.54 108	9.98 273	3	57
4	9.44 210	43	9.45 940	47	10.54 060	9.98 270	4	56
5	9.44 253	44	9.45 987	48	10.54 013	9.98 266	4	55
6	9.44 297	44	9.46 035	47	10.53 965	9.98 262	3	54
7	9.44 341	44	9.46 082	48	10.53 918	9.98 259	4	53
8	9.44 385	43	9.46 130	47	10.53 870	9.98 255	4	52
9	9.44 428	44	9.46 177	47	10.53 823	9.98 251	3	51
10	9.44 472	44	9.46 224	47	10.53 776	9.98 248	4	50
11	9.44 516	43	9.46 271	48	10.53 729	9.98 244	4	49
12	9.44 559	43	9.46 319	47	10.53 681	9.98 240	3	48
13	9.44 602	44	9.46 366	47	10.53 634	9.98 237	4	47
14	9.44 646	43	9.46 413	47	10.53 587	9.98 233	4	46
15	9.44 689	44	9.46 460	47	10.53 540	9.98 229	3	45
16	9.44 733	43	9.46 507	47	10.53 493	9.98 226	4	44
17	9.44 776	43	9.46 554	47	10.53 446	9.98 222	4	43
18	9.44 819	43	9.46 601	47	10.53 399	9.98 218	3	42
19	9.44 862	43	9.46 648	46	10.53 352	9.98 215	4	41
20	9.44 905	43	9.46 694	47	10.53 306	9.98 211	4	40
21	9.44 948	44	9.46 741	47	10.53 259	9.98 207	3	39
22	9.44 992	43	9.46 788	47	10.53 212	9.98 204	4	38
23	9.45 035	42	9.46 835	46	10.53 165	9.98 200	4	37
24	9.45 077	43	9.46 881	47	10.53 119	9.98 196	4	36
25	9.45 120	43	9.46 928	47	10.53 072	9.98 192	3	35
26	9.45 163	43	9.46 975	46	10.53 025	9.98 189	4	34
27	9.45 206	43	9.47 021	47	10.52 979	9.98 185	4	33
28	9.45 249	43	9.47 068	46	10.52 932	9.98 181	4	32
29	9.45 292	42	9.47 114	46	10.52 886	9.98 177	3	31
30	9.45 344	43	9.47 160	47	10.52 840	9.98 174	4	30
31	9.45 377	42	9.47 207	46	10.52 793	9.98 170	4	29
32	9.45 419	43	9.47 253	46	10.52 747	9.98 166	4	28
33	9.45 462	42	9.47 299	47	10.52 701	9.98 162	3	27
34	9.45 504	43	9.47 346	46	10.52 654	9.98 159	4	26
35	9.45 547	42	9.47 392	46	10.52 608	9.98 155	4	25
36	9.45 589	43	9.47 438	46	10.52 562	9.98 151	4	24
37	9.45 632	42	9.47 484	46	10.52 516	9.98 147	3	23
38	9.45 674	42	9.47 530	46	10.52 470	9.98 144	4	22
39	9.45 716	42	9.47 576	46	10.52 424	9.98 140	4	21
40	9.45 758	43	9.47 622	46	10.52 378	9.98 136	4	20
41	9.45 801	42	9.47 668	46	10.52 332	9.98 132	3	19
42	9.45 843	42	9.47 714	46	10.52 286	9.98 129	4	18
43	9.45 885	42	9.47 760	46	10.52 240	9.98 125	4	17
44	9.45 927	42	9.47 806	46	10.52 194	9.98 121	4	16
45	9.45 969	42	9.47 852	45	10.52 148	9.98 117	4	15
46	9.46 011	42	9.47 897	46	10.52 103	9.98 113	3	14
47	9.46 053	42	9.47 943	46	10.52 057	9.98 110	4	13
48	9.46 095	41	9.47 989	46	10.52 011	9.98 106	4	12
49	9.46 136	42	9.48 035	45	10.51 965	9.98 102	4	11
50	9.46 178	42	9.48 080	46	10.51 920	9.98 098	4	10
51	9.46 220	42	9.48 126	45	10.51 874	9.98 094	4	9
52	9.46 262	41	9.48 171	46	10.51 829	9.98 090	3	8
53	9.46 303	42	9.48 217	45	10.51 783	9.98 087	4	7
54	9.46 345	41	9.48 262	45	10.51 738	9.98 083	4	6
55	9.46 386	42	9.48 307	46	10.51 693	9.98 079	4	5
56	9.46 428	41	9.48 353	45	10.51 647	9.98 075	4	4
57	9.46 469	42	9.48 398	45	10.51 602	9.98 071	4	3
58	9.46 511	41	9.48 443	46	10.51 557	9.98 067	4	2
59	9.46 552	42	9.48 489	45	10.51 511	9.98 063	3	1
60	9.46 594		9.48 534		10.51 466	9.98 060		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	'

Prop. Pts.

	48	47	46
.1	4.8	4.7	4.6
.2	9.6	9.4	9.2
.3	14.4	14.1	13.8
.4	19.2	18.8	18.4
.5	24.0	23.5	23.0
.6	28.8	28.2	27.6
.7	33.6	32.9	32.2
.8	38.4	37.6	36.8
.9	43.2	42.3	41.4

	45	44	43
.1	4.5	4.4	4.3
.2	9.0	8.8	8.6
.3	13.5	13.2	12.9
.4	18.0	17.6	17.2
.5	22.5	22.0	21.5
.6	27.0	26.4	25.8
.7	31.5	30.8	30.1
.8	36.0	35.2	34.4
.9	40.5	39.6	38.7

	42	41
.1	4.2	4.1
.2	8.4	8.2
.3	12.6	12.3
.4	16.8	16.4
.5	21.0	20.5
.6	25.2	24.6
.7	29.4	28.7
.8	33.6	32.8
.9	37.8	36.9

	4	3
.1	0.4	0.3
.2	0.8	0.6
.3	1.2	0.9
.4	1.6	1.2
.5	2.0	1.5
.6	2.4	1.8
.7	2.8	2.1
.8	3.2	2.4
.9	3.6	2.7

Prop. Pts.

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.46 594	41	9.48 534	45	10.51 466	9.98 060	4	60
1	9.46 635	41	9.48 579	45	10.51 421	9.98 056	4	59
2	9.46 676	41	9.48 624	45	10.51 376	9.98 052	4	58
3	9.46 717	41	9.48 669	45	10.51 331	9.98 048	4	57
4	9.46 758	42	9.48 714	45	10.51 286	9.98 044	4	56
5	9.46 800	41	9.48 759	45	10.51 241	9.98 040	4	55
6	9.46 841	41	9.48 804	45	10.51 196	9.98 036	4	54
7	9.46 882	41	9.48 849	45	10.51 151	9.98 032	3	53
8	9.46 923	41	9.48 894	45	10.51 106	9.98 029	4	52
9	9.46 964	41	9.48 939	45	10.51 061	9.98 025	4	51
10	9.47 005	40	9.48 984	45	10.51 016	9.98 021	4	50
11	9.47 045	41	9.49 029	44	10.50 971	9.98 017	4	49
12	9.47 086	41	9.49 073	45	10.50 927	9.98 013	4	48
13	9.47 127	41	9.49 118	45	10.50 882	9.98 009	4	47
14	9.47 168	41	9.49 163	44	10.50 837	9.98 005	4	46
15	9.47 209	40	9.49 207	45	10.50 793	9.98 001	4	45
16	9.47 249	41	9.49 252	44	10.50 748	9.97 997	4	44
17	9.47 290	40	9.49 296	45	10.50 704	9.97 993	4	43
18	9.47 330	41	9.49 341	44	10.50 659	9.97 989	3	42
19	9.47 371	40	9.49 385	45	10.50 615	9.97 986	4	41
20	9.47 411	41	9.49 430	44	10.50 570	9.97 982	4	40
21	9.47 452	40	9.49 474	45	10.50 526	9.97 978	4	39
22	9.47 492	41	9.49 519	44	10.50 481	9.97 974	4	38
23	9.47 533	40	9.49 563	44	10.50 437	9.97 970	4	37
24	9.47 573	40	9.49 607	45	10.50 393	9.97 966	4	36
25	9.47 613	41	9.49 652	44	10.50 348	9.97 962	4	35
26	9.47 654	40	9.49 696	44	10.50 304	9.97 958	4	34
27	9.47 694	40	9.49 740	44	10.50 260	9.97 954	4	33
28	9.47 734	40	9.49 784	44	10.50 216	9.97 950	4	32
29	9.47 774	40	9.49 828	44	10.50 172	9.97 946	4	31
30	9.47 814	40	9.49 872	44	10.50 128	9.97 942	4	30
31	9.47 854	40	9.49 916	44	10.50 084	9.97 938	4	29
32	9.47 894	40	9.49 960	44	10.50 040	9.97 934	4	28
33	9.47 934	40	9.50 004	44	10.49 996	9.97 930	4	27
34	9.47 974	40	9.50 048	44	10.49 952	9.97 926	4	26
35	9.48 014	40	9.50 092	44	10.49 908	9.97 922	4	25
36	9.48 054	40	9.50 136	44	10.49 864	9.97 918	4	24
37	9.48 094	39	9.50 180	43	10.49 820	9.97 914	4	23
38	9.48 133	40	9.50 223	44	10.49 777	9.97 910	4	22
39	9.48 173	40	9.50 267	44	10.49 733	9.97 906	4	21
40	9.48 213	39	9.50 311	44	10.49 689	9.97 902	4	20
41	9.48 252	40	9.50 355	43	10.49 645	9.97 898	4	19
42	9.48 292	40	9.50 398	44	10.49 602	9.97 894	4	18
43	9.48 332	39	9.50 442	43	10.49 558	9.97 890	4	17
44	9.48 371	40	9.50 485	44	10.49 515	9.97 886	4	16
45	9.48 411	39	9.50 529	43	10.49 471	9.97 882	4	15
46	9.48 450	40	9.50 572	44	10.49 428	9.97 878	4	14
47	9.48 490	39	9.50 616	43	10.49 384	9.97 874	4	13
48	9.48 529	39	9.50 659	44	10.49 341	9.97 870	4	12
49	9.48 568	39	9.50 703	43	10.49 297	9.97 866	5	11
50	9.48 607	40	9.50 746	43	10.49 254	9.97 861	4	10
51	9.48 647	39	9.50 789	44	10.49 211	9.97 857	4	9
52	9.48 686	39	9.50 833	43	10.49 167	9.97 853	4	8
53	9.48 725	39	9.50 876	43	10.49 124	9.97 849	4	7
54	9.48 764	39	9.50 919	43	10.49 081	9.97 845	4	6
55	9.48 803	39	9.50 962	43	10.49 038	9.97 841	4	5
56	9.48 842	39	9.51 005	43	10.48 995	9.97 837	4	4
57	9.48 881	39	9.51 048	44	10.48 952	9.97 833	4	3
58	9.48 920	39	9.51 092	43	10.48 908	9.97 829	4	2
59	9.48 959	39	9.51 135	43	10.48 865	9.97 825	4	1
60	9.48 998		9.51 178		10.48 822	9.97 821		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

	45	44	43
.1	4.5	4.4	4.3
.2	9.0	8.8	8.6
.3	13.5	13.2	12.9
.4	18.0	17.6	17.2
.5	22.5	22.0	21.5
.6	27.0	26.4	25.8
.7	31.5	30.8	30.1
.8	36.0	35.2	34.4
.9	40.5	39.6	38.7

	42	41
.1	4.2	4.1
.2	8.4	8.2
.3	12.6	12.3
.4	16.8	16.4
.5	21.0	20.5
.6	25.2	24.6
.7	29.4	28.7
.8	33.6	32.8
.9	37 8	36.9

	40	39
.1	4.0	3.9
.2	8.0	7.8
.3	12.0	11.7
.4	16.0	15.6
.5	20.0	19.5
.6	24.0	23.4
.7	28.0	27.3
.8	32.0	31.2
.9	36.0	35.1

	5	4	3
.1	0.5	0.4	0.3
.2	1.0	0.8	0.6
.3	1.5	1.2	0.9
.4	2.0	1.6	1.2
.5	2.5	2.0	1.5
.6	3.0	2.4	1.8
.7	3.5	2.8	2.1
.8	4.0	3.2	2.4
.9	4.5	3.6	2.7

Prop. Pts.

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.48 998	39	9.51 178	43	10.48 822	9.97 821	4	60
1	9.49 037	39	9.51 221	43	10.48 779	9.97 817	5	59
2	9.49 076	39	9.51 264	42	10.48 736	9.97 812	4	58
3	9.49 115	38	9.51 306	43	10.48 694	9.97 808	4	57
4	9.49 153	39	9.51 349	43	10.48 651	9.97 804	4	56
5	9.49 192	39	9.51 392	43	10.48 608	9.97 800	4	55
6	9.49 231	38	9.51 435	43	10.48 565	9.97 796	4	54
7	9.49 269	39	9.51 478	42	10.48 522	9.97 792	4	53
8	9.49 308	39	9.51 520	43	10.48 480	9.97 788	4	52
9	9.49 347	38	9.51 563	43	10.48 437	9.97 784	5	51
10	9.49 385	39	9.51 606	42	10.48 394	9.97 779	4	50
11	9.49 424	38	9.51 648	43	10.48 352	9.97 775	4	49
12	9.49 462	38	9.51 691	43	10.48 309	9.97 771	4	48
13	9.49 500	39	9.51 734	42	10.48 266	9.97 767	4	47
14	9.49 539	38	9.51 776	43	10.48 224	9.97 763	4	46
15	9.49 577	38	9.51 819	42	10.48 181	9.97 759	5	45
16	9.49 615	39	9.51 861	42	10.48 139	9.97 754	4	44
17	9.49 654	38	9.51 903	43	10.48 097	9.97 750	4	43
18	9.49 692	38	9.51 946	42	10.48 054	9.97 746	4	42
19	9.49 730	38	9.51 988	43	10.48 012	9.97 742	4	41
20	9.49 768	38	9.52 031	42	10.47 969	9.97 738	4	40
21	9.49 806	38	9.52 073	42	10.47 927	9.97 734	5	39
22	9.49 844	38	9.52 115	42	10.47 885	9.97 729	4	38
23	9.49 882	38	9.52 157	43	10.47 843	9.97 725	4	37
24	9.49 920	38	9.52 200	42	10.47 800	9.97 721	4	36
25	9.49 958	38	9.52 242	42	10.47 758	9.97 717	4	35
26	9.49 996	38	9.52 284	42	10.47 716	9.97 713	5	34
27	9.50 034	38	9.52 326	42	10.47 674	9.97 708	4	33
28	9.50 072	38	9.52 368	42	10.47 632	9.97 704	4	32
29	9.50 110	38	9.52 410	42	10.47 590	9.97 700	4	31
30	9.50 148	37	9.52 452	42	10.47 548	9.97 696	5	30
31	9.50 185	38	9.52 494	42	10.47 506	9.97 691	4	29
32	9.50 223	38	9.52 536	42	10.47 464	9.97 687	4	28
33	9.50 261	37	9.52 578	42	10.47 422	9.97 683	4	27
34	9.50 298	38	9.52 620	41	10.47 380	9.97 679	5	26
35	9.50 336	38	9.52 661	42	10.47 339	9.97 674	4	25
36	9.50 374	37	9.52 703	42	10.47 297	9.97 670	4	24
37	9.50 411	38	9.52 745	42	10.47 255	9.97 666	4	23
38	9.50 449	37	9.52 787	42	10.47 213	9.97 662	5	22
39	9.50 486	37	9.52 829	41	10.47 171	9.97 657	4	21
40	9.50 523	38	9.52 870	42	10.47 130	9.97 653	4	20
41	9.50 561	37	9.52 912	41	10.47 088	9.97 649	4	19
42	9.50 598	37	9.52 953	42	10.47 047	9.97 645	5	18
43	9.50 635	38	9.52 995	42	10.47 005	9.97 640	4	17
44	9.50 673	37	9.53 037	41	10.46 963	9.97 636	4	16
45	9.50 710	37	9.53 078	42	10.46 922	9.97 632	4	15
46	9.50 747	37	9.53 120	41	10.46 880	9.97 628	5	14
47	9.50 784	37	9.53 161	41	10.46 839	9.97 623	4	13
48	9.50 821	37	9.53 202	42	10.46 798	9.97 619	4	12
49	9.50 858	38	9.53 244	41	10.46 756	9.97 615	5	11
50	9.50 896	37	9.53 285	42	10.46 715	9.97 610	4	10
51	9.50 933	37	9.53 327	41	10.46 673	9.97 606	4	9
52	9.50 970	37	9.53 368	41	10.46 632	9.97 602	5	8
53	9.51 007	36	9.53 409	41	10.46 591	9.97 597	4	7
54	9.51 043	37	9.53 450	42	10.46 550	9.97 593	4	6
55	9.51 080	37	9.53 492	41	10.46 508	9.97 589	5	5
56	9.51 117	37	9.53 533	41	10.46 467	9.97 584	4	4
57	9.51 154	37	9.53 574	41	10.46 426	9.97 580	4	3
58	9.51 191	36	9.53 615	41	10.46 385	9.97 576	5	2
59	9.51 227	37	9.53 656	41	10.46 344	9.97 571	4	1
60	9.51 264		9.53 697		10.46 303	9.97 567		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

	43	42	41
.1	4.3	4.2	4.1
.2	8.6	8.4	8.2
.3	12.9	12.6	12.3
.4	17.2	16.8	16.4
.5	21.5	21.0	20.5
.6	25.8	25.2	24.6
.7	30.1	29.4	28.7
.8	34.4	33.6	32.8
.9	38.7	37.8	36.9

	39	38	37
.1	3.9	3.8	3.7
.2	7.8	7.6	7.4
.3	11.7	11.4	11.1
.4	15.6	15.2	14.8
.5	19.5	19.0	18.5
.6	23.4	22.8	22.2
.7	27.3	26.6	25.9
.8	31.2	30.4	29.6
.9	35.1	34.2	33.3

	36	5	4
.1	3.6	0.5	0.4
.2	7.2	1.0	0.8
.3	10.8	1.5	1.2
.4	14.4	2.0	1.6
.5	18.0	2.5	2.0
.6	21.6	3.0	2.4
.7	25.2	3.5	2.8
.8	28.8	4.0	3.2
.9	32.4	4.5	3.6

Prop. Pts.

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.51 264	37	9.53 697	41	10.46 303	9.97 567	4	60
1	9.51 301	37	9.53 738	41	10.46 262	9.97 563	5	59
2	9.51 338	36	9.53 779	41	10.46 221	9.97 558	4	58
3	9.51 374	37	9.53 820	41	10.46 180	9.97 554	4	57
4	9.51 411	36	9.53 861	41	10.46 139	9.97 550	5	56
5	9.51 447	37	9.53 902	41	10.46 098	9.97 545	4	55
6	9.51 484	36	9.53 943	41	10.46 057	9.97 541	5	54
7	9.51 520	37	9.53 984	41	10.46 016	9.97 536	4	53
8	9.51 557	36	9.54 025	40	10.45 975	9.97 532	4	52
9	9.51 593	36	9.54 065	41	10.45 935	9.97 528	5	51
10	9.51 629	37	9.54 106	41	10.45 894	9.97 523	4	50
11	9.51 666	36	9.54 147	40	10.45 853	9.97 519	4	49
12	9.51 702	36	9.54 187	41	10.45 813	9.97 515	5	48
13	9.51 738	36	9.54 228	41	10.45 772	9.97 510	4	47
14	9.51 774	37	9.54 269	40	10.45 731	9.97 506	5	46
15	9.51 811	36	9.54 309	41	10.45 691	9.97 501	4	45
16	9.51 847	36	9.54 350	40	10.45 650	9.97 497	5	44
17	9.51 883	36	9.54 390	41	10.45 610	9.97 492	4	43
18	9.51 919	36	9.54 431	40	10.45 569	9.97 488	4	42
19	9.51 955	36	9.54 471	41	10.45 529	9.97 484	5	41
20	9.51 991	36	9.54 512	40	10.45 488	9.97 479	4	40
21	9.52 027	36	9.54 552	41	10.45 448	9.97 475	5	39
22	9.52 063	36	9.54 593	40	10.45 407	9.97 470	4	38
23	9.52 099	36	9.54 633	40	10.45 367	9.97 466	5	37
24	9.52 135	36	9.54 673	41	10.45 327	9.97 461	4	36
25	9.52 171	36	9.54 714	40	10.45 286	9.97 457	4	35
26	9.52 207	35	9.54 754	40	10.45 246	9.97 453	5	34
27	9.52 242	36	9.54 794	41	10.45 206	9.97 448	4	33
28	9.52 278	36	9.54 835	40	10.45 165	9.97 444	5	32
29	9.52 314	36	9.54 875	40	10.45 125	9.97 439	4	31
30	9.52 350	35	9.54 915	40	10.45 085	9.97 435	5	30
31	9.52 385	36	9.54 955	40	10.45 045	9.97 430	4	29
32	9.52 421	35	9.54 995	40	10.45 005	9.97 426	5	28
33	9.52 456	36	9.55 035	40	10.44 965	9.97 421	4	27
34	9.52 492	35	9.55 075	40	10.44 925	9.97 417	5	26
35	9.52 527	36	9.55 115	40	10.44 885	9.97 412	4	25
36	9.52 563	35	9.55 155	40	10.44 845	9.97 408	5	24
37	9.52 598	36	9.55 195	40	10.44 805	9.97 403	4	23
38	9.52 634	35	9.55 235	40	10.44 765	9.97 399	5	22
39	9.52 669	36	9.55 275	40	10.44 725	9.97 394	4	21
40	9.52 705	35	9.55 315	40	10.44 685	9.97 390	5	20
41	9.52 740	35	9.55 355	40	10.44 645	9.97 385	4	19
42	9.52 775	36	9.55 395	39	10.44 605	9.97 381	5	18
43	9.52 811	35	9.55 434	40	10.44 566	9.97 376	4	17
44	9.52 846	35	9.55 474	40	10.44 526	9.97 372	5	16
45	9.52 881	35	9.55 514	40	10.44 486	9.97 367	4	15
46	9.52 916	35	9.55 554	39	10.44 446	9.97 363	5	14
47	9.52 951	35	9.55 593	40	10.44 407	9.97 358	5	13
48	9.52 986	35	9.55 633	40	10.44 367	9.97 353	4	12
49	9.53 021	35	9.55 673	39	10.44 327	9.97 349	5	11
50	9.53 056	36	9.55 712	40	10.44 288	9.97 344	4	10
51	9.53 092	34	9.55 752	39	10.44 248	9.97 340	5	9
52	9.53 126	35	9.55 791	40	10.44 209	9.97 335	4	8
53	9.53 161	35	9.55 831	39	10.44 169	9.97 331	5	7
54	9.53 196	35	9.55 870	40	10.44 130	9.97 326	4	6
55	9.53 231	35	9.55 910	39	10.44 090	9.97 322	5	5
56	9.53 266	35	9.55 949	40	10.44 051	9.97 317	5	4
57	9.53 301	35	9.55 989	39	10.44 011	9.97 312	4	3
58	9.53 336	34	9.56 028	39	10.43 972	9.97 308	5	2
59	9.53 370	35	9.56 067	40	10.43 933	9.97 303	4	1
60	9.53 405		9.56 107		10.43 893	9.97 299		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

	41	40	39
.1	4.1	4.0	3.9
.2	8.2	8.0	7.8
.3	12.3	12.0	11.7
.4	16.4	16.0	15.6
.5	20.5	20.0	19.5
.6	24.6	24.0	23.4
.7	28.7	28.0	27.3
.8	32.8	32.0	31.2
.9	36.9	36.0	35.1

	37	36	35
.1	3.7	3.6	3.5
.2	7.4	7.2	7.0
.3	11.1	10.8	10.5
.4	14.8	14.4	14.0
.5	18.5	18.0	17.5
.6	22.2	21.6	21.0
.7	25.9	25.2	24.5
.8	29.6	28.8	28.0
.9	33.3	32.4	31.5

	34	5	4
.1	3.4	0.5	0.4
.2	6.8	1.0	0.8
.3	10.2	1.5	1.2
.4	13.6	2.0	1.6
.5	17.0	2.5	2.0
.6	20.4	3.0	2.4
.7	23.8	3.5	2.8
.8	27.2	4.0	3.2
.9	30.6	4.5	3.6

Prop. Pts.

70°

Prop. Pts.

	40	39
.1	4.0	3.9
.2	8.0	7.8
.3	12.0	11.7
.4	16.0	15.6
.5	20.0	19.5
.6	24.0	23.4
.7	28.0	27.3
.8	32.0	31.2
.9	36.0	35.1

	38	37
.1	3.8	3.7
.2	7.6	7.4
.3	11.4	11.1
.4	15.2	14.8
.5	19.0	18.5
.6	22.8	22.2
.7	26.6	25.9
.8	30.4	29.6
.9	34.2	33.3

	35	34
.1	3.5	3.4
.2	7.0	6.8
.3	10.5	10.2
.4	14.0	13.6
.5	17.5	17.0
.6	21.0	20.4
.7	24.5	23.8
.8	28.0	27.2
.9	31.5	30.6

	33	5	4
.1	3.3	0.5	0.4
.2	6.6	1.0	0.8
.3	9.9	1.5	1.2
.4	13.2	2.0	1.6
.5	16.5	2.5	2.0
.6	19.8	3.0	2.4
.7	23.1	3.5	2.8
.8	26.4	4.0	3.2
.9	29.7	4.5	3.6

Prop. Pts.

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.53 405	35	9.56 107	39	10.43 893	9.97 299	5	60
1	9.53 440	35	9.56 146	39	10.43 854	9.97 294	5	59
2	9.53 475	34	9.56 185	39	10.43 815	9.97 289	4	58
3	9.53 509	35	9.56 224	40	10.43 776	9.97 285	5	57
4	9.53 544	34	9.56 264	39	10.43 736	9.97 280	4	56
5	9.53 578	35	9.56 303	39	10.43 697	9.97 276	5	55
6	9.53 613	34	9.56 342	39	10.43 658	9.97 271	5	54
7	9.53 647	35	9.56 381	39	10.43 619	9.97 266	4	53
8	9.53 682	34	9.56 420	39	10.43 580	9.97 262	5	52
9	9.53 716	35	9.56 459	39	10.43 541	9.97 257	5	51
10	9.53 751	34	9.56 498	39	10.43 502	9.97 252	4	50
11	9.53 785	34	9.56 537	39	10.43 463	9.97 248	5	49
12	9.53 819	35	9.56 576	39	10.43 424	9.97 243	5	48
13	9.53 854	34	9.56 615	39	10.43 385	9.97 238	4	47
14	9.53 888	34	9.56 654	39	10.43 346	9.97 234	5	46
15	9.53 922	35	9.56 693	39	10.43 307	9.97 229	5	45
16	9.53 957	34	9.56 732	39	10.43 268	9.97 224	4	44
17	9.53 991	34	9.56 771	39	10.43 229	9.97 220	5	43
18	9.54 025	34	9.56 810	39	10.43 190	9.97 215	5	42
19	9.54 059	34	9.56 849	38	10.43 151	9.97 210	4	41
20	9.54 093	34	9.56 887	39	10.43 113	9.97 206	5	40
21	9.54 127	34	9.56 926	39	10.43 074	9.97 201	5	39
22	9.54 161	34	9.56 965	39	10.43 035	9.97 196	4	38
23	9.54 195	34	9.57 004	38	10.42 996	9.97 192	5	37
24	9.54 229	34	9.57 042	39	10.42 958	9.97 187	5	36
25	9.54 263	34	9.57 081	39	10.42 919	9.97 182	4	35
26	9.54 297	34	9.57 120	38	10.42 880	9.97 178	5	34
27	9.54 331	34	9.57 158	39	10.42 842	9.97 173	5	33
28	9.54 365	34	9.57 197	38	10.42 803	9.97 168	5	32
29	9.54 399	34	9.57 235	39	10.42 765	9.97 163	4	31
30	9.54 433	33	9.57 274	38	10.42 726	9.97 159	5	30
31	9.54 466	34	9.57 312	39	10.42 688	9.97 154	5	29
32	9.54 500	34	9.57 351	38	10.42 649	9.97 149	4	28
33	9.54 534	33	9.57 389	39	10.42 611	9.97 145	5	27
34	9.54 567	34	9.57 428	38	10.42 572	9.97 140	5	26
35	9.54 601	34	9.57 466	38	10.42 534	9.97 135	5	25
36	9.54 635	33	9.57 504	39	10.42 496	9.97 130	4	24
37	9.54 668	34	9.57 543	38	10.42 457	9.97 126	5	23
38	9.54 702	33	9.57 581	38	10.42 419	9.97 121	5	22
39	9.54 735	34	9.57 619	39	10.42 381	9.97 116	5	21
40	9.54 769	33	9.57 658	38	10.42 342	9.97 111	4	20
41	9.54 802	34	9.57 696	38	10.42 304	9.97 107	5	19
42	9.54 836	33	9.57 734	38	10.42 266	9.97 102	5	18
43	9.54 869	34	9.57 772	38	10.42 228	9.97 097	5	17
44	9.54 903	33	9.57 810	39	10.42 190	9.97 092	5	16
45	9.54 936	33	9.57 849	38	10.42 151	9.97 087	4	15
46	9.54 969	34	9.57 887	38	10.42 113	9.97 083	5	14
47	9.55 003	33	9.57 925	38	10.42 075	9.97 078	5	13
48	9.55 036	33	9.57 963	38	10.42 037	9.97 073	5	12
49	9.55 069	33	9.58 001	38	10.41 999	9.97 068	5	11
50	9.55 102	34	9.58 039	38	10.41 961	9.97 063	4	10
51	9.55 136	33	9.58 077	38	10.41 923	9.97 059	5	9
52	9.55 169	33	9.58 115	38	10.41 885	9.97 054	5	8
53	9.55 202	33	9.58 153	38	10.41 847	9.97 049	5	7
54	9.55 235	33	9.58 191	38	10.41 809	9.97 044	5	6
55	9.55 268	33	9.58 229	38	10.41 771	9.97 039	4	5
56	9.55 301	33	9.58 267	37	10.41 733	9.97 035	5	4
57	9.55 334	33	9.58 304	38	10.41 696	9.97 030	5	3
58	9.55 367	33	9.58 342	38	10.41 658	9.97 025	5	2
59	9.55 400	33	9.58 380	38	10.41 620	9.97 020	5	1
60	9.55 433		9.58 418		10.41 582	9.97 015		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	′
0	9.55 433	33	9.58 418	37	10.41 582	9.97 015	5	60
1	9.55 466	33	9.58 455	38	10.41 545	9.97 010	5	59
2	9.55 499	33	9.58 493	38	10.41 507	9.97 005	4	58
3	9.55 532	32	9.58 531	38	10.41 469	9.97 001	5	57
4	9.55 564	33	9.58 569	37	10.41 431	9.96 996	5	56
5	9.55 597	33	9.58 606	38	10.41 394	9.96 991	5	55
6	9.55 630	33	9.58 644	37	10.41 356	9.96 986	5	54
7	9.55 663	32	9.58 681	38	10.41 319	9.96 981	5	53
8	9.55 695	33	9.58 719	38	10.41 281	9.96 976	5	52
9	9.55 728	33	9.58 757	37	10.41 243	9.96 971	5	51
10	9.55 761	32	9.58 794	38	10.41 206	9.96 966	4	50
11	9.55 793	33	9.58 832	37	10.41 168	9.96 962	5	49
12	9.55 826	32	9.58 869	38	10.41 131	9.96 957	5	48
13	9.55 858	33	9.58 907	37	10.41 093	9.96 952	5	47
14	9.55 891	32	9.58 944	37	10.41 056	9.96 947	5	46
15	9.55 923	33	9.58 981	38	10.41 019	9.96 942	5	45
16	9.55 956	32	9.59 019	37	10.40 981	9.96 937	5	44
17	9.55 988	33	9.59 056	38	10.40 944	9.96 932	5	43
18	9.56 021	32	9.59 094	37	10.40 906	9.96 927	5	42
19	9.56 053	32	9.59 131	37	10.40 869	9.96 922	5	41
20	9.56 085	33	9.59 168	37	10.40 832	9.96 917	5	40
21	9.56 118	32	9.59 205	38	10.40 795	9.96 912	5	39
22	9.56 150	32	9.59 243	37	10.40 757	9.96 907	4	38
23	9.56 182	33	9.59 280	37	10.40 720	9.96 903	5	37
24	9.56 215	32	9.59 317	37	10.40 683	9.96 898	5	36
25	9.56 247	32	9.59 354	37	10.40 646	9.96 893	5	35
26	9.56 279	32	9.59 391	38	10.40 609	9.96 888	5	34
27	9.56 311	32	9.59 429	37	10.40 571	9.96 883	5	33
28	9.56 343	32	9.59 466	37	10.40 534	9.96 878	5	32
29	9.56 375	33	9.59 503	37	10.40 497	9.96 873	5	31
30	9.56 408	32	9.59 540	37	10.40 460	9.96 868	5	30
31	9.56 440	32	9.59 577	37	10.40 423	9.96 863	5	29
32	9.56 472	32	9.59 614	37	10.40 386	9.96 858	5	28
33	9.56 504	32	9.59 651	37	10.40 349	9.96 853	5	27
34	9.56 536	32	9.59 688	37	10.40 312	9.96 848	5	26
35	9.56 568	31	9.59 725	37	10.40 275	9.96 843	5	25
36	9.56 599	32	9.59 762	37	10.40 238	9.96 838	5	24
37	9.56 631	32	9.59 799	36	10.40 201	9.96 833	5	23
38	9.56 663	32	9.59 835	37	10.40 165	9.96 828	5	22
39	9.56 695	32	9.59 872	37	10.40 128	9.96 823	5	21
40	9.56 727	32	9.59 909	37	10.40 091	9.96 818	5	20
41	9.56 759	31	9.59 946	37	10.40 054	9.96 813	5	19
42	9.56 790	32	9.59 983	36	10.40 017	9.96 808	5	18
43	9.56 822	32	9.60 019	37	10.39 981	9.96 803	5	17
44	9.56 854	32	9.60 056	37	10.39 944	9.96 798	5	16
45	9.56 886	31	9.60 093	37	10.39 907	9.96 793	5	15
46	9.56 917	32	9.60 130	36	10.39 870	9.96 788	5	14
47	9.56 949	31	9.60 166	37	10.39 834	9.96 783	5	13
48	9.56 980	32	9.60 203	37	10.39 797	9.96 778	6	12
49	9.57 012	32	9.60 240	36	10.39 760	9.96 772	5	11
50	9.57 044	31	9.60 276	37	10.39 724	9.96 767	5	10
51	9.57 075	32	9.60 313	36	10.39 687	9.96 762	5	9
52	9.57 107	31	9.60 349	37	10.39 651	9.96 757	5	8
53	9.57 138	31	9.60 386	36	10.39 614	9.96 752	5	7
54	9.57 169	32	9.60 422	37	10.39 578	9.96 747	5	6
55	9.57 201	31	9.60 459	36	10.39 541	9.96 742	5	5
56	9.57 232	32	9.60 495	37	10.39 505	9.96 737	6	4
57	9.57 264	31	9.60 532	36	10.39 468	9.96 732	5	3
58	9.57 295	31	9.60 568	37	10.39 432	9.96 727	5	2
59	9.57 326	32	9.60 605	36	10.39 395	9.96 722	5	1
60	9.57 358		9.60 641		10.39 359	9.96 717		0
′	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

	38	37	36
.1	3.8	3.7	3.6
.2	7.6	7.4	7.2
.3	11.4	11.1	10.8
.4	15.2	14.8	14.4
.5	19.0	18.5	18.0
.6	22.8	22.2	21.6
.7	26.6	25.9	25.2
.8	30.4	29.6	28.8
.9	34.2	33.3	32.4

	33	32	31
.1	3.3	3.2	3.1
.2	6.6	6.4	6.2
.3	9.9	9.6	9.3
.4	13.2	12.8	12.4
.5	16.5	16.0	15.5
.6	19.8	19.2	18.6
.7	23.1	22.4	21.7
.8	26.4	25.6	24.8
.9	29.7	28.8	27.9

	6	5	4
.1	0.6	0.5	0.4
.2	1.2	1.0	0.8
.3	1.8	1.5	1.2
.4	2.4	2.0	1.6
.5	3.0	2.5	2.0
.6	3.6	3.0	2.4
.7	4.2	3.5	2.8
.8	4.8	4.0	3.2
.9	5.4	4.5	3.6

Prop. Pts.

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.57 358	31	9.60 641	36	10.39 359	9.96 717	6	60
1	9.57 389	31	9.60 677	37	10.39 323	9.96 711	5	59
2	9.57 420	31	9.60 714	36	10.39 286	9.96 706	5	58
3	9.57 451	31	9.60 750	36	10.39 250	9.96 701	5	57
4	9.57 482	32	9.60 786	37	10.39 214	9.96 696	5	56
5	9.57 514	31	9.60 823	36	10.39 177	9.96 691	5	55
6	9.57 545	31	9.60 859	36	10.39 141	9.96 686	5	54
7	9.57 576	31	9.60 895	36	10.39 105	9.96 681	5	53
8	9.57 607	31	9.60 931	36	10.39 069	9.96 676	6	52
9	9.57 638	31	9.60 967	37	10.39 033	9.96 670	5	51
10	9.57 669	31	9.61 004	36	10.38 996	9.96 665	5	50
11	9.57 700	31	9.61 040	36	10.38 960	9.96 660	5	49
12	9.57 731	31	9.61 076	36	10.38 924	9.96 655	5	48
13	9.57 762	31	9.61 112	36	10.38 888	9.96 650	5	47
14	9.57 793	31	9.61 148	36	10.38 852	9.96 645	5	46
15	9.57 824	31	9.61 184	36	10.38 816	9.96 640	6	45
16	9.57 855	30	9.61 220	36	10.38 780	9.96 634	5	44
17	9.57 885	31	9.61 256	36	10.38 744	9.96 629	5	43
18	9.57 916	31	9.61 292	36	10.38 708	9.96 624	5	42
19	9.57 947	31	9.61 328	36	10.38 672	9.96 619	5	41
20	9.57 978	30	9.61 364	36	10.38 636	9.96 614	6	40
21	9.58 008	31	9.61 400	36	10.38 600	9.96 608	5	39
22	9.58 039	31	9.61 436	36	10.38 564	9.96 603	5	38
23	9.58 070	31	9.61 472	36	10.38 528	9.96 598	5	37
24	9.58 101	30	9.61 508	36	10.38 492	9.96 593	5	36
25	9.58 131	31	9.61 544	35	10.38 456	9.96 588	6	35
26	9.58 162	30	9.61 579	36	10.38 421	9.96 582	5	34
27	9.58 192	31	9.61 615	36	10.38 385	9.96 577	5	33
28	9.58 223	30	9.61 651	36	10.38 349	9.96 572	5	32
29	9.58 253	31	9.61 687	35	10.38 313	9.96 567	5	31
30	9.58 284	30	9.61 722	36	10.38 278	9.96 562	6	30
31	9.58 314	31	9.61 758	36	10.38 242	9.96 556	5	29
32	9.58 345	30	9.61 794	36	10.38 206	9.96 551	5	28
33	9.58 375	31	9.61 830	35	10.38 170	9.96 546	5	27
34	9.58 406	30	9.61 865	36	10.38 135	9.96 541	6	26
35	9.58 436	31	9.61 901	35	10.38 099	9.96 535	5	25
36	9.58 467	30	9.61 936	36	10.38 064	9.96 530	5	24
37	9.58 497	30	9.61 972	36	10.38 028	9.96 525	5	23
38	9.58 527	30	9.62 008	35	10.37 992	9.96 520	6	22
39	9.58 557	31	9.62 043	36	10.37 957	9.96 514	5	21
40	9.58 588	30	9.62 079	35	10.37 921	9.96 509	5	20
41	9.58 618	30	9.62 114	36	10.37 886	9.96 504	6	19
42	9.58 648	30	9.62 150	35	10.37 850	9.96 498	5	18
43	9.58 678	31	9.62 185	36	10.37 815	9.96 493	5	17
44	9.58 709	30	9.62 221	35	10.37 779	9.96 488	5	16
45	9.58 739	30	9.62 256	36	10.37 744	9.96 483	6	15
46	9.58 769	30	9.62 292	35	10.37 708	9.96 477	5	14
47	9.58 799	30	9.62 327	35	10.37 673	9.96 472	5	13
48	9.58 829	30	9.62 362	36	10.37 638	9.96 467	6	12
49	9.58 859	30	9.62 398	35	10.37 602	9.96 461	5	11
50	9.58 889	30	9.62 433	35	10.37 567	9.96 456	5	10
51	9.58 919	30	9.62 468	36	10.37 532	9.96 451	6	9
52	9.58 949	30	9.62 504	35	10.37 496	9.96 445	5	8
53	9.58 979	30	9.62 539	35	10.37 461	9.96 440	5	7
54	9.59 009	30	9.62 574	35	10.37 426	9.96 435	6	6
55	9.59 039	30	9.62 609	36	10.37 391	9.96 429	5	5
56	9.59 069	29	9.62 645	35	10.37 355	9.96 424	5	4
57	9.59 098	30	9.62 680	35	10.37 320	9.96 419	6	3
58	9.59 128	30	9.62 715	35	10.37 285	9.96 413	5	2
59	9.59 158	30	9.62 750	35	10.37 250	9.96 408	5	1
60	9.59 188		9.62 785		10.37 215	9.96 403		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

	37	36	35
.1	3.7	3.6	3.5
.2	7.4	7.2	7.0
.3	11.1	10.8	10.5
.4	14.8	14.4	14.0
.5	18.5	18.0	17.5
.6	22.2	21.6	21.0
.7	25.9	25.2	24.5
.8	29.6	28.8	28.0
.9	33.3	32.4	31.5

	32	31	30
.1	3.2	3.1	3.0
.2	6.4	6.2	6.0
.3	9.6	9.3	9.0
.4	12.8	12.4	12.0
.5	16.0	15.5	15.0
.6	19.2	18.6	18.0
.7	22.4	21.7	21.0
.8	25.6	24.8	24.0
.9	28.8	27.9	27.0

	29	6	5
.1	2.9	0.6	0.5
.2	5.8	1.2	1.0
.3	8.7	1.8	1.5
.4	11.6	2.4	2.0
.5	14.5	3.0	2.5
.6	17 4	3.6	3.0
.7	20.3	4.2	3.5
.8	23.2	4.8	4.0
.9	26.1	5.4	4.5

Prop. Pts.

67°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.59 188	30	9.62 785	35	10.37 215	9.96 403	6	60
1	9.59 218	29	9.62 820	35	10.37 180	9.96 397	5	59
2	9.59 247	30	9.62 855	35	10.37 145	9.96 392	5	58
3	9.59 277	30	9.62 890	36	10.37 110	9.96 387	6	57
4	9.59 307	29	9.62 926	35	10.37 074	9.96 381	5	56
5	9.59 336	30	9.62 961	35	10.37 039	9.96 376	6	55
6	9.59 366	30	9.62 996	35	10.37 004	9.96 370	5	54
7	9.59 396	29	9.63 031	35	10.36 960	9.96 365	5	53
8	9.59 425	30	9.63 066	35	10.36 934	9.96 360	6	52
9	9.59 455	29	9.63 101	34	10.36 899	9.96 354	5	51
10	9.59 484	30	9.63 135	35	10.36 865	9.96 349	6	50
11	9.59 514	29	9.63 170	35	10.36 830	9.96 343	5	49
12	9.59 543	30	9.63 205	35	10.36 795	9.96 338	5	48
13	9.59 573	29	9.63 240	35	10.36 760	9.96 333	6	47
14	9.59 602	30	9.63 275	35	10.36 725	9.96 327	5	46
15	9.59 632	29	9.63 310	35	10.36 690	9.96 322	6	45
16	9.59 661	29	9.63 345	34	10.36 655	9.96 316	5	44
17	9.59 690	30	9.63 379	35	10.36 621	9.96 311	6	43
18	9.59 720	29	9.63 414	35	10.36 586	9.96 305	5	42
19	9.59 749	29	9.63 449	35	10.36 551	9.96 300	6	41
20	9.59 778	30	9.63 484	35	10.36 516	9.96 294	5	40
21	9.59 808	29	9.63 519	34	10.36 481	9.96 289	5	39
22	9.59 837	29	9.63 553	35	10.36 447	9.96 284	6	38
23	9.59 866	29	9.63 588	35	10.46 412	9.96 278	5	37
24	9.59 895	29	9.63 623	34	10.36 377	9.96 273	6	36
25	9.59 924	30	9.63 657	35	10.36 343	9.96 267	5	35
26	9.59 954	29	9.63 692	34	10.36 308	9.96 262	6	34
27	9.59 983	29	9.63 726	35	10.36 274	9.96 256	5	33
28	9.60 012	29	9.63 761	35	10.36 239	9.96 251	6	32
29	9.60 041	29	9.63 796	34	10.36 204	9.96 245	5	31
30	9.60 070	29	9.63 830	35	10.36 170	9.96 240	6	30
31	9.60 099	29	9.63 865	34	10.36 135	9.96 234	5	29
32	9.60 128	29	9.63 899	35	10.36 101	9.96 229	6	28
33	9.60 157	29	9.63 934	34	10.36 066	9.96 223	5	27
34	9.60 186	29	9.63 968	35	10.36 032	9.96 218	6	26
35	9.60 215	29	9.64 003	34	10.35 997	9.96 212	5	25
36	9.60 244	29	9.64 037	35	10.35 963	9.96 207	6	24
37	9.60 273	29	9.64 072	34	10.35 928	9.96 201	5	23
38	9.60 302	29	9.64 106	34	10.35 894	9.96 196	6	22
39	9.60 331	28	9.64 140	35	10.35 860	9.96 190	5	21
40	9.60 359	29	9.64 175	34	10.35 825	9.96 185	6	20
41	9.60 388	29	9.64 209	34	10.35 791	9.96 179	5	19
42	9.60 417	29	9.64 243	35	10.35 757	9.96 174	6	18
43	9.60 446	28	9.64 278	34	10.35 722	9.96 168	6	17
44	9.60 474	29	9.64 312	34	10.35 688	9.96 162	5	16
45	9.60 503	29	9.64 346	35	10.35 654	9.96 157	6	15
46	9.60 532	29	9.64 381	34	10.35 619	9.96 151	5	14
47	9.60 561	28	9.64 415	34	10.35 585	9.96 146	6	13
48	9.60 589	29	9.64 449	34	10.35 551	9.96 140	5	12
49	9.60 618	28	9.64 483	34	10.35 517	9.96 135	6	11
50	9.60 646	29	9.64 517	35	10.35 483	9.96 129	6	10
51	9.60 675	29	9.64 552	34	10.35 448	9.96 123	5	9
52	9.60 704	28	9.64 586	34	10.35 414	9.96 118	6	8
53	9.60 732	29	9.64 620	34	10.35 380	9.96 112	5	7
54	9.60 761	28	9.64 654	34	10.35 346	9.96 107	6	6
55	9.60 789	29	9.64 688	34	10.35 312	9.96 101	6	5
56	9.60 818	28	9.64 722	34	10.35 278	9.96 095	5	4
57	9.60 846	29	9.64 756	34	10.35 244	9.96 090	6	3
58	9.60 875	28	9.64 790	34	10.35 210	9.96 084	5	2
59	9.60 903	28	9.64 824	34	10.35 176	9.96 079	6	1
60	9.60 931		9.64 858		10.35 142	9.96 073		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

	36	35	34
.1	3.6	3.5	3.4
.2	7.2	7.0	6.8
.3	10.8	10.5	10.2
.4	14.4	14.0	13.6
.5	18.0	17.5	17.0
.6	21.6	21.0	20.4
.7	25.2	24.5	23.8
.8	28.8	28.0	27.2
.9	32.4	31.5	30.6

	30	29	28
.1	3.0	2.9	2.8
.2	6.0	5.8	5.6
.3	9.0	8.7	8.4
.4	12.0	11.6	11.2
.5	15.0	14.5	14.0
.6	18.0	17.4	16.8
.7	21.0	20.3	19.6
.8	24.0	23.2	22.4
.9	27.0	26.1	25.2

	6	5
.1	0.6	0.5
.2	1.2	1.0
.3	1.8	1.5
.4	2.4	2.0
.5	3.0	2.5
.6	3.6	3.0
.7	4.2	3.5
.8	4.8	4.0
.9	5.4	4.5

Prop. Pts.

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.60 931	29	9.64 858	34	10.35 142	9.96 073	6	60
1	9.60 960	28	9.64 892	34	10.35 108	9.96 067	5	59
2	9.60 088	28	9.64 926	34	10.35 074	9.96 062	6	58
3	9.61 016	29	9.64 960	34	10.35 040	9.96 056	6	57
4	9.61 045	28	9.64 994	34	10.35 006	9.96 050	5	56
5	9.61 073	28	9.65 028	34	10.34 972	9.96 045	6	55
6	9.61 101	28	9.65 062	34	10.34 938	9.96 039	5	54
7	9.61 129	29	9.65 096	34	10.34 904	9.96 034	6	53
8	9.61 158	28	9.65 130	34	10.34 870	9.96 028	6	52
9	9.61 186	28	9.65 164	33	10.34 836	9.96 022	5	51
10	9.61 214	28	9.65 197	34	10.34 803	9.96 017	6	50
11	9.61 242	28	9.65 231	34	10.34 769	9.96 011	6	49
12	9.61 270	28	9.65 265	34	10.34 735	9.96 005	5	48
13	9.61 298	28	9.65 299	34	10.34 701	9.96 000	6	47
14	9.61 326	28	9.65 333	33	10.34 667	9.95 994	6	46
15	9.61 354	28	9.65 366	34	10.34 634	9.95 988	6	45
16	9.61 382	29	9.65 400	34	10.34 600	9.95 982	5	44
17	9.61 411	27	9.65 434	33	10.34 566	9.95 977	6	43
18	9.61 438	28	9.65 467	34	10.34 533	9.95 971	6	42
19	9.61 466	28	9.65 501	34	10.34 499	9.95 965	5	41
20	9.61 494	28	9.65 535	33	10.34 465	9.95 960	6	40
21	9.61 522	28	9.65 568	34	10.34 432	9.95 954	6	39
22	9.61 550	28	9.65 602	34	10.34 398	9.95 948	6	38
23	9.61 578	28	9.65 636	33	10.34 364	9.95 942	5	37
24	9.61 606	28	9.65 669	34	10.34 331	9.95 937	6	36
25	9.61 634	28	9.65 703	33	10.34 297	9.95 931	6	35
26	9.61 662	27	9.65 736	34	10.34 264	9.95 925	5	34
27	9.61 689	28	9.65 770	33	10.34 230	9.95 920	6	33
28	9.61 717	28	9.65 803	34	10.34 197	9.95 914	6	32
29	9.61 745	28	9.65 837	33	10.34 163	9.95 908	6	31
30	9.61 773	27	9.65 870	34	10.34 130	9.95 902	5	30
31	9.61 800	28	9.65 904	33	10.34 096	9.95 897	6	29
32	9.61 828	28	9.65 937	34	10.34 063	9.95 891	6	28
33	9.61 856	27	9.65 971	33	10.34 029	9.95 885	6	27
34	9.61 883	28	9.66 004	34	10.33 996	9.95 879	6	26
35	9.61 911	28	9.66 038	33	10.33 962	9.95 873	5	25
36	9.61 939	27	9.66 071	33	10.33 929	9.95 868	6	24
37	9.61 966	28	9.66 104	34	10.33 896	9.95 862	6	23
38	9.61 994	27	9.66 138	33	10.33 862	9.95 856	6	22
39	9.62 021	28	9.66 171	33	10.33 829	9.95 850	6	21
40	9.62 049	27	9.66 204	34	10.33 796	9.95 844	5	20
41	9.62 076	28	9.66 238	33	10.33 762	9.95 839	6	19
42	9.62 104	27	9.66 271	33	10.33 729	9.95 833	6	18
43	9.62 131	28	9.66 304	33	10.33 696	9.95 827	6	17
44	9.62 159	27	9.66 337	34	10.33 663	9.95 821	6	16
45	9.62 186	28	9.66 371	33	10.33 629	9.95 815	5	15
46	9.62 214	27	9.66 404	33	10.33 596	9.95 810	6	14
47	9.62 241	27	9.66 437	33	10.33 563	9.95 804	6	13
48	9.62 268	28	9.66 470	33	10.33 530	9.95 798	6	12
49	9.62 296	27	9.66 503	34	10.33 497	9.95 792	6	11
50	9.62 323	27	9.66 537	33	10.33 463	9.95 786	6	10
51	9.62 350	27	9.66 570	33	10.33 430	9.95 780	5	9
52	9.62 377	28	9.66 603	33	10.33 397	9.95 775	6	8
53	9.62 405	27	9.66 636	33	10.33 364	9.95 769	6	7
54	9.62 432	27	9.66 669	33	10.33 331	9.95 763	6	6
55	9.62 459	27	9.66 702	33	10.33 298	9.95 757	6	5
56	9.62 486	27	9.66 735	33	10.33 265	9.95 751	6	4
57	9.62 513	28	9.66 768	33	10.33 232	9.95 745	6	3
58	9.62 541	27	9.66 801	33	10.33 199	9.95 739	6	2
59	9.62 568	27	9.66 834	33	10.33 166	9.95 733	5	1
60	9.62 595		9.66 867		10.33 133	9.95 728		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

	34	33
.1	3.4	3.3
.2	6.8	6.6
.3	10.2	9.9
.4	13.6	13.2
.5	17.0	16.5
.6	20.4	19.8
.7	23.8	23.1
.8	27.2	26.4
.9	30.6	29.7

	29	28	27
.1	2.9	2.8	2.7
.2	5.8	5.6	5.4
.3	8.7	8.4	8.1
.4	11.6	11.2	10.8
.5	14.5	14.0	13.5
.6	17.4	16.8	16.2
.7	20.3	19.6	18.9
.8	23.2	22.4	21.6
.9	26.1	25.2	24.3

	6	5
.1	0.6	0.5
.2	1.2	1.0
.3	1.8	1.5
.4	2.4	2.0
.5	3.0	2.5
.6	3.6	3.0
.7	4.2	3.5
.8	4.8	4.0
.9	5.4	4.5

Prop. Pts.

65°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.62 595	27	9.66 867	33	10.33 133	9.95 728	6	60
1	9.62 622	27	9.66 900	33	10.33 100	9.95 722	6	59
2	9.62 649	27	9.66 933	33	10.33 067	9.95 716	6	58
3	9.62 676	27	9.66 966	33	10.33 034	9.95 710	6	57
4	9.62 703	27	9.66 999	33	10.33 001	9.95 704	6	56
5	9.62 730	27	9.67 032	33	10.32 968	9.95 698	6	55
6	9.62 757	27	9.67 065	33	10.32 935	9.95 692	6	54
7	9.62 784	27	9.67 098	33	10.32 902	9.95 686	6	53
8	9.62 811	27	9.67 131	32	10.32 869	9.95 680	6	52
9	9.62 838	27	9.67 163	33	10.32 837	9.95 674	6	51
10	9.62 865	27	9.67 196	33	10.32 804	9.95 668	5	50
11	9.62 892	26	9.67 229	33	10.32 771	9.95 663	6	49
12	9.62 918	27	9.67 262	33	10.32 738	9.95 657	6	48
13	9.62 945	27	9.67 295	32	10.32 705	9.95 651	6	47
14	9.62 972	27	9.67 327	33	10.32 673	9.95 645	6	46
15	9.62 999	27	9.67 360	33	10.32 640	9.95 639	6	45
16	9.63 026	26	9.67 393	33	10.32 607	9.95 633	6	44
17	9.63 052	27	9.67 426	32	10.32 574	9.95 627	6	43
18	9.63 079	27	9.67 458	33	10.32 542	9.95 621	6	42
19	9.63 106	27	9.67 491	33	10.32 509	9.95 615	6	41
20	9.63 133	26	9.67 524	32	10.32 476	9.95 609	6	40
21	9.63 159	27	9.67 556	33	10.32 444	9.95 603	6	39
22	9.63 186	27	9.67 589	33	10.32 411	9.95 597	6	38
23	9.63 213	26	9.67 622	32	10.32 378	9.95 591	6	37
24	9.63 239	27	9.67 654	33	10.32 346	9.95 585	6	36
25	9.63 266	26	9.67 687	32	10.32 313	9.95 579	6	35
26	9.63 292	27	9.67 719	33	10.32 281	9.95 573	6	34
27	9.63 319	26	9.67 752	33	10.32 248	9.95 567	6	33
28	9.63 345	27	9.67 785	32	10.32 215	9.95 561	6	32
29	9.63 372	26	9.67 817	33	10.32 183	9.95 555	6	31
30	9.63 398	27	9.67 850	32	10.32 150	9.95 549	6	30
31	9.63 425	26	9.67 882	33	10.32 118	9.95 543	6	29
32	9.63 451	27	9.67 915	32	10.32 085	9.95 537	6	28
33	9.63 478	26	9.67 947	33	10.32 053	9.95 531	6	27
34	9.63 504	27	9.67 980	32	10.32 020	9.95 525	6	26
35	9.63 531	26	9.68 012	32	10.31 988	9.95 519	6	25
36	9.63 557	26	9.68 044	33	10.31 956	9.95 513	6	24
37	9.63 583	27	9.68 077	32	10.31 923	9.95 507	7	23
38	9.63 610	26	9.68 109	33	10.31 891	9.95 500	6	22
39	9.63 636	26	9.68 142	32	10.31 858	9.95 494	6	21
40	9.63 662	27	9.68 174	32	10.31 826	9.95 488	6	20
41	9.63 689	26	9.68 206	33	10.31 794	9.95 482	6	19
42	9.63 715	26	9.68 239	32	10.31 761	9.95 476	6	18
43	9.63 741	26	9.68 271	32	10.31 729	9.95 470	6	17
44	9.63 767	27	9.68 303	33	10.31 697	9.95 464	6	16
45	9.63 794	26	9.68 336	32	10.31 664	9.95 458	6	15
46	9.63 820	26	9.68 368	32	10.31 632	9.95 452	6	14
47	9.63 846	26	9.68 400	32	10.31 600	9.95 446	6	13
48	9.63 872	26	9.68 432	33	10.31 568	9.95 440	6	12
49	9.63 898	26	9.68 465	32	10.31 535	9.95 434	7	11
50	9.63 924	26	9.68 497	32	10.31 503	9.95 427	6	10
51	9.63 950	26	9.68 529	32	10.31 471	9.95 421	6	9
52	9.63 976	26	9.68 561	32	10.31 439	9.95 415	6	8
53	9.64 002	26	9.68 593	33	10.31 407	9.95 409	6	7
54	9.64 028	26	9.68 626	32	10.31 374	9.95 403	6	6
55	9.64 054	26	9.68 658	32	10.31 342	9.95 397	6	5
56	9.64 080	26	9.68 690	32	10.31 310	9.95 391	7	4
57	9.64 106	26	9.68 722	32	10.31 278	9.95 384	6	3
58	9.64 132	26	9.68 754	32	10.31 246	9.95 378	6	2
59	9.64 158	26	9.68 786	32	10.31 214	9.95 372	6	1
60	9.64 184		9.68 818		10.31 182	9.95 366		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

	33	32
.1	3.3	3.2
.2	6.6	6.4
.3	9.9	9.6
.4	13.2	12.8
.5	16.5	16.0
.6	19.8	19.2
.7	23.1	22.4
.8	26.4	25.6
.9	29.7	28.8

	27	26
.1	2.7	2.6
.2	5.4	5.2
.3	8.1	7.8
.4	10.8	10.4
.5	13.5	13.0
.6	16.2	15.6
.7	18.9	18.2
.8	21.6	20.8
.9	24.3	23.4

	7	6	5
.1	0.7	0.6	0.5
.2	1.4	1.2	1.0
.3	2.1	1.8	1.5
.4	2.8	2.4	2.0
.5	3.5	3.0	2.5
.6	4.2	3.6	3.0
.7	4.9	4.2	3.5
.8	5.6	4.8	4.0
.9	6.3	5.4	4.5

Prop. Pts.

Prop. Pts.

	32	31
.1	3.2	3.1
.2	6.4	6.2
.3	9.6	9.3
.4	12.8	12.4
.5	16.0	15.5
.6	19.2	18.6
.7	22.4	21.7
.8	25.6	24.8
.9	28.8	27.9

	26	25	24
.1	2.6	2.5	2.4
.2	5.2	5.0	4.8
.3	7.8	7.5	7.2
.4	10.4	10.0	9.6
.5	13.0	12.5	12.0
.6	15.6	15.0	14.4
.7	18.2	17.5	16.8
.8	20.8	20.0	19.2
.9	23.4	22.5	21.6

	7	6
.1	0.7	0.6
.2	1.4	1.2
.3	2.1	1.8
.4	2.8	2.4
.5	3.5	3.0
.6	4.2	3.6
.7	4.9	4.2
.8	5.6	4.8
.9	6.3	5.4

Prop. Pts.

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.64 184	26	9.68 818	32	10.31 182	9.95 366	6	60
1	9.64 210	26	9.68 850	32	10.31 150	9.95 360	6	59
2	9.64 236	26	9.68 882	32	10.31 118	9.95 354	6	58
3	9.64 262	26	9.68 914	32	10.31 086	9.95 348	7	57
4	9.64 288	25	9.68 946	32	10.31 054	9.95 341	6	56
5	9.64 313	26	9.68 978	32	10.31 022	9.95 335	6	55
6	9.64 339	26	9.69 010	32	10.30 990	9.95 329	6	54
7	9.64 365	26	9.69 042	32	10.30 958	9.95 323	6	53
8	9.64 391	26	9.69 074	32	10.30 926	9.95 317	7	52
9	9.64 417	25	9.69 106	32	10.30 894	9.95 310	6	51
10	9.64 442	26	9.69 138	32	10.30 862	9.95 304	6	50
11	9.64 468	26	9.69 170	32	10.30 830	9.95 298	6	49
12	9.64 494	25	9.69 202	32	10.30 798	9.95 292	6	48
13	9.64 519	26	9.69 234	32	10.30 766	9.95 286	7	47
14	9.64 545	26	9.69 266	32	10.30 734	9.95 279	6	46
15	9.64 571	25	9.69 298	31	10.30 702	9.95 273	6	45
16	9.64 596	26	9.69 329	32	10.30 671	9.95 267	6	44
17	9.64 622	25	9.69 361	32	10.30 639	9.95 261	7	43
18	9.64 647	26	9.69 393	32	10.30 607	9.95 254	6	42
19	9.64 673	25	9.69 425	32	10.30 575	9.95 248	6	41
20	9.64 698	26	9.69 457	31	10.30 543	9.95 242	6	40
21	9.64 724	25	9.69 488	32	10.30 512	9.95 236	7	39
22	9.64 749	26	9.69 520	32	10.30 480	9.95 229	6	38
23	9.64 775	25	9.69 552	32	10.30 448	9.95 223	6	37
24	9.64 800	26	9.69 584	31	10.30 416	9.95 217	6	36
25	9.64 826	25	9.69 615	32	10.30 385	9.95 211	7	35
26	9.64 851	26	9.69 647	32	10.30 353	9.95 204	6	34
27	9.64 877	25	9.69 679	31	10.30 321	9.95 198	6	33
28	9.64 902	25	9.69 710	32	10.30 290	9.95 192	7	32
29	9.64 927	26	9.69 742	32	10.30 258	9.95 185	6	31
30	9.64 953	25	9.69 774	31	10.30 226	9.95 179	6	30
31	9.64 978	25	9.69 805	32	10.30 195	9.95 173	6	29
32	9.65 003	26	9.69 837	31	10.30 163	9.95 167	7	28
33	9.65 029	25	9.69 868	32	10.30 132	9.95 160	6	27
34	9.65 054	25	9.69 900	32	10.30 100	9.95 154	6	26
35	9.65 079	25	9.69 932	31	10.30 068	9.95 148	7	25
36	9.65 104	26	9.69 963	32	10.30 037	9.95 141	6	24
37	9.65 130	25	9.69 995	31	10.30 005	9.95 135	6	23
38	9.65 155	25	9.70 026	32	10.29 974	9.95 129	7	22
39	9.65 180	25	9.70 058	31	10.29 942	9.95 122	6	21
40	9.65 205	25	9.70 089	32	10.29 911	9.95 116	6	20
41	9.65 230	25	9.70 121	31	10.29 879	9.95 110	7	19
42	9.65 255	26	9.70 152	32	10.29 848	9.95 103	6	18
43	9.65 281	25	9.70 184	31	10.29 816	9.95 097	7	17
44	9.65 306	25	9.70 215	32	10.29 785	9.95 090	6	16
45	9.65 331	25	9.70 247	31	10.29 753	9.95 084	6	15
46	9.65 356	25	9.70 278	31	10.29 722	9.95 078	7	14
47	9.65 381	25	9.70 309	32	10.29 691	9.95 071	6	13
48	9.65 406	25	9.70 341	31	10.29 659	9.95 065	6	12
49	9.65 431	25	9.70 372	32	10.29 628	9.95 059	7	11
50	9.65 456	25	9.70 404	31	10.29 596	9.95 052	6	10
51	9.65 481	25	9.70 435	31	10.29 565	9.95 046	7	9
52	9.65 506	25	9.70 466	32	10.29 534	9.95 039	6	8
53	9.65 531	25	9.70 498	31	10.29 502	9.95 033	6	7
54	9.65 556	24	9.70 529	31	10.29 471	9.95 027	7	6
55	9.65 580	25	9.70 560	32	10.29 440	9.95 020	6	5
56	9.65 605	25	9.70 592	31	10.29 408	9.95 014	7	4
57	9.65 630	25	9.70 623	31	10.29 377	9.95 007	6	3
58	9.65 655	25	9.70 654	31	10.29 346	9.95 001	6	2
59	9.65 680	25	9.70 685	32	10.29 315	9.94 995	7	1
60	9.65 705		9.70 717		10.29 283	9.94 988		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

63°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	′
0	9.65 705	24	9.70 717	31	10.29 283	9.94 988	6	60
1	9.65 729	25	9.70 748	31	10.29 252	9.94 982	7	59
2	9.65 754	25	9.70 779	31	10.29 221	9.94 975	6	58
3	9.65 779	25	9.70 810	31	10.29 190	9.94 969	7	57
4	9.65 804	24	9.70 841	32	10.29 159	9.94 962	6	56
5	9.65 828	25	9.70 873	31	10.29 127	9.94 956	7	55
6	9.65 853	25	9.70 904	31	10.29 096	9.94 949	6	54
7	9.65 878	24	9.70 935	31	10.29 065	9.94 943	7	53
8	9.65 902	25	9.70 966	31	10.29 034	9.94 936	6	52
9	9.65 927	25	9.70 997	31	10.29 003	9.94 930	7	51
10	9.65 952	24	9.71 028	31	10.28 972	9.94 923	6	50
11	9.65 976	25	9.71 059	31	10.28 941	9.94 917	6	49
12	9.66 001	24	9.71 090	31	10.28 910	9.94 911	7	48
13	9.66 025	25	9.71 121	32	10.28 879	9.94 904	6	47
14	9.66 050	25	9.71 153	31	10.28 847	9.94 898	7	46
15	9.66 075	24	9.71 184	31	10.28 816	9.94 891	6	45
16	9.66 099	25	9.71 215	31	10.28 785	9.94 885	7	44
17	9.66 124	24	9.71 246	31	10.28 754	9.94 878	7	43
18	9.66 148	25	9.71 277	31	10.28 723	9.94 871	6	42
19	9.66 173	24	9.71 308	31	10.28 692	9.94 865	7	41
20	9.66 197	24	9.71 339	31	10.28 661	9.94 858	6	40
21	9.66 221	25	9.71 370	31	10.28 630	9.94 852	7	39
22	9.66 246	24	9.71 401	30	10.28 599	9.94 845	6	38
23	9.66 270	25	9.71 431	31	10.28 569	9.94 839	7	37
24	9.66 295	24	9.71 462	31	10.28 538	9.94 832	6	36
25	9.66 319	24	9.71 493	31	10.28 507	9.94 826	7	35
26	9.66 343	25	9.71 524	31	10.28 476	9.94 819	6	34
27	9.66 368	24	9.71 555	31	10.28 445	9.94 813	7	33
28	9.66 392	24	9.71 586	31	10.28 414	9.94 806	7	32
29	9.66 416	25	9.71 617	31	10.28 383	9.94 799	6	31
30	9.66 441	24	9.71 648	31	10.28 352	9.94 793	7	30
31	9.66 465	24	9.71 679	30	10.28 321	9.94 786	6	29
32	9.66 489	24	9.71 709	31	10.28 291	9.94 780	7	28
33	9.66 513	24	9.71 740	31	10.28 260	9.94 773	6	27
34	9.66 537	25	9.71 771	31	10.28 229	9.94 767	7	26
35	9.66 562	24	9.71 802	31	10.28 198	9.94 760	7	25
36	9.66 586	24	9.71 833	30	10.28 167	9.94 753	6	24
37	9.66 610	24	9.71 863	31	10.28 137	9.94 747	7	23
38	9.66 634	24	9.71 894	31	10.28 106	9.94 740	6	22
39	9.66 658	24	9.71 925	30	10.28 075	9.94 734	7	21
40	9.66 682	24	9.71 955	31	10.28 045	9.94 727	7	20
41	9.66 706	25	9.71 986	31	10.28 014	9.94 720	6	19
42	9.66 731	24	9.72 017	31	10.27 983	9.94 714	7	18
43	9.66 755	24	9.72 048	30	10.27 952	9.94 707	7	17
44	9.66 779	24	9.72 078	31	10.27 922	9.94 700	6	16
45	9.66 803	24	9.72 109	31	10.27 891	9.94 694	7	15
46	9.66 827	24	9.72 140	30	10.27 860	9.94 687	7	14
47	9.66 851	24	9.72 170	31	10.27 830	9.94 680	6	13
48	9.66 875	24	9.72 201	30	10.27 799	9.94 674	7	12
49	9.66 899	23	9.72 231	31	10.27 769	9.94 667	7	11
50	9.66 922	24	9.72 262	31	10.27 738	9.94 660	6	10
51	9.66 946	24	9.72 293	30	10.27 707	9.94 654	7	9
52	9.66 970	24	9.72 323	31	10.27 677	9.94 647	7	8
53	9.66 994	24	9.72 354	30	10.27 646	9.94 640	6	7
54	9.67 018	24	9.72 384	31	10.27 616	9.94 634	7	6
55	9.67 042	24	9.72 415	30	10.27 585	9.94 627	7	5
56	9.67 066	24	9.72 445	31	10.27 555	9.94 620	6	4
57	9.67 090	23	9.72 476	30	10.27 524	9.94 614	7	3
58	9.67 113	24	9.72 506	31	10.27 494	9.94 607	7	2
59	9.67 137	24	9.72 537	30	10.27 463	9.94 600	7	1
60	9.67 161		9.72 567		10.27 433	9.94 593		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

	32	31	30
.1	3.2	3.1	3.0
.2	6.4	6.2	6.0
.3	9.6	9.3	9.0
.4	12.8	12.4	12.0
.5	16.0	15.5	15.0
.6	19.2	18.6	18.0
.7	22.4	21.7	21.0
.8	25.6	24.8	24.0
.9	28.8	27.9	27.0

	25	24	23
.1	2.5	2.4	2.3
.2	5.0	4.8	4.6
.3	7.5	7.2	6.9
.4	10.0	9.6	9.2
.5	12.5	12.0	11.5
.6	17.0	14.4	13.8
.7	17.5	16.8	16.1
.8	20.0	19.2	18.4
.9	22.5	21.6	20.7

	7	6
.1	0.7	0.6
.2	1.4	1.2
.3	2.1	1.8
.4	2.8	2.4
.5	3.5	3.0
.6	4.2	3.6
.7	4.9	4.2
.8	5.6	4.8
.9	6.3	5.4

Prop. Pts.

Prop. Pts.

	31	30	29
.1	3.1	3.0	2.9
.2	6.2	6.0	5.8
.3	9.3	9.0	8.7
.4	12.4	12.0	11.6
.5	15.5	15.0	14.5
.6	18.6	18.0	17.4
.7	21.7	21.0	20.3
.8	24.8	24.0	23.2
.9	27.9	27.0	26.1

	24	23	22
.1	2.4	2.3	2.2
.2	4.8	4.6	4.4
.3	7.2	6.9	6.6
.4	9.6	9.2	8.8
.5	12.0	11.5	11.0
.6	14.4	13.8	13.2
.7	16.8	16.1	15.4
.8	19.2	18.4	17.6
.9	21.6	20.7	19.8

	7	6
.1	0.7	0.6
.2	1.4	1.2
.3	2.1	1.8
.4	2.8	2.4
.5	3.5	3.0
.6	4.2	3.6
.7	4.9	4.2
.8	5.6	4.8
.9	6.3	5.4

Prop. Pts.

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.67 161	24	9.72 567	31	10.27 433	9.94 593	6	60
1	9.67 185	23	9.72 598	30	10.27 402	9.94 587	7	59
2	9.67 208	24	9.72 628	31	10.27 372	9.94 580	7	58
3	9.67 232	24	9.72 659	30	10.27 341	9.94 573	6	57
4	9.67 256	24	9.72 689	31	10.27 311	9.94 567	7	56
5	9.67 280	23	9.72 720	30	10.27 280	9.94 560	7	55
6	9.67 303	24	9.72 750	30	10.27 250	9.94 553	7	54
7	9.67 327	23	9.72 780	31	10.27 220	9.94 546	6	53
8	9.67 350	24	9.72 811	30	10.27 189	9.94 540	7	52
9	9.67 374	24	9.72 841	31	10.27 159	9.94 533	7	51
10	9.67 398	23	9.72 872	30	10.27 128	9.94 526	7	50
11	9.67 421	24	9.72 902	30	10.27 098	9.94 519	6	49
12	9.67 445	23	9.72 932	31	10.27 068	9.94 513	7	48
13	9.67 468	24	9.72 963	30	10.27 037	9.94 506	7	47
14	9.67 492	23	9.72 993	30	10.27 007	9.94 499	7	46
15	9.67 515	24	9.73 023	31	10.26 977	9.94 492	7	45
16	9.67 539	23	9.73 054	30	10.26 946	9.94 485	6	44
17	9.67 562	24	9.73 084	30	10.26 916	9.94 479	7	43
18	9.67 586	23	9.73 114	30	10.26 886	9.94 472	7	42
19	9.67 609	24	9.73 144	31	10.26 856	9.94 465	7	41
20	9.67 633	23	9.73 175	30	10.26 825	9.94 458	7	40
21	9.67 656	24	9.73 205	30	10.26 795	9.94 451	6	39
22	9.67 680	23	9.73 235	30	10.26 765	9.94 445	7	38
23	9.67 703	23	9.73 265	30	10.26 735	9.94 438	7	37
24	9.67 726	24	9.73 295	31	10.26 705	9.94 431	7	36
25	9.67 750	23	9.73 326	30	10.26 674	9.94 424	7	35
26	9.67 773	23	9.73 356	30	10.26 644	9.94 417	7	34
27	9.67 796	24	9.73 386	30	10.26 614	9.94 410	6	33
28	9.67 820	23	9.73 416	30	10.26 584	9.94 404	7	32
29	9.67 843	23	9.73 446	30	10.26 554	9.94 397	7	31
30	9.67 866	24	9.73 476	31	10.26 524	9.94 390	7	30
31	9.67 890	23	9.73 507	30	10.26 493	9.94 383	7	29
32	9.67 913	23	9.73 537	30	10.26 463	9.94 376	7	28
33	9.67 936	23	9.73 567	30	10.26 433	9.94 369	7	27
34	9.67 959	23	9.73 597	30	10.26 403	9.94 362	7	26
35	9.67 982	24	9.73 627	30	10.26 373	9.94 355	6	25
36	9.68 006	23	9.73 657	30	10.26 343	9.94 349	7	24
37	9.68 029	23	9.73 687	30	10.26 313	9.94 342	7	23
38	9.68 052	23	9.73 717	30	10.26 283	9.94 335	7	22
39	9.68 075	23	9.73 747	30	10.26 253	9.94 328	7	21
40	9.68 098	23	9.73 777	30	10.26 223	9.94 321	7	20
41	9.68 121	23	9.73 807	30	10.26 193	9.94 314	7	19
42	9.68 144	23	9.73 837	30	10.26 163	9.94 307	7	18
43	9.68 167	23	9.73 867	30	10.26 133	9.94 300	7	17
44	9.68 190	23	9.73 897	30	10.26 103	9.94 293	7	16
45	9.68 213	24	9.73 927	30	10.26 073	9.94 286	7	15
46	9.68 237	23	9.73 957	30	10.26 043	9.94 279	6	14
47	9.68 260	23	9.73 987	30	10.26 013	9.94 273	7	13
48	9.68 283	22	9.74 017	30	10.25 983	9.94 266	7	12
49	9.68 305	23	9.74 047	30	10.25 953	9.94 259	7	11
50	9.68 328	23	9.74 077	30	10.25 923	9.94 252	7	10
51	9.68 351	23	9.74 107	30	10.25 893	9.94 245	7	9
52	9.68 374	23	9.74 137	29	10.25 863	9.94 238	7	8
53	9.68 397	23	9.74 166	30	10.25 834	9.94 231	7	7
54	9.68 420	23	9.74 196	30	10.25 804	9.94 224	7	6
55	9.68 443	23	9.74 226	30	10.25 774	9.94 217	7	5
56	9.68 466	23	9.74 256	30	10.25 744	9.94 210	7	4
57	9.68 489	23	9.74 286	30	10.25 714	9.94 203	7	3
58	9.68 512	22	9.74 316	29	10.25 684	9.94 196	7	2
59	9.68 534	23	9.74 345	30	10.25 655	9.94 189	7	1
60	9.68 557		9.74 375		10.25 625	9.94 182		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

61°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	′
0	9.68 557	23	9.74 375	30	10.25 625	9.94 182	7	60
1	9.68 580	23	9.74 405	30	10.25 595	9.94 175	7	59
2	9.68 603	22	9.74 435	30	10.25 565	9.94 168	7	58
3	9.68 625	23	9.74 465	29	10.25 535	9.94 161	7	57
4	9.68 648	23	9.74 494	30	10.25 506	9.94 154	7	56
5	9.68 671	23	9.74 524	30	10.25 476	9.94 147	7	55
6	9.68 694	22	9.74 554	29	10.25 446	9.94 140	7	54
7	9.68 716	23	9.74 583	30	10.25 417	9.94 133	7	53
8	9.68 739	23	9.74 613	30	10.25 387	9.94 126	7	52
9	9.68 762	22	9.74 643	30	10.25 357	9.94 119	7	51
10	9.68 784	23	9.74 673	29	10.25 327	9.94 112	7	50
11	9.68 807	22	9.74 702	30	10.25 298	9.94 105	7	49
12	9.68 829	23	9.74 732	30	10.25 268	9.94 098	8	48
13	9.68 852	23	9.74 762	29	10.25 238	9.94 090	7	47
14	9.68 875	22	9.74 791	30	10.25 209	9.94 083	7	46
15	9.68 897	23	9.74 821	30	10.25 179	9.94 076	7	45
16	9.68 920	22	9.74 851	29	10.25 149	9.94 069	7	44
17	9.68 942	23	9.74 880	30	10.25 120	9.94 062	7	43
18	9.68 965	22	9.74 910	29	10.25 090	9.94 055	7	42
19	9.68 987	23	9.74 939	30	10.25 061	9.94 048	7	41
20	9.69 010	22	9.74 969	29	10.25 031	9.94 041	7	40
21	9.69 032	23	9.74 998	30	10.25 002	9.94 034	7	39
22	9.69 055	22	9.75 028	30	10.24 972	9.94 027	7	38
23	9.69 077	23	9.75 058	29	10.24 942	9.94 020	8	37
24	9.69 100	22	9.75 087	30	10.24 913	9.94 012	7	36
25	9.69 122	22	9.75 117	29	10.24 883	9.94 005	7	35
26	9.69 144	23	9.75 146	30	10.24 854	9.93 998	7	34
27	9.69 167	22	9.75 176	29	10.24 824	9.93 991	7	33
28	9.69 189	23	9.75 205	30	10.24 795	9.93 984	7	32
29	9.69 212	22	9.75 235	29	10.24 765	9.93 977	7	31
30	9.69 234	22	9.75 264	30	10.24 736	9.93 970	7	30
31	9.69 256	23	9.75 294	29	10.24 706	9.93 963	8	29
32	9.69 279	22	9.75 323	30	10.24 677	9.93 955	7	28
33	9.69 301	22	9.75 353	29	10.24 647	9.93 948	7	27
34	9.69 323	22	9.75 382	29	10.24 618	9.93 941	7	26
35	9.69 345	23	9.75 411	30	10.24 589	9.93 934	7	25
36	9.69 368	22	9.75 441	29	10.24 559	9.93 927	7	24
37	9.69 390	22	9.75 470	30	10.24 530	9.93 920	8	23
38	9.69 412	22	9.75 500	29	10.24 500	9.93 912	7	22
39	9.69 434	22	9.75 529	29	10.24 471	9.93 905	7	21
40	9.69 456	23	9.75 558	30	10.24 442	9.93 898	7	20
41	9.69 479	22	9.75 588	29	10.24 412	9.93 891	7	19
42	9.69 501	22	9.75 617	30	10.24 383	9.93 884	8	18
43	9.69 523	22	9.75 647	29	10.24 353	9.93 876	7	17
44	9.69 545	22	9.75 676	29	10.24 324	9.93 869	7	16
45	9.69 567	22	9.75 705	30	10.24 295	9.93 862	7	15
46	9.69 589	22	9.75 735	29	10.24 265	9.93 855	8	14
47	9.69 611	22	9.75 764	29	10.24 236	9.93 847	7	13
48	9.69 633	22	9.75 793	29	10.24 207	9.93 840	7	12
49	9.69 655	22	9.75 822	30	10.24 178	9.93 833	7	11
50	9.69 677	22	9.75 852	29	10.24 148	9.93 826	7	10
51	9.69 699	22	9.75 881	29	10.24 119	9.93 819	8	9
52	9.69 721	22	9.75 910	29	10.24 090	9.93 811	7	8
53	9.69 743	22	9.75 939	30	10.24 061	9.93 804	7	7
54	9.69 765	22	9.75 969	29	10.24 031	9.93 797	8	6
55	9.69 787	22	9.75 998	29	10.24 002	9.93 789	7	5
56	9.69 809	22	9.76 027	29	10.23 973	9.93 782	7	4
57	9.69 831	22	9.76 056	30	10.23 944	9.93 775	7	3
58	9.69 853	22	9.76 086	29	10.23 914	9.93 768	8	2
59	9.69 875	22	9.76 115	29	10.23 885	9.93 760	7	1
60	9.69 897		9.76 144		10.23 856	9.93 753		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

	30	29
.1	3.0	2.9
.2	6.0	5.8
.3	9.0	8.7
.4	12.0	11.6
.5	15.0	14.5
.6	18.0	17.4
.7	21.0	20.3
.8	24.0	23.2
.9	27.0	26.1

	23	22
.1	2.3	2.2
.2	4.6	4.4
.3	6.9	6.6
.4	9.2	8.8
.5	11.5	11.0
.6	13.8	13.2
.7	16.1	15.4
.8	18.4	17.6
.9	20.7	19.8

	8	7
.1	0.8	0.7
.2	1.6	1.4
.3	2.4	2.1
.4	3.2	2.8
.5	4.0	3.5
.6	4.8	4.2
.7	5.6	4.9
.8	6.4	5.6
.9	7.2	6.3

Prop. Pts.

Prop. Pts.

	30	29	28
.1	3.0	2.9	2.8
.2	6.0	5.8	5.6
.3	9.0	8.7	8.4
.4	12.0	11.6	11.2
.5	15.0	14.5	14.0
.6	18.0	17.4	16.8
.7	21.0	20.3	19.6
.8	24.0	23.2	22.4
.9	27.0	26.1	25.2

	22	21
.1	2.2	2.1
.2	4.4	4.2
.3	6.6	6.3
.4	8.8	8.4
.5	11.0	10.5
.6	13.2	12.6
.7	15.4	14.7
.8	17.6	16.8
.9	19.8	18.9

	8	7
.1	0.8	0.7
.2	1.6	1.4
.3	2.4	2.1
.4	3.2	2.8
.5	4.0	3.5
.6	4.8	4.2
.7	5.6	4.9
.8	6.4	5.6
.9	7.2	6.3

Prop. Pts.

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.69 897	22	9.76 144	29	10.23 856	9.93 753	7	60
1	9.69 919	22	9.76 173	29	10.23 827	9.93 746	8	59
2	9.69 941	22	9.76 202	29	10.23 798	9.93 738	7	58
3	9.69 963	21	9.76 231	30	10.23 769	9.93 731	7	57
4	9.69 984	22	9.76 261	29	10.23 739	9.93 724	7	56
5	9.70 006	22	9.76 290	29	10.23 710	9.93 717	8	55
6	9.70 028	22	9.76 319	29	10.23 681	9.93 709	7	54
7	9.70 050	22	9.76 348	29	10.23 652	9.93 702	7	53
8	9.70 072	21	9.76 377	29	10.23 623	9.93 695	8	52
9	9.70 093	22	9.76 406	29	10.23 594	9.93 687	7	51
10	9.70 115	22	9.76 435	29	10.23 565	9.93 680	7	50
11	9.70 137	22	9.76 464	29	10.23 536	9.93 673	8	49
12	9.70 159	21	9.76 493	29	10.23 507	9.93 665	7	48
13	9.70 180	22	9.76 522	29	10.23 478	9.93 658	8	47
14	9.70 202	22	9.76 551	29	10.23 449	9.93 650	7	46
15	9.70 224	21	9.76 580	29	10.23 420	9.93 643	7	45
16	9.70 245	22	9.76 609	30	10.23 391	9.93 636	8	44
17	9.70 267	21	9.76 639	29	10.23 361	9.93 628	7	43
18	9.70 288	22	9.76 668	29	10.23 332	9.93 621	7	42
19	9.70 310	22	9.76 697	28	10.23 303	9.93 614	8	41
20	9.70 332	21	9.76 725	29	10.23 275	9.93 606	7	40
21	9.70 353	22	9.76 754	29	10.23 246	9.93 599	8	39
22	9.70 375	21	9.76 783	29	10.23 217	9.93 591	7	38
23	9.70 396	22	9.76 812	29	10.23 188	9.93 584	7	37
24	9.70 418	21	9.76 841	29	10.23 159	9.93 577	8	36
25	9.70 439	22	9.76 870	29	10.23 130	9.93 569	7	35
26	9.70 461	21	9.76 899	29	10.23 101	9.93 562	8	34
27	9.70 482	22	9.76 928	29	10.23 072	9.93 554	7	33
28	9.70 504	21	9.76 957	29	10.23 043	9.93 547	8	32
29	9.70 525	22	9.76 986	29	10.23 014	9.93 539	7	31
30	9.70 547	21	9.77 015	29	10.22 985	9.93 532	7	30
31	9.70 568	22	9.77 044	29	10.22 956	9.93 525	8	29
32	9.70 590	21	9.77 073	28	10.22 927	9.93 517	7	28
33	9.70 611	22	9.77 101	29	10.22 899	9.93 510	8	27
34	9.70 633	21	9.77 130	29	10.22 870	9.93 502	7	26
35	9.70 654	21	9.77 159	29	10.22 841	9.93 495	8	25
36	9.70 675	22	9.77 188	29	10.22 812	9.93 487	7	24
37	9.70 697	21	9.77 217	29	10.22 783	9.93 480	8	23
38	9.70 718	21	9.77 246	28	10.22 754	9.93 472	7	22
39	9.70 739	22	9.77 274	29	10.22 726	9.93 465	8	21
40	9.70 761	21	9.77 303	29	10.22 697	9.93 457	7	20
41	9.70 782	21	9.77 332	29	10.22 668	9.93 450	8	19
42	9.70 803	21	9.77 361	29	10.22 639	9.93 442	7	18
43	9.70 824	22	9.77 390	28	10.22 610	9.93 435	8	17
44	9.70 846	21	9.77 418	29	10.22 582	9.93 427	7	16
45	9.70 867	21	9.77 447	29	10.22 553	9.93 420	8	15
46	9.70 888	21	9.77 476	29	10.22 524	9.93 412	7	14
47	9.70 909	22	9.77 505	28	10.22 495	9.93 405	8	13
48	9.70 931	21	9.77 533	29	10.22 467	9.93 397	7	12
49	9.70 952	21	9.77 562	29	10.22 438	9.93 390	8	11
50	9.70 973	21	9.77 591	28	10.22 409	9.93 382	7	10
51	9.70 994	21	9.77 619	29	10.22 381	9.93 375	8	9
52	9.71 015	21	9.77 648	29	10.22 352	9.93 367	7	8
53	9.71 036	22	9.77 677	29	10.22 323	9.93 360	8	7
54	9.71 058	21	9.77 706	28	10.22 294	9.93 352	8	6
55	9.71 079	21	9.77 734	29	10.22 266	9.93 344	7	5
56	9.71 100	21	9.77 763	28	10.22 237	9.93 337	8	4
57	9.71 121	21	9.77 791	29	10.22 209	9.93 329	7	3
58	9.71 142	21	9.77 820	29	10.22 180	9.93 322	8	2
59	9.71 163	21	9.77 849	28	10.22 151	9.93 314	7	1
60	9.71 184		9.77 877		10.22 123	9.93 307		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

'	L Sin	d	L Tan	c d	L Cot	L Cos	d	'
0	9.71 184	21	9.77 877	29	10.22 123	9.93 307	8	60
1	9.71 205	21	9.77 906	29	10.22 094	9.93 299	8	59
2	9.71 226	21	9.77 935	28	10.22 065	9.93 291	7	58
3	9.71 247	21	9.77 963	29	10.22 037	9.93 284	8	57
4	9.71 268	21	9.77 992	28	10.22 008	9.93 276	7	56
5	9.71 289	21	9.78 020	29	10.21 980	9.93 269	8	55
6	9.71 310	21	9.78 049	28	10.21 951	9.93 261	8	54
7	9.71 331	21	9.78 077	29	10.21 923	9.93 253	7	53
8	9.71 352	21	9.78 106	29	10.21 894	9.93 246	8	52
9	9.71 373	20	9.78 135	28	10.21 865	9.93 238	8	51
10	9.71 393	21	9.78 163	29	10.21 837	9.93 230	7	50
11	9.71 414	21	9.78 192	28	10.21 808	9.93 223	8	49
12	9.71 435	21	9.78 220	29	10.21 780	9.93 215	8	48
13	9.71 456	21	9.78 249	28	10.21 751	9.93 207	7	47
14	9.71 477	21	9.78 277	29	10.21 723	9.93 200	8	46
15	9.71 498	21	9.78 306	28	10.21 694	9.93 192	8	45
16	9.71 519	20	9.78 334	29	10.21 666	9.93 184	7	44
17	9.71 539	21	9.78 363	28	10.21 637	9.93 177	8	43
18	9.71 560	21	9.78 391	28	10.21 609	9.93 169	8	42
19	9.71 581	21	9.78 419	29	10.21 581	9.93 161	7	41
20	9.71 602	20	9.78 448	28	10.21 552	9.93 154	8	40
21	9.71 622	21	9.78 476	29	10.21 524	9.93 146	8	39
22	9.71 643	21	9.78 505	28	10.21 495	9.93 138	7	38
23	9.71 664	21	9.78 533	29	10.21 467	9.93 131	8	37
24	9.71 685	20	9.78 562	28	10.21 438	9.93 123	8	36
25	9.71 705	21	9.78 590	28	10.21 410	9.93 115	7	35
26	9.71 726	21	9.78 618	29	10.21 382	9.93 108	8	34
27	9.71 747	20	9.78 647	28	10.21 353	9.93 100	8	33
28	9.71 767	21	9.78 675	29	10.21 325	9.93 092	8	32
29	9.71 788	21	9.78 704	28	10.21 296	9.93 084	7	31
30	9.71 809	20	9.78 732	28	10.21 268	9.93 077	8	30
31	9.71 829	21	9.78 760	29	10.21 240	9.93 069	8	29
32	9.71 850	20	9.78 789	28	10.21 211	9.93 061	8	28
33	9.71 870	21	9.78 817	28	10.21 183	9.93 053	7	27
34	9.71 891	20	9.78 845	29	10.21 155	9.93 046	8	26
35	9.71 911	21	9.78 874	28	10.21 126	9.93 038	8	25
36	9.71 932	20	9.78 902	28	10.21 098	9.93 030	8	24
37	9.71 952	21	9.78 930	29	10.21 070	9.93 022	8	23
38	9.71 973	21	9.78 959	28	10.21 041	9.93 014	7	22
39	9.71 994	20	9.78 987	28	10.21 013	9.93 007	8	21
40	9.72 014	20	9.79 015	28	10.20 985	9.92 999	8	20
41	9.72 034	21	9.79 043	29	10.20 957	9.92 991	8	19
42	9.72 055	20	9.79 072	28	10.20 928	9.92 983	7	18
43	9.72 075	21	9.79 100	28	10.20 900	9.92 976	8	17
44	9.72 096	20	9.79 128	28	10.20 872	9.92 968	8	16
45	9.72 116	21	9.79 156	29	10.20 844	9.92 960	8	15
46	9.72 137	20	9.79 185	28	10.20 815	9.92 952	8	14
47	9.72 157	20	9.79 213	28	10.20 787	9.92 944	8	13
48	9.72 177	21	9.79 241	28	10.20 759	9.92 936	7	12
49	9.72 198	20	9.79 269	28	10.20 731	9.92 929	8	11
50	9.72 218	20	9.79 297	29	10.20 703	9.92 921	8	10
51	9.72 238	21	9.79 326	28	10.20 674	9.92 913	8	9
52	9.72 259	20	9.79 354	28	10.20 646	9.92 905	8	8
53	9.72 279	20	9.79 382	28	10.20 618	9.92 897	8	7
54	9.72 299	21	9.79 410	28	10.20 590	9.92 889	8	6
55	9.72 320	20	9.79 438	28	10.20 562	9.92 881	7	5
56	9.72 340	20	9.79 466	29	10.20 534	9.92 874	8	4
57	9.72 360	21	9.79 495	28	10.20 505	9.92 866	8	3
58	9.72 381	20	9.79 523	28	10.20 477	9.92 858	8	2
59	9.72 401	20	9.79 551	28	10.20 449	9.92 850	8	1
60	9.72 421		9.79 579		10.20 421	9.92 842		0
'	L Cos	d	L Cot	c d	L Tan	L Sin	d	'

Prop. Pts.

	29	28
.1	2.9	2.8
.2	5.8	5.6
.3	8.7	8.4
.4	11.6	11.2
.5	14.5	14.0
.6	17.4	16.8
.7	20.3	19.6
.8	23.2	22.4
.9	26.1	25.2

	21	20
.1	2.1	2.0
.2	4.2	4.0
.3	6.3	6.0
.4	8.4	8.0
.5	10.5	10.0
.6	12.6	12.0
.7	14.7	14.0
.8	16.8	16.0
.9	18.9	18.0

	8	7
.1	0.8	0.7
.2	1.6	1.4
.3	2.4	2.1
.4	3.2	2.8
.5	4.0	3.5
.6	4.8	4.2
.7	5.6	4.9
.8	6.4	5.6
.9	7.2	6.3

Prop. Pts.

Prop. Pts.

	29	28	27
.1	2.9	2.8	2.7
.2	5.8	5.6	5.4
.3	8.7	8.4	8.1
.4	11.6	11.2	10.8
.5	14.5	14.0	13.5
.6	17.4	16.8	16.2
.7	20.3	19.6	18.9
.8	23.2	22.4	21.6
.9	26.1	25.2	24.3

	21	20	19
.1	2.1	2.0	1.9
.2	4.2	4.0	3.8
.3	6.3	6.0	5.7
.4	8.4	8.0	7.6
.5	10.5	10.0	9.5
.6	12.6	12.0	11.4
.7	14.7	14.0	13.3
.8	16.8	16.0	15.2
.9	18.9	18.0	17.1

	9	8	7
.1	0.9	0.8	0.7
.2	1.8	1.6	1.4
.3	2.7	2.4	2.1
.4	3.6	3.2	2.8
.5	4.5	4.0	3.5
.6	5.4	4.8	4.2
.7	6.3	5.6	4.9
.8	7.2	6.4	5.6
.9	8.1	7.2	6.3

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.72 421	20	9.79 579	28	10.20 421	9.92 842	8	60
1	9.72 441	20	9.79 607	28	10.20 393	9.92 834	8	59
2	9.72 461	21	9.79 635	28	10.20 365	9.92 826	8	58
3	9.72 482	20	9.79 663	28	10.20 337	9.92 818	8	57
4	9.72 502	20	9.79 691	28	10.20 309	9.92 810	7	56
5	9.72 522	20	9.79 719	28	10.20 281	9.92 803	8	55
6	9.72 542	20	9.79 747	29	10.20 253	9.92 795	8	54
7	9.72 562	20	9.79 776	28	10.20 224	9.92 787	8	53
8	9.72 582	20	9.79 804	28	10.20 196	9.92 779	8	52
9	9.72 602	20	9.79 832	28	10.20 168	9.92 771	8	51
10	9.72 622	21	9.79 860	28	10.20 140	9.92 763	8	50
11	9.72 643	20	9.79 888	28	10.20 112	9.92 755	8	49
12	9.72 663	20	9.79 916	28	10.20 084	9.92 747	8	48
13	9.72 683	20	9.79 944	28	10.20 056	9.92 739	8	47
14	9.72 703	20	9.79 972	28	10.20 028	9.92 731	8	46
15	9.72 723	20	9.80 000	28	10.20 000	9.92 723	8	45
16	9.72 743	20	9.80 028	28	10.19 972	9.92 715	8	44
17	9.72 763	20	9.80 056	28	10.19 944	9.92 707	8	43
18	9.72 783	20	9.80 084	28	10.19 916	9.92 699	8	42
19	9.72 803	20	9.80 112	28	10.19 888	9.92 691	8	41
20	9.72 823	20	9.80 140	28	10.19 860	9.92 683	8	40
21	9.72 843	20	9.80 168	27	10.19 832	9.92 675	8	39
22	9.72 863	20	9.80 195	28	10.19 805	9.92 667	8	38
23	9.72 883	19	9.80 223	28	10.19 777	9.92 659	8	37
24	9.72 902	20	9.80 251	28	10.19 749	9.92 651	8	36
25	9.72 922	20	9.80 279	28	10.19 721	9.92 643	8	35
26	9.72 942	20	9.80 307	28	10.19 693	9.92 635	8	34
27	9.72 962	20	9.80 335	28	10.19 665	9.92 627	8	33
28	9.72 982	20	9.80 363	28	10.19 637	9.92 619	8	32
29	9.73 002	20	9.80 391	28	10.19 609	9.92 611	8	31
30	9.73 022	19	9.80 419	28	10.19 581	9.92 603	8	30
31	9.73 041	20	9.80 447	27	10.19 553	9.92 595	8	29
32	9.73 061	20	9.80 474	28	10.19 526	9.92 587	8	28
33	9.73 081	20	9.80 502	28	10.19 498	9.92 579	8	27
34	9.73 101	20	9.80 530	28	10.19 470	9.92 571	8	26
35	9.73 121	19	9.80 558	28	10.19 442	9.92 563	8	25
36	9.73 140	20	9.80 586	28	10.19 414	9.92 555	9	24
37	9.73 160	20	9.80 614	28	10.19 386	9.92 546	8	23
38	9.73 180	20	9.80 642	27	10.19 358	9.92 538	8	22
39	9.73 200	19	9.80 669	28	10.19 331	9.92 530	8	21
40	9.73 219	20	9.80 697	28	10.19 303	9.92 522	8	20
41	9.73 239	20	9.80 725	28	10.19 275	9.92 514	8	19
42	9.73 259	19	9.80 753	28	10.19 247	9.92 506	8	18
43	9.73 278	20	9.80 781	27	10.19 219	9.92 498	8	17
44	9.73 298	20	9.80 808	28	10.19 192	9.92 490	8	16
45	9.73 318	19	9.80 836	28	10.19 164	9.92 482	9	15
46	9.73 337	20	9.80 864	28	10.19 136	9.92 473	8	14
47	9.73 357	20	9.80 892	27	10.19 108	9.92 465	8	13
48	9.73 377	19	9.80 919	28	10.19 081	9.92 457	8	12
49	9.73 396	20	9.80 947	28	10.19 053	9.92 449	8	11
50	9.73 416	19	9.80 975	28	10.19 025	9.92 441	8	10
51	9.73 435	20	9.81 003	27	10.18 997	9.92 433	8	9
52	9.73 455	19	9.81 030	28	10.18 970	9.92 425	9	8
53	9.73 474	20	9.81 058	28	10.18 942	9.92 416	8	7
54	9.73 494	19	9.81 086	27	10.18 914	9.92 408	8	6
55	9.73 513	20	9.81 113	28	10.18 887	9.92 400	8	5
56	9.73 533	19	9.81 141	28	10.18 859	9.92 392	8	4
57	9.73 552	20	9.81 169	27	10.18 831	9.92 384	8	3
58	9.73 572	19	9.81 196	28	10.18 804	9.92 376	9	2
59	9.73 591	20	9.81 224	28	10.18 776	9.92 367	8	1
60	9.73 611		9.81 252		10.18 748	9.92 359		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

57°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.73 611	19	9.81 252	27	10.18 748	9.92 359	8	60
1	9.73 630	20	9.81 279	28	10.18 721	9.92 351	8	59
2	9.73 650	19	9.81 307	28	10.18 693	9.92 343	8	58
3	9.73 669	20	9.81 335	27	10.18 665	9.92 335	9	57
4	9.73 689	19	9.81 362	28	10.18 638	9.92 326	8	56
5	9.73 708	19	9.81 390	28	10.18 610	9.92 318	8	55
6	9.73 727	20	9.81 418	27	10.18 582	9.92 310	8	54
7	9.73 747	19	9.81 445	28	10.18 555	9.92 302	9	53
8	9.73 766	19	9.81 473	27	10.18 527	9.92 293	8	52
9	9.73 785	20	9.81 500	28	10.18 500	9.92 285	8	51
10	9.73 805	19	9.81 528	28	10.18 472	9.92 277	8	50
11	9.73 824	19	9.81 556	27	10.18 444	9.92 269	9	49
12	9.73 843	20	9.81 583	28	10.18 417	9.92 260	8	48
13	9.73 863	19	9.81 611	27	10.18 389	9.92 252	8	47
14	9.73 882	19	9.81 638	28	10.18 362	9.92 244	9	46
15	9.73 901	20	9.81 666	27	10.18 334	9.92 235	8	45
16	9.73 921	19	9.81 693	28	10.18 307	9.92 227	8	44
17	9.73 940	19	9.81 721	27	10.18 279	9.92 219	8	43
18	9.73 959	19	9.81 748	28	10.18 252	9.92 211	9	42
19	9.73 978	19	9.81 776	27	10.18 224	9.92 202	8	41
20	9.73 997	20	9.81 803	28	10.18 197	9.92 194	8	40
21	9.74 017	19	9.81 831	27	10.18 169	9.92 186	9	39
22	9.74 036	19	9.81 858	28	10.18 142	9.92 177	8	38
23	9.74 055	19	9.81 886	27	10.18 114	9.92 169	8	37
24	9.74 074	19	9.81 913	28	10.18 087	9.92 161	9	36
25	9.74 093	20	9.81 941	27	10.18 059	9.92 152	8	35
26	9.74 113	19	9.81 968	28	10.18 032	9.92 144	8	34
27	9.74 132	19	9.81 996	27	10.18 004	9.92 136	9	33
28	9.74 151	19	9.82 023	28	10.17 977	9.92 127	8	32
29	9.74 170	19	9.82 051	27	10.17 949	9.92 119	8	31
30	9.74 189	19	9.82 078	28	10.17 922	9.92 111	9	30
31	9.74 208	19	9.82 106	27	10.17 894	9.92 102	8	29
32	9.74 227	19	9.82 133	28	10.17 867	9.92 094	8	28
33	9.74 246	19	9.82 161	27	10.17 839	9.92 086	9	27
34	9.74 265	19	9.82 188	27	10.17 812	9.92 077	8	26
35	9.74 284	19	9.82 215	28	10.17 785	9.92 069	9	25
36	9.74 303	19	9.82 243	27	10.17 757	9.92 060	8	24
37	9.74 322	19	9.82 270	28	10.17 730	9.92 052	8	23
38	9.74 341	19	9.82 298	27	10.17 702	9.92 044	9	22
39	9.74 360	19	9.82 325	27	10.17 675	9.92 035	8	21
40	9.74 379	19	9.82 352	28	10.17 648	9.92 027	9	20
41	9.74 398	19	9.82 380	27	10.17 620	9.92 018	8	19
42	9.74 417	19	9.82 407	28	10.17 593	9.92 010	8	18
43	9.74 436	19	9.82.435	27	10.17 565	9.92 002	9	17
44	9.74 455	19	9.82 462	27	10.17 538	9.91 993	8	16
45	9.74 474	19	9.82 489	28	10.17 511	9.91 985	9	15
46	9.74 493	19	9.82 517	27	10.17 483	9.91 976	8	14
47	9.74 512	19	9.82 544	27	10.17 456	9.91 968	9	13
48	9.74 531	18	9.82 571	28	10.17 429	9.91 959	8	12
49	9.74 549	19	9.82 599	27	10.17 401	9.91 951	9	11
50	9.74 568	19	9.82 626	27	10.17 374	9.91 942	8	10
51	9.74 587	19	9.82 653	28	10.17 347	9.91 934	9	9
52	9.74 606	19	9.82 681	27	10.17 319	9.91 925	8	8
53	9.74 625	19	9.82 708	27	10.17 292	9.91 917	9	7
54	9.74 644	18	9.82 735	27	10.17 265	9.91 908	8	6
55	9.74 662	19	9.82 762	28	10.17 238	9.91 900	9	5
56	9.74 681	19	9.82 790	27	10.17 210	9.91 891	8	4
57	9.74 700	19	9.82 817	27	10.17 183	9.91 883	9	3
58	9.74 719	18	9.82 844	27	10.17 156	9.91 874	8	2
59	9.74 737	19	9.82 871	28	10.17 129	9.91 866	9	1
60	9.74 756		9.82 899		10.17 101	9.91 857		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

	28	27	20
.1	2.8	2.7	2.0
.2	5.6	5.4	4.0
.3	8.4	8.1	6.0
.4	11.2	10.8	8.0
.5	14.0	13.5	10.0
.6	16.8	16.2	12.0
.7	19.6	18.9	14.0
.8	22.4	21.6	16.0
.9	25.2	24.3	18.0

	19	18
.1	1.9	1.8
.2	3.8	3.6
.3	5.7	5.4
.4	7.6	7.2
.5	9.5	9.0
.6	11.4	10.8
.7	13.3	12.6
.8	15.2	14.4
.9	17.1	16.2

	9	8
.1	0.9	0.8
.2	1.8	1.6
.3	2.7	2.4
.4	3.6	3.2
.5	4.5	4.0
.6	5.4	4.8
.7	6.3	5.6
.8	7.2	6.4
.9	8.1	7.2

Prop. Pts.

Prop. Pts.

	28	27	26
.1	2.8	2.7	2.6
.2	5.6	5.4	5.2
.3	8.4	8.1	7.8
.4	11.2	10.8	10.4
.5	14.0	13.5	13.0
.6	16.8	16.2	15.6
.7	19.6	18.9	18.2
.8	22.4	21.6	20.8
.9	25.2	24.3	23.4

	19	18
.1	1.9	1.8
.2	3.8	3.6
.3	5.7	5.4
.4	7.6	7.2
.5	9.5	9.0
.6	11.4	10.8
.7	13.3	12.6
.8	15.2	14.4
.9	17.1	16.2

	9	8
.1	0.9	0.8
.2	1.8	1.6
.3	2.7	2.4
.4	3.6	3.2
.5	4.5	4.0
.6	5.4	4.8
.7	6.3	5.6
.8	7.2	6.4
.9	8.1	7.2

Prop. Pts.

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.74 756	19	9.82 899	27	10.17 101	9.91 857	8	60
1	9.74 775	19	9.82 926	27	10.17 074	9.91 849	9	59
2	9.74 794	18	9.82 953	27	10.17 047	9.91 840	8	58
3	9.74 812	19	9.82 980	28	10.17 020	9.91 832	9	57
4	9.74 831	19	9.83 008	27	10.16 992	9.91 823	8	56
5	9.74 850	18	9.83 035	27	10.16 965	9.91 815	9	55
6	9.74 868	19	9.83 062	27	10.16 938	9.91 806	8	54
7	9.74 887	19	9.83 089	28	10.16 911	9.91 798	9	53
8	9.74 906	18	9.83 117	27	10.16 883	9.91 789	8	52
9	9.74 924	19	9.83 144	27	10.16 856	9.91 781	9	51
10	9.74 943	18	9.83 171	27	10.16 829	9.91 772	9	50
11	9.74 961	19	9.83 198	27	10.16 802	9.91 763	8	49
12	9.74 980	19	9.83 225	27	10.16 775	9.91 755	9	48
13	9.74 999	18	9.83 252	28	10.16 748	9.91 746	8	47
14	9.75 017	19	9.83 280	27	10.16 720	9.91 738	9	46
15	9.75 036	18	9.83 307	27	10.16 693	9.91 729	9	45
16	9.75 054	19	9.83 334	27	10.16 666	9.91 720	8	44
17	9.75 073	18	9.83 361	27	10.16 639	9.91 712	9	43
18	9.75 091	19	9.83 388	27	10.16 612	9.91 703	8	42
19	9.75 110	18	9.83 415	27	10.16 585	9.91 695	9	41
20	9.75 128	19	9.83 442	28	10.16 558	9.91 686	9	40
21	9.75 147	18	9.83 470	27	10.16 530	9.91 677	8	39
22	9.75 165	19	9.83 497	27	10.16 503	9.91 669	9	38
23	9.75 184	18	9.83 524	27	10.16 476	9.91 660	9	37
24	9.75 202	19	9.83 551	27	10.16 449	9.91 651	8	36
25	9.75 221	18	9.83 578	27	10.16 422	9.91 643	9	35
26	9.75 239	19	9.83 605	27	10.16 395	9.91 634	9	34
27	9.75 258	18	9.83 632	27	10.16 368	9.91 625	8	33
28	9.75 276	18	9.83 659	27	10.16 341	9.91 617	9	32
29	9.75 294	19	9.83 686	27	10.16 314	9.91 608	9	31
30	9.75 313	18	9.83 713	27	10.16 287	9.91 599	8	30
31	9.75 331	19	9.83 740	28	10.16 260	9.91 591	9	29
32	9.75 350	18	9.83 768	27	10.16 232	9.91 582	9	28
33	9.75 368	18	9.83 795	27	10.16 205	9.91 573	8	27
34	9.75 386	19	9.83 822	27	10.16 178	9.91 565	9	26
35	9.75 405	18	9.83 849	27	10.16 151	9.91 556	9	25
36	9.75 423	18	9.83 876	27	10.16 124	9.91 547	9	24
37	9.75 441	18	9.83 903	27	10.16 097	9.91 538	8	23
38	9.75 459	19	9.83 930	27	10.16 070	9.91 530	9	22
39	9.75 478	18	9.83 957	27	10.16 043	9.91 521	9	21
40	9.75 496	18	9.83 984	27	10.16 016	9.91 512	8	20
41	9.75 514	19	9.84 011	27	10.15 989	9.91 504	9	19
42	9.75 533	18	9.84 038	27	10.15 962	9.91 495	9	18
43	9.75 551	18	9.84 065	27	10.15 935	9.91 486	9	17
44	9.75 569	18	9.84 092	27	10.15 908	9.91 477	8	16
45	9.75 587	18	9.84 119	27	10.15 881	9.91 469	9	15
46	9.75 605	19	9.84 146	27	10.15 854	9.91 460	9	14
47	9.75 624	18	9.84 173	27	10.15 827	9.91 451	9	13
48	9.75 642	18	9.84 200	27	10.15 800	9.91 442	9	12
49	9.75 660	18	9.84 227	27	10.15 773	9.91 433	8	11
50	9.75 678	18	9.84 254	26	10.15 746	9.91 425	9	10
51	9.75 696	18	9.84 280	27	10.15 720	9.91 416	9	9
52	9.75 714	19	9.84 307	27	10.15 693	9.91 407	9	8
53	9.75 733	18	9.84 334	27	10.15 666	9.91 398	9	7
54	9.75 751	18	9.84 361	27	10.15 639	9.91 389	8	6
55	9.75 769	18	9.84 388	27	10.15 612	9.91 381	9	5
56	9.75 787	18	9.84 415	27	10.15 585	9.91 372	9	4
57	9.75 805	18	9.84 442	27	10.15 558	9.91 363	9	3
58	9.75 823	18	9.84 469	27	10.15 531	9.91 354	9	2
59	9.75 841	18	9.84 496	27	10.15 504	9.91 345	9	1
60	9.75 859		9.84 523		10.15 477	9.91 336		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.75 859	18	9.84 523	27	10.15 477	9.91 336	8	60
1	9.75 877	18	9.84 550	26	10.15 450	9.91 328	9	59
2	9.75 895	18	9.84 576	27	10.15 424	9.91 319	9	58
3	9.75 913	18	9.84 603	27	10.15 397	9.91 310	9	57
4	9.75 931	18	9.84 630	27	10.15 370	9.91 301	9	56
5	9.75 949	18	9.84 657	27	10.15 343	9.91 292	9	55
6	9.75 967	18	9.84 684	27	10.15 316	9.91 283	9	54
7	9.75 985	18	9.84 711	27	10.15 289	9.91 274	8	53
8	9.76 003	18	9.84 738	26	10.15 262	9.91 266	9	52
9	9.76 021	18	9.84 764	27	10.15 236	9.91 257	9	51
10	9.76 039	18	9.84 791	27	10.15 209	9.91 248	9	50
11	9.76 057	18	9.84 818	27	10.15 182	9.91 239	9	49
12	9.76 075	18	9.84 845	27	10.15 155	9.91 230	9	48
13	9.76 093	18	9.84 872	27	10.15 128	9.91 221	9	47
14	9.76 111	18	9.84 899	26	10.15 101	9.91 212	9	46
15	9.76 129	17	9.84 925	27	10.15 075	9.91 203	9	45
16	9.76 146	18	9.84 952	27	10.15 048	9.91 194	9	44
17	9.76 164	18	9.84 979	27	10.15 021	9.91 185	9	43
18	9.76 182	18	9.85 006	27	10.14 994	9.91 176	9	42
19	9.76 200	18	9.85 033	26	10.14 967	9.91 167	9	41
20	9.76 218	18	9.85 059	27	10.14 941	9.91 158	9	40
21	9.76 236	17	9.85 086	27	10.14 914	9.91 149	8	39
22	9.76 253	18	9.85 113	27	10.14 887	9.91 141	9	38
23	9.76 271	18	9.85 140	26	10.14 860	9.91 132	9	37
24	9.76 289	18	9.85 166	27	10.14 834	9.91 123	9	36
25	9.76 307	17	9.85 193	27	10.14 807	9.91 114	9	35
26	9.76 324	18	9.85 220	27	10.14 780	9.91 105	9	34
27	9.76 342	18	9.85 247	26	10.14 753	9.91 096	9	33
28	9.76 360	18	9.85 273	27	10.14 727	9.91 087	9	32
29	9.76 378	17	9.85 300	27	10.14 700	9.91 078	9	31
30	9.76 395	18	9.85 327	27	10.14 673	9.91 069	9	30
31	9.76 413	18	9.85 354	26	10.14 646	9.91 060	9	29
32	9.76 431	17	9.85 380	27	10.14 620	9.91 051	9	28
33	9.76 448	18	9.85 407	27	10.14 593	9.91 042	9	27
34	9.76 466	18	9.85 434	26	10.14 566	9.91 033	10	26
35	9.76 484	17	9.85 460	27	10.14 540	9.91 023	9	25
36	9.76 501	18	9.85 487	27	10.14 513	9.91 014	9	24
37	9.76 519	18	9.85 514	26	10.14 486	9.91 005	9	23
38	9.76 537	17	9.85 540	27	10.14 460	9.90 996	9	22
39	9.76 554	18	9.85 567	27	10.14 433	9.90 987	9	21
40	9.76 572	18	9.85 594	26	10.14 406	9.90 978	9	20
41	9.76 590	17	9.85 620	27	10.14 380	9.90 969	9	19
42	9.76 607	18	9.85 647	27	10.14 353	9.90 960	9	18
43	9.76 625	17	9.85 674	26	10.14 326	9.90 951	9	17
44	9.76 642	18	9.85 700	27	10.14 300	9.90 942	9	16
45	9.76 660	17	9.85 727	27	10.14 273	9.90 933	9	15
46	9.76 677	18	9.85 754	26	10.14 246	9.90 924	9	14
47	9.76 695	17	9.85 780	27	10.14 220	9.90 915	9	13
48	9.76 712	18	9.85 807	27	10.14 193	9.90 906	10	12
49	9.76 730	17	9.85 834	26	10.14 166	9.90 896	9	11
50	9.76 747	18	9.85 860	27	10.14 140	9.90 887	9	10
51	9.76 765	17	9.85 887	26	10.14 113	9.90 878	9	9
52	9.76 782	18	9.85 913	27	10.14 087	9.90 869	9	8
53	9.76 800	17	9.85 940	27	10.14 060	9.90 860	9	7
54	9.76 817	18	9.85 967	26	10.14 033	9.90 851	9	6
55	9.76 835	17	9.85 993	27	10.14 007	9.90 842	10	5
56	9.76 852	18	9.86 020	26	10.13 980	9.90 832	9	4
57	9.76 870	17	9.86 046	27	10.13 954	9.90 823	9	3
58	9.76 887	17	9.86 073	27	10.13 927	9.90 814	9	2
59	9.76 904	18	9.86 100	26	10.13 900	9.90 805	9	1
60	9.76 922		9.86 126		10.13 874	9.90 796		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

	27	26	18
.1	2.7	2.6	1.8
.2	5.4	5.2	3.6
.3	8.1	7.8	5.4
.4	10.8	10.4	7.2
.5	13.5	13.0	9.0
.6	16.2	15.6	10.8
.7	18.9	18.2	12.6
.8	21.6	20.8	14.4
.9	24.3	23.4	16.2

	17	10
.1	1.7	1.0
.2	3.4	2.0
.3	5.1	3.0
.4	6.8	4.0
.5	8.5	5.0
.6	10.2	6.0
.7	11.9	7.0
.8	13.6	8.0
.9	15.3	9.0

	9	8
.1	0.9	0.8
.2	1.8	1.6
.3	2.7	2.4
.4	3.6	3.2
.5	4.5	4.0
.6	5.4	4.8
.7	6.3	5.6
.8	7.2	6.4
.9	8.1	7.2

Prop. Pts.

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.76 922	17	9.86 126	27	10.13 874	9.90 796	9	60
1	9.76 939	18	9.86 153	26	10.13 847	9.90 787	10	59
2	9.76 957	17	9.86 179	27	10.13 821	9.90 777	9	58
3	9.76 974	17	9.86 206	26	10.13 794	9.90 768	9	57
4	9.76 991	18	9.86 232	27	10.13 768	9.90 759	9	56
5	9.77 009	17	9.86 259	26	10.13 741	9.90 750	9	55
6	9.77 026	17	9.86 285	27	10.13 715	9.90 741	10	54
7	9.77 043	18	9.86 312	26	10.13 688	9.90 731	9	53
8	9.77 061	17	9.86 338	27	10.13 662	9.90 722	9	52
9	9.77 078	17	9.86 365	27	10.13 635	9.90 713	9	51
10	9.77 095	17	9.86 392	26	10.13 608	9.90 704	10	50
11	9.77 112	18	9.86 418	27	10.13 582	9.90 694	9	49
12	9.77 130	17	9.86 445	26	10.13 555	9.90 685	9	48
13	9.77 147	17	9.86 471	27	10.13 529	9.90 676	9	47
14	9.77 164	17	9.86 498	26	10.13 502	9.90 667	10	46
15	9.77 181	18	9.86 524	27	10.13 476	9.90 657	9	45
16	9.77 199	17	9.86 551	26	10.13 449	9.90 648	9	44
17	9.77 216	17	9.86 577	26	10.13 423	9.90 639	9	43
18	9.77 233	17	9.86 603	27	10.13 397	9.90 630	10	42
19	9.77 250	18	9.86 630	26	10.13 370	9.90 620	9	41
20	9.77 268	17	9.86 656	27	10.13 344	9.90 611	9	40
21	9.77 285	17	9.86 683	26	10.13 317	9.90 602	10	39
22	9.77 302	17	9.86 709	27	10.13 291	9.90 592	9	38
23	9.77 319	17	9.86 736	26	10.13 264	9.90 583	9	37
24	9.77 336	17	9.86 762	27	10.13 238	9.90 574	9	36
25	9.77 353	17	9.86 789	26	10.13 211	9.90 565	10	35
26	9.77 370	17	9.86 815	27	10.13 185	9.90 555	9	34
27	9.77 387	18	9.86 842	26	10.13 158	9.90 546	9	33
28	9.77 405	17	9.86 868	26	10.13 132	9.90 537	10	32
29	9.77 422	17	9.86 894	27	10.13 106	9.90 527	9	31
30	9.77 439	17	9.86 921	26	10.13 079	9.90 518	9	30
31	9.77 456	17	9.86 947	27	10.13 053	9.90 509	10	29
32	9.77 473	17	9.86 974	26	10.13 026	9.90 499	9	28
33	9.77 490	17	9.87 000	27	10.13 000	9.90 490	10	27
34	9.77 507	17	9.87 027	26	10.12 973	9.90 480	9	26
35	9.77 524	17	9.87 053	26	10.12 947	9.90 471	9	25
36	9.77 541	17	9.87 079	27	10.12 921	9.90 462	10	24
37	9.77 558	17	9.87 106	26	10.12 894	9.90 452	9	23
38	9.77 575	17	9.87 132	26	10.12 868	9.90 443	9	22
39	9.77 592	17	9.87 158	27	10.12 842	9.90 434	10	21
40	9.77 609	17	9.87 185	26	10.12 815	9.90 424	9	20
41	9.77 626	17	9.87 211	27	10.12 789	9.90 415	10	19
42	9.77 643	17	9.87 238	26	10.12 762	9.90 405	9	18
43	9.77 660	17	9.87 264	26	10.12 736	9.90 396	10	17
44	9.77 677	17	9.87 290	27	10.12 710	9.90 386	9	16
45	9.77 694	17	9.87 317	26	10.12 683	9.90 377	9	15
46	9.77 711	17	9.87 343	26	10.12 657	9.90 368	10	14
47	9.77 728	16	9.87 369	27	10.12 631	9.90 358	9	13
48	9.77 744	17	9.87 396	26	10.12 604	9.90 349	10	12
49	9.77 761	17	9.87 422	26	10.12 578	9.90 339	9	11
50	9.77 778	17	9.87 448	27	10.12 552	9.90 330	10	10
51	9.77 795	17	9.87 475	26	10.12 525	9.90 320	9	9
52	9.77 812	17	9.87 501	26	10.12 499	9.90 311	10	8
53	9.77 829	17	9.87 527	27	10.12 473	9.90 301	9	7
54	9.77 846	16	9.87 554	26	10.12 446	9.90 292	10	6
55	9.77 862	17	9.87 580	26	10.12 420	9.90 282	9	5
56	9.77 879	17	9.87 606	27	10.12 394	9.90 273	10	4
57	9.77 896	17	9.87 633	26	10.12 367	9.90 263	9	3
58	9.77 913	17	9.87 659	26	10.12 341	9.90 254	10	2
59	9.77 930	16	9.87 685	26	10.12 315	9.90 244	9	1
60	9.77 946		9.87 711		10.12 289	9.90 235		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

	27	26	18
.1	2.7	2.6	1.8
.2	5.4	5.2	3.6
.3	8.1	7.8	5.4
.4	10.8	10.4	7.2
.5	13.5	13.0	9.0
.6	16.2	15.6	10.8
.7	18.9	18.2	12.6
.8	21.6	20.8	14.4
.9	24.3	23.4	16.2

	17	16
.1	1.7	1.6
.2	3.4	3.2
.3	5.1	4.8
.4	6.8	6.4
.5	8.5	8.0
.6	10.2	9.6
.7	11.9	11.2
.8	13.6	12.8
.9	15.3	14.4

	10	9
.1	1.0	0.9
.2	2.0	1.8
.3	3.0	2.7
.4	4.0	3.6
.5	5.0	4.5
.6	6.0	5.4
.7	7.0	6.3
.8	8.0	7.2
.9	9.0	8.1

Prop. Pts.

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	′
0	9.77 946	17	9.87 711	27	10.12 289	9.90 235	10	60
1	9.77 963	17	9.87 738	26	10.12 262	9.90 225	9	59
2	9.77 980	17	9.87 764	26	10.12 236	9.90 216	10	58
3	9.77 997	16	9.87 790	27	10.12 210	9.90 206	9	57
4	9.78 013	17	9.87 817	26	10.12 183	9.90 197	10	56
5	9.78 030	17	9.87 843	26	10.12 157	9.90 187	9	55
6	9.78 047	16	9.87 869	26	10.12 131	9.90 178	10	54
7	9.78 063	17	9.87 895	27	10.12 105	9.90 168	9	53
8	9.78 080	17	9.87 922	26	10.12 078	9.90 159	10	52
9	9.78 097	16	9.87 948	26	10.12 052	9.90 149	10	51
10	9.78 113	17	9.87 974	26	10.12 026	9.90 139	9	50
11	9.78 130	17	9.88 000	27	10.12 000	9.90 130	10	49
12	9.78 147	16	9.88 027	26	10.11 973	9.90 120	9	48
13	9.78 163	17	9.88 053	26	10.11 947	9.90 111	10	47
14	9.78 180	17	9.88 079	26	10.11 921	9.90 101	10	46
15	9.78 197	16	9.88 105	26	10.11 895	9.90 091	9	45
16	9.78 213	17	9.88 131	27	10.11 869	9.90 082	10	44
17	9.78 230	16	9.88 158	26	10.11 842	9.90 072	9	43
18	9.78 246	17	9.88 184	26	10.11 816	9.90 063	10	42
19	9.78 263	17	9.88 210	26	10.11 790	9.90 053	10	41
20	9.78 280	16	9.88 236	26	10.11 764	9.90 043	9	40
21	9.78 296	17	9.88 262	27	10.11 738	9.90 034	10	39
22	9.78 313	16	9.88 289	26	10.11 711	9.90 024	10	38
23	9.78 329	17	9.88 315	26	10.11 685	9.90 014	9	37
24	9.78 346	16	9.88 341	26	10.11 659	9.90 005	10	36
25	9.78 362	17	9.88 367	26	10.11 633	9.89 995	10	35
26	9.78 379	16	9.88 393	27	10.11 607	9.89 985	9	34
27	9.78 395	17	9.88 420	26	10.11 580	9.89 976	10	33
28	9.78 412	16	9.88 446	26	10.11 554	9.89 966	10	32
29	9.78 428	17	9.88 472	26	10.11 528	9.89 956	9	31
30	9.78 445	16	9.88 498	26	10.11 502	9.89 947	10	30
31	9.78 461	17	9.88 524	26	10.11 476	9.89 937	10	29
32	9.78 478	16	9.88 550	27	10.11 450	9.89 927	9	28
33	9.78 494	16	9.88 577	26	10.11 423	9.89 918	10	27
34	9.78 510	17	9.88 603	26	10.11 397	9.89 908	10	26
35	9.78 527	16	9.88 629	26	10.11 371	9.89 898	10	25
36	9.78 543	17	9.88 655	26	10.11 345	9.89 888	9	24
37	9.78 560	16	9.88 681	26	10.11 319	9.89 879	10	23
38	9.78 576	16	9.88 707	26	10.11 293	9.89 869	10	22
39	9.78 592	17	9.88 733	26	10.11 267	9.89 859	10	21
40	9.78 609	16	9.88 759	27	10.11 241	9.89 849	9	20
41	9.78 625	17	9.88 786	26	10.11 214	9.89 840	10	19
42	9.78 642	16	9.88 812	26	10.11 188	9.89 830	10	18
43	9.78 658	16	9.88 838	26	10.11 162	9.89 820	10	17
44	9.78 674	17	9.88 864	26	10.11 136	9.89 810	9	16
45	9.78 691	16	9.88 890	26	10.11 110	9.89 801	10	15
46	9.78 707	16	9.88 916	26	10.11 084	9.89 791	10	14
47	9.78 723	16	9.88 942	26	10.11 058	9.89 781	10	13
48	9.78 739	17	9.88 968	26	10.11 032	9.89 771	10	12
49	9.78 756	16	9.88 994	26	10.11 006	9.89 761	9	11
50	9.78 772	16	9.89 020	26	10.10 980	9.89 752	10	10
51	9.78 788	17	9.89 046	27	10.10 954	9.89 742	10	9
52	9.78 805	16	9.89 073	26	10.10 927	9.89 732	10	8
53	9.78 821	16	9.89 099	26	10.10 901	9.89 722	10	7
54	9.78 837	16	9.89 125	26	10.10 875	9.89 712	10	6
55	9.78 853	16	9.89 151	26	10.10 849	9.89 702	9	5
56	9.78 869	17	9.89 177	26	10.10 823	9.89 693	10	4
57	9.78 886	16	9.89 203	26	10.10 797	9.89 683	10	3
58	9.78 902	16	9.89 229	26	10.10 771	9.89 673	10	2
59	9.78 918	16	9.89 255	26	10.10 745	9.89 663	10	1
60	9.78 934		9.89 281		10.10 719	9.89 653		0
′	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

	27	26	17
.1	2.7	2.6	1.7
.2	5.4	5.2	3.4
.3	8.1	7.8	5.1
.4	10.8	10.4	6.8
.5	13.5	13.0	8.5
.6	16.2	15.6	10.2
.7	18.9	18.2	11.9
.8	21.6	20.8	13.6
.9	24.3	23.4	15.3

	16	10	9
.1	1.6	1.0	0.9
.2	3.2	2.0	1.8
.3	4.8	3.0	2.7
.4	6.4	4.0	3.6
.5	8.0	5.0	4.5
.6	9.6	6.0	5.4
.7	11.2	7.0	6.3
.8	12.8	8.0	7.2
.9	14.4	9.0	8.1

Prop. Pts.

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.78 934	16	9.89 281	26	10.10 719	9.89 653	10	60
1	9.78 950	17	9.89 307	26	10.10 693	9.89 643	10	59
2	9.78 967	16	9.89 333	26	10.10 667	9.89 633	9	58
3	9.78 983	16	9.89 359	26	10.10 641	9.89 624	10	57
4	9.78 999	16	9.89 385	26	10.10 615	9.89 614	10	56
5	9.79 015	16	9.89 411	26	10.10 589	9.89 604	10	55
6	9.79 031	16	9.89 437	26	10.10 563	9.89 594	10	54
7	9.79 047	16	9.89 463	26	10.10 537	9.89 584	10	53
8	9.79 063	16	9.89 489	26	10.10 511	9.89 574	10	52
9	9.79 079	16	9.89 515	26	10.10 485	9.89 564	10	51
10	9.79 095	16	9.89 541	26	10.10 459	9.89 554	10	50
11	9.79 111	17	9.89 567	26	10.10 433	9.89 544	10	49
12	9.79 128	16	9.89 593	26	10.10 407	9.89 534	10	48
13	9.79 144	16	9.89 619	26	10.10 381	9.89 524	10	47
14	9.79 160	16	9.89 645	26	10.10 355	9.89 514	10	46
15	9.79 176	16	9.89 671	26	10.10 329	9.89 504	9	45
16	9.79 192	16	9.89 697	26	10.10 303	9.89 495	10	44
17	9.79 208	16	9.89 723	26	10.10 277	9.89 485	10	43
18	9.79 224	16	9.89 749	26	10.10 251	9.89 475	10	42
19	9.79 240	16	9.89 775	26	10.10 225	9.89 465	10	41
20	9.79 256	16	9.89 801	26	10.10 199	9.89 455	10	40
21	9.79 272	16	9.89 827	26	10.10 173	9.89 445	10	39
22	9.79 288	16	9.89 853	26	10.10 147	9.89 435	10	38
23	9.79 304	15	9.89 879	26	10.10 121	9.89 425	10	37
24	9.79 319	16	9.89 905	26	10.10 095	9.89 415	10	36
25	9.79 335	16	9.89 931	26	10.10 069	9.89 405	10	35
26	9.79 351	16	9.89 957	26	10.10 043	9.89 395	10	34
27	9.79 367	16	9.89 983	26	10.10 017	9.89 385	10	33
28	9.79 383	16	9.90 009	26	10.09 991	9.89 375	11	32
29	9.79 399	16	9.90 035	26	10.09 965	9.89 364	10	31
30	9.79 415	16	9.90 061	25	10.09 939	9.89 354	10	30
31	9.79 431	16	9.90 086	26	10.09 914	9.89 344	10	29
32	9.79 447	16	9.90 112	26	10.09 888	9.89 334	10	28
33	9.79 463	15	9.90 138	26	10.09 862	9.89 324	10	27
34	9.79 478	16	9.90 164	26	10.09 836	9.89 314	10	26
35	9.79 494	16	9.90 190	26	10.09 810	9.89 304	10	25
36	9.79 510	16	9.90 216	26	10.09 784	9.89 294	10	24
37	9.79 526	16	9.90 242	26	10.09 758	9.89 284	10	23
38	9.79 542	16	9.90 268	26	10.09 732	9.89 274	10	22
39	9.79 558	15	9.90 294	26	10.09 706	9.89 264	10	21
40	9.79 573	16	9.90 320	26	10.09 680	9.89 254	10	20
41	9.79 589	16	9.90 346	25	10.09 654	9.89 244	11	19
42	9.79 605	16	9.90 371	26	10.09 629	9.89 233	10	18
43	9.79 621	15	9.90 397	26	10.09 603	9.89 223	10	17
44	9.79 636	16	9.90 423	26	10.09 577	9.89 213	10	16
45	9.79 652	16	9.90 449	26	10.09 551	9.89 203	10	15
46	9.79 668	16	9.90 475	26	10.09 525	9.89 193	10	14
47	9.79 684	15	9.90 501	26	10.09 499	9.89 183	10	13
48	9.79 699	16	9.90 527	26	10.09 473	9.89 173	11	12
49	9.79 715	16	9.90 553	25	10.09 447	9.89 162	10	11
50	9.79 731	15	9.90 578	26	10.09 422	9.89 152	10	10
51	9.79 746	16	9.90 604	26	10.09 396	9.89 142	10	9
52	9.79 762	16	9.90 630	26	10.09 370	9.89 132	10	8
53	9.79 778	15	9.90 656	26	10.09 344	9.89 122	10	7
54	9.79 793	16	9.90 682	26	10.09 318	9.89 112	11	6
55	9.79 809	16	9.90 708	26	10.09 292	9.89 101	10	5
56	9.79 825	15	9.90 734	25	10.09 266	9.89 091	10	4
57	9.79 840	16	9.90 759	26	10.09 241	9.89 081	10	3
58	9.79 856	16	9.90 785	26	10.09 215	9.89 071	11	2
59	9.79 872	15	9.90 811	26	10.09 189	9.89 060	10	1
60	9.79 887		9.90 837		10.09 163	9.89 050		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

	26	25
.1	2.6	2.5
.2	5.2	5.0
.3	7.8	7.5
.4	10.4	10.0
.5	13.0	12.5
.6	15.6	15.0
.7	18.2	17.5
.8	20.8	20.0
.9	23.4	22.5

	17	16
.1	1.7	1.6
.2	3.4	3.2
.3	5.1	4.8
.4	6.8	6.4
.5	8.5	8.0
.6	10.2	9.6
.7	11.9	11.2
.8	13.6	12.8
.9	15.3	14.4

	15	11
.1	1.5	1.1
.2	3.0	2.2
.3	4.5	3.3
.4	6.0	4.4
.5	7.5	5.5
.6	9.0	6.6
.7	10.5	7.7
.8	12.0	8.8
.9	13.5	9.9

	10	9
.1	1.0	0.9
.2	2.0	1.8
.3	3.0	2.7
.4	4.0	3.6
.5	5.0	4.5
.6	6.0	5.4
.7	7.0	6.3
.8	8.0	7.2
.9	9.0	8.1

Prop. Pts.

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	′
0	9.79 887	16	9.90 837	26	10.09 163	9.89 050	10	60
1	9.79 903	15	9.90 863	26	10.09 137	9.89 040	10	59
2	9.79 918	16	9.90 889	25	10.09 111	9.89 030	10	58
3	9.79 934	16	9.90 914	26	10.09 086	9.89 020	11	57
4	9.79 950	15	9.90 940	26	10.09 060	9.89 009	10	56
5	9.79 965	16	9.90 966	26	10.09 034	9.88 999	10	55
6	9.79 981	15	9.90 992	26	10.09 008	9.88 989	11	54
7	9.79 996	16	9.91 018	25	10.08 982	9.88 978	10	53
8	9.80 012	15	9.91 043	26	10.08 957	9.88 968	10	52
9	9.80 027	16	9.91 069	26	10.08 931	9.88 958	10	51
10	9.80 043	15	9.91 095	26	10.08 905	9.88 948	11	50
11	9.80 058	16	9.91 121	26	10.08 879	9.88 937	10	49
12	9.80 074	15	9.91 147	25	10.08 853	9.88 927	10	48
13	9.80 089	16	9.91 172	26	10.08 828	9.88 917	11	47
14	9.80 105	15	9.91 198	26	10.08 802	9.88 906	10	46
15	9.80 120	16	9.91 224	26	10.08 776	9.88 896	10	45
16	9.80 136	15	9.91 250	26	10.08 750	9.88 886	11	44
17	9.80 151	15	9.91 276	25	10.08 724	9.88 875	10	43
18	9.80 166	16	9.91 301	26	10.08 699	9.88 865	10	42
19	9.80 182	15	9.91 327	26	10.08 673	9.88 855	11	41
20	9.80 197	16	9.91 353	26	10.08 647	9.88 844	10	40
21	9.80 213	15	9.91 379	25	10.08 621	9.88 834	10	39
22	9.80 228	16	9.91 404	26	10.08 596	9.88 824	11	38
23	9.80 244	15	9.91 430	26	10.08 570	9.88 813	10	37
24	9.80 259	15	9.91 456	26	10.08 544	9.88 803	10	36
25	9.80 274	16	9.91 482	25	10.08 518	9.88 793	11	35
26	9.80 290	15	9.91 507	26	10.08 493	9.88 782	10	34
27	9.80 305	15	9.91 533	26	10.08 467	9.88 772	11	33
28	9.80 320	16	9.91 559	26	10.08 441	9.88 761	10	32
29	9.80 336	15	9.91 585	25	10.08 415	9.88 751	10	31
30	9.80 351	15	9.91 610	26	10.08 390	9.88 741	11	30
31	9.80 366	16	9.91 636	26	10.08 364	9.88 730	10	29
32	9.80 382	15	9.91 662	26	10.08 338	9.88 720	11	28
33	9.80 397	15	9.91 688	25	10.08 312	9.88 709	10	27
34	9.80 412	16	9.91 713	26	10.08 287	9.88 699	11	26
35	9.80 428	15	9.91 739	26	10.08 261	9.88 688	10	25
36	9.80 443	15	9.91 765	26	10.08 235	9.88 678	10	24
37	9.80 458	15	9.91 791	25	10.08 209	9.88 668	11	23
38	9.80 473	16	9.91 816	26	10.08 184	9.88 657	10	22
39	9.80 489	15	9.91 842	26	10.08 158	9.88 647	11	21
40	9.80 504	15	9.91 868	25	10.08 132	9.88 636	10	20
41	9.80 519	15	9.91 893	26	10.08 107	9.88 626	11	19
42	9.80 534	16	9.91 919	26	10.08 081	9.88 615	10	18
43	9.80 550	15	9.91 945	26	10.08 055	9.88 605	11	17
44	9.80 565	15	9.91 971	25	10.08 029	9.88 594	10	16
45	9.80 580	15	9.91 996	26	10.08 004	9.88 584	11	15
46	9.80 595	15	9.92 022	26	10.07 978	9.88 573	10	14
47	9.80 610	15	9.92 048	25	10.07 952	9.88 563	11	13
48	9.80 625	16	9.92 073	26	10.07 927	9.88 552	10	12
49	9.80 641	15	9.92 099	26	10.07 901	9.88 542	11	11
50	9.80 656	15	9.92 125	25	10.07 875	9.88 531	10	10
51	9.80 671	15	9.92 150	26	10.07 850	9.88 521	11	9
52	9.80 686	15	9.92 176	26	10.07 824	9.88 510	11	8
53	9.80 701	15	9.92 202	25	10.07 798	9.88 499	10	7
54	9.80 716	15	9.92 227	26	10.07 773	9.88 489	11	6
55	9.80 731	15	9.92 253	26	10.07 747	9.88 478	10	5
56	9.80 746	16	9.92 279	25	10.07 721	9.88 468	11	4
57	9.80 762	15	9.92 304	26	10.07 696	9.88 457	10	3
58	9.80 777	15	9.92 330	26	10.07 670	9.88 447	11	2
59	9.80 792	15	9.92 356	25	10.07 644	9.88 436	11	1
60	9.80 807		9.92 381		10.07 619	9.88 425		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

	26	25
.1	2.6	2.5
.2	5.2	5.0
.3	7.8	7.5
.4	10.4	10.0
.5	13.0	12.5
.6	15.6	15.0
.7	18.2	17.5
.8	20.8	20.0
.9	23.4	22.5

	16	15
.1	1.6	1.5
.2	3.2	3.0
.3	4.8	4.5
.4	6.4	6.0
.5	8.0	7.5
.6	9.6	9.0
.7	11.2	10.5
.8	12.8	12.0
.9	14.4	13.5

	11	10
.1	1.1	1.0
.2	2.2	2.0
.3	3.3	3.0
.4	4.4	4.0
.5	5.5	5.0
.6	6.6	6.0
.7	7.7	7.0
.8	8.8	8.0
.9	9.9	9.0

Prop. Pts.

'	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.80 807	15	9.92 381	26	10.07 619	9.88 425	10	60
1	9.80 822	15	9.92 407	26	10.07 593	9.88 415	11	59
2	9.80 837	15	9.92 433	25	10.07 567	9.88 404	10	58
3	9.80 852	15	9.92 458	26	10.07 542	9.88 394	11	57
4	9.80 867	15	9.92 484	26	10.07 516	9.88 383	11	56
5	9.80 882	15	9.92 510	25	10.07 490	9.88 372	10	55
6	9.80 897	15	9.92 535	26	10.07 465	9.88 362	11	54
7	9.80 912	15	9.92 561	26	10.07 439	9.88 351	11	53
8	9.80 927	15	9.92 587	25	10.07 413	9.88 340	10	52
9	9.80 942	15	9.92 612	26	10.07 388	9.88 330	11	51
10	9.80 957	15	9.92 638	25	10.07 362	9.88 319	11	50
11	9.80 972	15	9.92 663	26	10.07 337	9.88 308	10	49
12	9.80 987	15	9.92 689	26	10.07 311	9.88 298	11	48
13	9.81 002	15	9.92 715	25	10.07 285	9.88 287	11	47
14	9.81 017	15	9.92 740	26	10.07 260	9.88 276	10	46
15	9.81 032	15	9.92 766	26	10.07 234	9.88 266	11	45
16	9.81 047	14	9.92 792	25	10.07 208	9.88 255	11	44
17	9.81 061	15	9.92 817	26	10.07 183	9.88 244	10	43
18	9.81 076	15	9.92 843	25	10.07 157	9.88 234	11	42
19	9.81 091	15	9.92 868	26	10.07 132	9.88 223	11	41
20	9.81 106	15	9.92 894	26	10.07 106	9.88 212	11	40
21	9.81 121	15	9.92 920	25	10.07 080	9.88 201	10	39
22	9.81 136	15	9.92 945	26	10.07 055	9.88 191	11	38
23	9.81 151	15	9.92 971	25	10.07 029	9.88 180	11	37
24	9.81 166	14	9.92 996	26	10.07 004	9.88 169	11	36
25	9.81 180	15	9.93 022	26	10.06 978	9.88 158	10	35
26	9.81 195	15	9.93 048	25	10.06 952	9.88 148	11	34
27	9.81 210	15	9.93 073	26	10.06 927	9.88 137	11	33
28	9.81 225	15	9.93 099	25	10.06 901	9.88 126	11	32
29	9.81 240	14	9.93 124	26	10.06 876	9.88 115	10	31
30	9.81 254	15	9.93 150	25	10.06 850	9.88 105	11	30
31	9.81 269	15	9.93 175	26	10.06 825	9.88 094	11	29
32	9.81 284	15	9.93 201	26	10.06 799	9.88 083	11	28
33	9.81 299	15	9.93 227	25	10.06 773	9.88 072	11	27
34	9.81 314	14	9.93 252	26	10.06 748	9.88 061	10	26
35	9.81 328	15	9.93 278	25	10.06 722	9.88 051	11	25
36	9.81 343	15	9.93 303	26	10.06 697	9.88 040	11	24
37	9.81 358	14	9.93 329	25	10.06 671	9.88 029	11	23
38	9.81 372	15	9.93 354	26	10.06 646	9.88 018	11	22
39	9.81 387	15	9.93 380	26	10.06 620	9.88 007	11	21
40	9.81 402	15	9.93 406	25	10.06 594	9.87 996	11	20
41	9.81 417	14	9.93 431	26	10.06 569	9.87 985	10	19
42	9.81 431	15	9.93 457	25	10.06 543	9.87 975	11	18
43	9.81 446	15	9.93 482	26	10.06 518	9.87 964	11	17
44	9.81 461	14	9.93 508	25	10.06 492	9.87 953	11	16
45	9.81 475	15	9.93 533	26	10.06 467	9.87 942	11	15
46	9.81 490	15	9.93 559	25	10.06 441	9.87 931	11	14
47	9.81 505	14	9.93 584	26	10.06 416	9.87 920	11	13
48	9.81 519	15	9.93 610	26	10.06 390	9.87 909	11	12
49	9.81 534	15	9.93 636	25	10.06 364	9.87 898	11	11
50	9.81 549	14	9.93 661	26	10.06 339	9.87 887	10	10
51	9.81 563	15	9.93 687	25	10.06 313	9.87 877	11	9
52	9.81 578	14	9.93 712	26	10.06 288	9.87 866	11	8
53	9.81 592	15	9.93 738	25	10.06 262	9.87 855	11	7
54	9.81 607	15	9.93 763	26	10.06 237	9.87 844	11	6
55	9.81 622	14	9.93 789	25	10.06 211	9.87 833	11	5
56	9.81 636	15	9.93 814	26	10.06 186	9.87 822	11	4
57	9.81 651	14	9.93 840	25	10.06 160	9.87 811	11	3
58	9.81 665	15	9.93 865	26	10.06 135	9.87 800	11	2
59	9.81 680	14	9.93 891	25	10.06 109	9.87 789	11	1
60	9.81 694		9.93 916		10.06 084	9.87 778		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	'

Prop. Pts.

	26	25
.1	2.6	2.5
.2	5.2	5.0
.3	7.8	7.5
.4	10.4	10.0
.5	13.0	12.5
.6	15.6	15.0
.7	18.2	17.5
.8	20.8	20.0
.9	23.4	22.5

	15	14
.1	1.5	1.4
.2	3.0	2.8
.3	4.5	4.2
.4	6.0	5.6
.5	7.5	7.0
.6	9.0	8.4
.7	10.5	9.8
.8	12.0	11.2
.9	13.5	12.6

	11	10
.1	1.1	1.0
.2	2.2	2.0
.3	3.3	3.0
.4	4.4	4.0
.5	5.5	5.0
.6	6.6	6.0
.7	7.7	7.0
.8	8.8	8.0
.9	9.9	9.0

Prop. Pts.

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.81 694	15	9.93 916	26	10.06 084	9.87 778	11	60
1	9.81 709	14	9.93 942	25	10.06 058	9.87 767	11	59
2	9.81 723	15	9.93 967	26	10.06 033	9.87 756	11	58
3	9.81 738	14	9.93 993	25	10.06 007	9.87 745	11	57
4	9.81 752	15	9.94 018	26	10.05 982	9.87 734	11	56
5	9.81 767	14	9.94 044	25	10.05 956	9.87 723	11	55
6	9.81 781	15	9.94 069	26	10.05 931	9.87 712	11	54
7	9.81 796	14	9.94 095	25	10.05 905	9.87 701	11	53
8	9.81 810	15	9.94 120	26	10.05 880	9.87 690	11	52
9	9.81 825	14	9.94 146	25	10.05 854	9.87 679	11	51
10	9.81 839	15	9.94 171	26	10.05 829	9.87 668	11	50
11	9.81 854	14	9.94 197	25	10.05 803	9.87 657	11	49
12	9.81 868	14	9.94 222	26	10.05 778	9.87 646	11	48
13	9.81 882	15	9.94 248	25	10.05 752	9.87 635	11	47
14	9.81 897	14	9.94 273	26	10.05 727	9.87 624	11	46
15	9.81 911	15	9.94 299	25	10.05 701	9.87 613	12	45
16	9.81 926	14	9.94 324	26	10.05 676	9.87 601	11	44
17	9.81 940	15	9.94 350	25	10.05 650	9.87 590	11	43
18	9.81 955	14	9.94 375	26	10.05 625	9.87 579	11	42
19	9.81 969	14	9.94 401	25	10.05 599	9.87 568	11	41
20	9.81 983	15	9.94 426	26	10.05 574	9.87 557	11	40
21	9.81 998	14	9.94 452	25	10.05 548	9.87 546	11	39
22	9.82 012	14	9.94 477	26	10.05 523	9.87 535	11	38
23	9.82 026	15	9.94 503	25	10.05 497	9.87 524	11	37
24	9.82 041	14	9.94 528	26	10.05 472	9.87 513	12	36
25	9.82 055	14	9.94 554	25	10.05 446	9.87 501	11	35
26	9.82 069	15	9.94 579	25	10.05 421	9.87 490	11	34
27	9.82 084	14	9.94 604	26	10.05 396	9.87 479	11	33
28	9.82 098	14	9.94 630	25	10.05 370	9.87 468	11	32
29	9.82 112	14	9.94 655	26	10.05 345	9.87 457	11	31
30	9.82 126	15	9.94 681	25	10.05 319	9.87 446	12	30
31	9.82 141	14	9.94 706	26	10.05 294	9.87 434	11	29
32	9.82 155	14	9.94 732	25	10.05 268	9.87 423	11	28
33	9.82 169	15	9.94 757	26	10.05 243	9.87 412	11	27
34	9.82 184	14	9.94 783	25	10.05 217	9.87 401	11	26
35	9.82 198	14	9.94 808	26	10.05 192	9.87 390	12	25
36	9.82 212	14	9.94 834	25	10.05 166	9.87 378	11	24
37	9.82 226	14	9.94 859	25	10.05 141	9.87 367	11	23
38	9.82 240	15	9.94 884	26	10.05 116	9.87 356	11	22
39	9.82 255	14	9.94 910	25	10.05 090	9.87 345	11	21
40	9.82 269	14	9.94 935	26	10.05 065	9.87 334	12	20
41	9.82 283	14	9.94 961	25	10.05 039	9.87 322	11	19
42	9.82 297	14	9.94 986	26	10.05 014	9.87 311	11	18
43	9.82 311	15	9.95 012	25	10.04 988	9.87 300	12	17
44	9.82 326	14	9.95 037	25	10.04 963	9.87 288	11	16
45	9.82 340	14	9.95 062	26	10.04 938	9.87 277	11	15
46	9.82 354	14	9.95 088	25	10.04 912	9.87 266	11	14
47	9.82 368	14	9.95 113	26	10.04 887	9.87 255	12	13
48	9.82 382	14	9.95 139	25	10.04 861	9.87 243	11	12
49	9.82 396	14	9.95 164	26	10.04 836	9.87 232	11	11
50	9.82 410	14	9.95 190	25	10.04 810	9.87 221	12	10
51	9.82 424	15	9.95 215	25	10.04 785	9.87 209	11	9
52	9.82 439	14	9.95 240	26	10.04 760	9.87 198	11	8
53	9.82 453	14	9.95 266	25	10.04 734	9.87 187	12	7
54	9.82 467	14	9.95 291	26	10.04 709	9.87 175	11	6
55	9.82 481	14	9.95 317	25	10.04 683	9.87 164	11	5
56	9.82 495	14	9.95 342	26	10.04 658	9.87 153	12	4
57	9.82 509	14	9.95 368	25	10.04 632	9.87 141	11	3
58	9.82 523	14	9.95 393	25	10.04 607	9.87 130	11	2
59	9.82 537	14	9.95 418	26	10.04 582	9.87 119	12	1
60	9.82 551		9.95 444		10.04 556	9.87 107		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

	26	25
.1	2.6	2.5
.2	5.2	5.0
.3	7.8	7.5
.4	10.4	10.0
.5	13.0	12.5
.6	15.6	15.0
.7	18.2	17.5
.8	20.8	20.0
.9	23.4	22.5

	15	14
.1	1.5	1.4
.2	3.0	2.8
.3	4.5	4.2
.4	6.0	5.6
.5	7.5	7.0
.6	9.0	8.4
.7	10.5	9.8
.8	12.0	11.2
.9	13.5	12.6

	12	11
.1	1.2	1.1
.2	2.4	2.2
.3	3.6	3.3
.4	4.8	4.4
.5	6.0	5.5
.6	7.2	6.6
.7	8.4	7.7
.8	9.6	8.8
.9	10.8	9.9

Prop. Pts.

48°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.82 551	14	9.95 444	25	10.04 556	9.87 107	11	60
1	9.82 565	14	9.95 469	26	10.04 531	9.87 096	11	59
2	9.82 579	14	9.95 495	25	10.04 505	9.87 085	12	58
3	9.82 593	14	9.95 520	25	10.04 480	9.87 073	11	57
4	9.82 607	14	9.95 545	26	10.04 455	9.87 062	12	56
5	9.82 621	14	9.95 571	25	10.04 429	9.87 050	11	55
6	9.82 635	14	9.95 596	26	10.04 404	9.87 039	11	54
7	9.82 649	14	9.95 622	25	10.04 378	9.87 028	12	53
8	9.82 663	14	9.95 647	25	10.04 353	9.87 016	11	52
9	9.82 677	14	9.95 672	26	10.04 328	9.87 005	12	51
10	9.82 691	14	9.95 698	25	10.04 302	9.86 993	11	50
11	9.82 705	14	9.95 723	25	10.04 277	9.86 982	12	49
12	9.82 719	14	9.95 748	26	10.04 252	9.86 970	11	48
13	9.82 733	14	9.95 774	25	10.04 226	9.86 959	12	47
14	9.82 747	14	9.95 799	26	10.04 201	9.86 947	11	46
15	9.82 761	14	9.95 825	25	10.04 175	9.86 936	12	45
16	9.82 775	13	9.95 850	25	10.04 150	9.86 924	11	44
17	9.82 788	14	9.95 875	26	10.04 125	9.86 913	11	43
18	9.82 802	14	9.95 901	25	10.04 099	9.86 902	12	42
19	9.82 816	14	9.95 926	26	10.04 074	9.86 890	11	41
20	9.82 830	14	9.95 952	25	10.04 048	9.86 879	12	40
21	9.82 844	14	9.95 977	25	10.04 023	9.86 867	12	39
22	9.82 858	14	9.96 002	26	10.03 998	9.86 855	11	38
23	9.82 872	13	9.96 028	25	10.03 972	9.86 844	12	37
24	9.82 885	14	9.96 053	25	10.03 947	9.86 832	11	36
25	9.82 899	14	9.96 078	26	10.03 922	9.86 821	12	35
26	9.82 913	14	9.96 104	25	10.03 896	9.86 809	11	34
27	9.82 927	14	9.96 129	26	10.03 871	9.86 798	12	33
28	9.82 941	14	9.96 155	25	10.03 845	9.86 786	11	32
29	9.82 955	13	9.96 180	25	10.03 820	9.86 775	12	31
30	9.82 968	14	9.96 205	26	10.03 795	9.86 763	11	30
31	9.82 982	14	9.96 231	25	10.03 769	9.86 752	12	29
32	9.82 996	14	9.96 256	25	10.03 744	9.86 740	12	28
33	9.83 010	13	9.96 281	26	10.03 719	9.86 728	11	27
34	9.83 023	14	9.96 307	25	10.03 693	9.86 717	12	26
35	9.83 037	14	9.96 332	25	10.03 668	9.86 705	11	25
36	9.83 051	14	9.96 357	26	10.03 643	9.86 694	12	24
37	9.83 065	13	9.96 383	25	10.03 617	9.86 682	12	23
38	9.83 078	14	9.96 408	25	10.03 592	9.86 670	11	22
39	9.83 092	14	9.96 433	26	10.03 567	9.86 659	12	21
40	9.83 106	14	9.96 459	25	10.03 541	9.86 647	12	20
41	9.83 120	13	9.96 484	26	10.03 516	9.86 635	11	19
42	9.83 133	14	9.96 510	25	10.03 490	9.86 624	12	18
43	9.83 147	14	9.96 535	25	10.03 465	9.86 612	12	17
44	9.83 161	13	9.96 560	26	10.03 440	9.86 600	11	16
45	9.83 174	14	9.96 586	25	10.03 414	9.86 589	12	15
46	9.83 188	14	9.96 611	25	10.03 389	9.86 577	12	14
47	9.83 202	13	9.96 636	26	10.03 364	9.86 565	11	13
48	9.83 215	14	9.96 662	25	10.03 338	9.86 554	12	12
49	9.83 229	13	9.96 687	25	10.03 313	9.86 542	12	11
50	9.83 242	14	9.96 712	26	10.03 288	9.86 530	12	10
51	9.83 256	14	9.96 738	25	10.03 262	9.86 518	11	9
52	9.83 270	13	9.96 763	25	10.03 237	9.86 507	12	8
53	9.83 283	14	9.96 788	26	10.03 212	9.86 495	12	7
54	9.83 297	13	9.96 814	25	10.03 186	9.86 483	11	6
55	9.83 310	14	9.96 839	25	10.03 161	9.86 472	12	5
56	9.83 324	14	9.96 864	26	10.03 136	9.86 460	12	4
57	9.83 338	13	9.96 890	25	10.03 110	9.86 448	12	3
58	9.83 351	14	9.96 915	25	10.03 085	9.86 436	11	2
59	9.83 365	13	9.96 940	26	10.03 060	9.86 425	12	1
60	9.83 378		9.96 966		10.03 034	9.86 413		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

	26	25
.1	2.6	2.5
.2	5.2	5.0
.3	7.8	7.5
.4	10.4	10.0
.5	13.0	12.5
.6	15.6	15.0
.7	18.2	17.5
.8	20.8	20.0
.9	23.4	22.5

	14	13
.1	1.4	1.3
.2	2.8	2.6
.3	4.2	3.9
.4	5.6	5.2
.5	7.0	6.5
.6	8.4	7.8
.7	9.8	9.1
.8	11.2	10.4
.9	12.6	11.7

	12	11
.1	1.2	1.1
.2	2.4	2.2
.3	3.6	3.3
.4	4.8	4.4
.5	6.0	5.5
.6	7.2	6.6
.7	8.4	7.7
.8	9.6	8.8
.9	10.8	9.9

Prop. Pts.

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.83 378	14	9.96 966	25	10.03 034	9.86 413	12	**60**
1	9.83 392	13	9.96 991	25	10.03 009	9.86 401	12	59
2	9.83 405	14	9.97 016	26	10.02 984	9.86 389	12	58
3	9.83 419	13	9.97 042	25	10.02 958	9.86 377	11	57
4	9.83 432	14	9.97 067	25	10.02 933	9.86 366	12	56
5	9.83 446	13	9.97 092	26	10.02 908	9.86 354	12	55
6	9.83 459	14	9.97 118	25	10.02 882	9.86 342	12	54
7	9.83 473	13	9.97 143	25	10.02 857	9.86 330	12	53
8	9.83 486	14	9.97 168	25	10.02 832	9.86 318	12	52
9	9.83 500	13	9.97 193	26	10.02 807	9.86 306	11	51
10	9.83 513	14	9.97 219	25	10.02 781	9.86 295	12	**50**
11	9.83 527	13	9.97 244	25	10.02 756	9.86 283	12	49
12	9.83 540	14	9.97 269	26	10.02 731	9.86 271	12	48
13	9.83 554	13	9.97 295	25	10.02 705	9.86 259	12	47
14	9.83 567	14	9.97 320	25	10.02 680	9.86 247	12	46
15	9.83 581	13	9.97 345	26	10.02 655	9.86 235	12	45
16	9.83 594	14	9.97 371	25	10.02 629	9.86 223	12	44
17	9.83 608	13	9.97 396	25	10.02 604	9.86 211	11	43
18	9.83 621	13	9.97 421	26	10.02 579	9.86 200	12	42
19	9.83 634	14	9.97 447	25	10.02 553	9.86 188	12	41
20	9.83 648	13	9.97 472	25	10.02 528	9.86 176	12	**40**
21	9.83 661	13	9.97 497	26	10.02 503	9.86 164	12	39
22	9.83 674	14	9.97 523	25	10.02 477	9.86 152	12	38
23	9.83 688	13	9.97 548	25	10.02 452	9.86 140	12	37
24	9.83 701	14	9.97 573	25	10.02 427	9.86 128	12	36
25	9.83 715	13	9.97 598	26	10.02 402	9.86 116	12	35
26	9.83 728	13	9.97 624	25	10.02 376	9.86 104	12	34
27	9.83 741	14	9.97 649	25	10.02 351	9.86 092	12	33
28	9.83 755	13	9.97 674	26	10.02 326	9.86 080	12	32
29	9.83 768	13	9.97 700	25	10.02 300	9.86 068	12	31
30	9.83 781	14	9.97 725	25	10.02 275	9.86 056	12	**30**
31	9.83 795	13	9.97 750	26	10.02 250	9.86 044	12	29
32	9.83 808	13	9.97 776	25	10.02 224	9.86 032	12	28
33	9.83 821	13	9.97 801	25	10.02 199	9.86 020	12	27
34	9.83 834	14	9.97 826	25	10.02 174	9.86 008	12	26
35	9.83 848	13	9.97 851	26	10.02 149	9.85 996	12	25
36	9.83 861	13	9.97 877	25	10.02 123	9.85 984	12	24
37	9.83 874	13	9.97 902	25	10.02 098	9.85 972	12	23
38	9.83 887	14	9.97 927	26	10.02 073	9.85 960	12	22
39	9.83 901	13	9.97 953	25	10.02 047	9.85 948	12	21
40	9.83 914	13	9.97 978	25	10.02 022	9.85 936	12	**20**
41	9.83 927	13	9.98 003	26	10.01 997	9.85 924	12	19
42	9.83 940	14	9.98 029	25	10.01 971	9.85 912	12	18
43	9.83 954	13	9.98 054	25	10.01 946	9.85 900	12	17
44	9.83 967	13	9.98 079	25	10.01 921	9.85 888	12	16
45	9.83 980	13	9.98 104	26	10.01 896	9.85 876	12	15
46	9.83 993	13	9.98 130	25	10.01 870	9.85 864	13	14
47	9.84 006	14	9.98 155	25	10.01 845	9.85 851	12	13
48	9.84 020	13	9.98 180	26	10.01 820	9.85 839	12	12
49	9.84 033	13	9.98 206	25	10.01 794	9.85 827	12	11
50	9.84 046	13	9.98 231	25	10.01 769	9.85 815	12	**10**
51	9.84 059	13	9.98 256	25	10.01 744	9.85 803	12	9
52	9.84 072	13	9.98 281	26	10.01 719	9.85 791	12	8
53	9.84 085	13	9.98 307	25	10.01 693	9.85 779	13	7
54	9.84 098	14	9.98 332	25	10.01 668	9.85 766	12	6
55	9.84 112	13	9.98 357	26	10.01 643	9.85 754	12	5
56	9.84 125	13	9.98 383	25	10.01 617	9.85 742	12	4
57	9.84 138	13	9.98 408	25	10.01 592	9.85 730	12	3
58	9.84 151	13	9.98 433	25	10.01 567	9.85 718	12	2
59	9.84 164	13	9.98 458	26	10.01 542	9.85 706	13	1
60	9.84 177		9.98 484		10.01 516	9.85 693		**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

Prop. Pts.

	26	25
.1	2.6	2.5
.2	5.2	5.0
.3	7.8	7.5
.4	10.4	10.0
.5	13.0	12.5
.6	15.6	15.0
.7	18.2	17.5
.8	20.8	20.0
.9	23 4	22.5

	14	13
.1	1.4	1.3
.2	2.8	2.6
.3	4.2	3.9
.4	5.6	5.2
.5	7.0	6.5
.6	8.4	7.8
.7	9.8	9.1
.8	11.2	10.4
.9	12.6	11.7

	12	11
.1	1.2	1.1
.2	2.4	2.2
.3	3.6	3.3
.4	4.8	4.4
.5	6.0	5.5
.6	7.2	6.6
.7	8.4	7.7
.8	9.6	8.8
.9	10.8	9.9

Prop. Pts.

Prop. Pts.

	26	25	14
.1	2.6	2.5	1.4
.2	5.2	5.0	2.8
.3	7.8	7.5	4.2
.4	10.4	10.0	5.6
.5	13.0	12.5	7.0
.6	15.6	15.0	8.4
.7	18.2	17.5	9.8
.8	20.8	20.0	11.2
.9	23.4	22.5	12.6

	13	12
.1	1.3	1.2
.2	2.6	2.4
.3	3.9	3.6
.4	5.2	4.8
.5	6.5	6.0
.6	7.8	7.2
.7	9.1	8.4
.8	10.4	9.6
.9	11.7	10.8

Prop. Pts.

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.84 177	13	9.98 484	25	10.01 516	9.85 693	12	60
1	9.84 190	13	9.98 509	25	10.01 491	9.85 681	12	59
2	9.84 203	13	9.98 534	26	10.01 466	9.85 669	12	58
3	9.84 216	13	9.98 560	25	10.01 440	9.85 657	12	57
4	9.84 229	13	9.98 585	25	10.01 415	9.85 645	13	56
5	9.84 242	13	9.98 610	25	10.01 390	9.85 632	12	55
6	9.84 255	14	9.98 635	26	10.01 365	9.85 620	12	54
7	9.84 269	13	9.98 661	25	10.01 339	9.85 608	12	53
8	9.84 282	13	9.98 686	25	10.01 314	9.85 596	13	52
9	9.84 295	13	9.98 711	26	10.01 289	9.85 583	12	51
10	9.84 308	13	9.98 737	25	10.01 263	9.85 571	12	50
11	9.84 321	13	9.98 762	25	10.01 238	9.85 559	12	49
12	9.84 334	13	9.98 787	25	10.01 213	9.85 547	13	48
13	9.84 347	13	9.98 812	26	10.01 188	9.85 534	12	47
14	9.84 360	13	9.98 838	25	10.01 162	9.85 522	12	46
15	9.84 373	12	9.98 863	25	10.01 137	9.85 510	13	45
16	9.84 385	13	9.98 888	25	10.01 112	9.85 497	12	44
17	9.84 398	13	9.98 913	26	10.01 087	9.85 485	12	43
18	9.84 411	13	9.98 939	25	10.01 061	9.85 473	13	42
19	9.84 424	13	9.98 964	25	10.01 036	9.85 460	12	41
20	9.84 437	13	9.98 989	26	10.01 011	9.85 448	12	40
21	9.84 450	13	9.99 015	25	10.00 985	9.85 436	13	39
22	9.84 463	13	9.99 040	25	10.00 960	9.85 423	12	38
23	9.84 476	13	9.99 065	25	10.00 935	9.85 411	12	37
24	9.84 489	13	9.99 090	26	10.00 910	9.85 399	13	36
25	9.84 502	13	9.99 116	25	10.00 884	9.85 386	12	35
26	9.84 515	13	9.99 141	25	10.00 859	9.85 374	13	34
27	9.84 528	12	9.99 166	25	10.00 834	9.85 361	12	33
28	9.84 540	13	9.99 191	26	10.00 809	9.85 349	12	32
29	9.84 553	13	9.99 217	25	10.00 783	9.85 337	13	31
30	9.84 566	13	9.99 242	25	10.00 758	9.85 324	12	30
31	9.84 579	13	9.99 267	26	10.00 733	9.85 312	13	29
32	9.84 592	13	9.99 293	25	10.00 707	9.85 299	12	28
33	9.84 605	13	9.99 318	25	10.00 682	9.85 287	13	27
34	9.84 618	12	9.99 343	25	10.00 657	9.85 274	12	26
35	9.84 630	13	9.99 368	26	10.00 632	9.85 262	12	25
36	9.84 643	13	9.99 394	25	10.00 606	9.85 250	13	24
37	9.84 656	13	9.99 419	25	10.00 581	9.85 237	12	23
38	9.84 669	13	9.99 444	25	10.00 556	9.85 225	13	22
39	9.84 682	12	9.99 469	26	10.00 531	9.85 212	12	21
40	9.84 694	13	9.99 495	25	10.00 505	9.85 200	13	20
41	9.84 707	13	9.99 520	25	10.00 480	9.85 187	12	19
42	9.84 720	13	9.99 545	25	10.00 455	9.85 175	13	18
43	9.84 733	12	9.99 570	26	10.00 430	9.85 162	12	17
44	9.84 745	13	9.99 596	25	10.00 404	9.85 150	13	16
45	9.84 758	13	9.99 621	25	10.00 379	9.85 137	12	15
46	9.84 771	13	9.99 646	26	10.00 354	9.85 125	13	14
47	9.84 784	12	9.99 672	25	10.00 328	9.85 112	12	13
48	9.84 796	13	9.99 697	25	10.00 303	9.85 100	13	12
49	9.84 809	13	9.99 722	25	10.00 278	9.85 087	13	11
50	9.84 822	13	9.99 747	26	10.00 253	9.85 074	12	10
51	9.84 835	12	9.99 773	25	10.00 227	9.85 062	13	9
52	9.84 847	13	9.99 798	25	10.00 202	9.85 049	12	8
53	9.84 860	13	9.99 823	25	10.00 177	9.85 037	13	7
54	9.84 873	12	9.99 848	26	10.00 152	9.85 024	12	6
55	9.84 885	13	9.99 874	25	10.00 126	9.85 012	13	5
56	9.84 898	13	9.99 899	25	10.00 101	9.84 999	13	4
57	9.84 911	12	9.99 924	25	10.00 076	9.84 986	12	3
58	9.84 923	13	9.99 949	26	10.00 051	9.84 974	13	2
59	9.84 936	13	9.99 975	25	10.00 025	9.84 961	12	1
60	9.84 949		10.00 000		10.00 000	9.84 949		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

45°

III. Natural Trigonometric Functions

0°

′	Sin	Tan	Cot	Cos	
0	0.0000	0.0000	∞	1.0000	60
1	0.0003	0.0003	3437.75	1.0000	59
2	0.0006	0.0006	1718.87	1.0000	58
3	0.0009	0.0009	1145.92	1.0000	57
4	0.0012	0.0012	859.436	1.0000	56
5	0.0015	0.0015	687.549	1.0000	55
6	0.0017	0.0017	572.957	1.0000	54
7	0.0020	0.0020	491.106	1.0000	53
8	0.0023	0.0023	429.718	1.0000	52
9	0.0026	0.0026	381.971	1.0000	51
10	0.0029	0.0029	343.774	1.0000	50
11	0.0032	0.0032	312.521	1.0000	49
12	0.0035	0.0035	286.478	1.0000	48
13	0.0038	0.0038	264.441	1.0000	47
14	0.0041	0.0041	245.552	1.0000	46
15	0.0044	0.0044	229.182	1.0000	45
16	0.0047	0.0047	214.858	1.0000	44
17	0.0049	0.0049	202.219	1.0000	43
18	0.0052	0.0052	190.984	1.0000	42
19	0.0055	0.0055	180.932	1.0000	41
20	0.0058	0.0058	171.885	1.0000	40
21	0.0061	0.0061	163.700	1.0000	39
22	0.0064	0.0064	156.259	1.0000	38
23	0.0067	0.0067	149.465	1.0000	37
24	0.0070	0.0070	143.237	1.0000	36
25	0.0073	0.0073	137.507	1.0000	35
26	0.0076	0.0076	132.219	1.0000	34
27	0.0079	0.0079	127.321	1.0000	33
28	0.0081	0.0081	122.774	1.0000	32
29	0.0084	0.0084	118.540	1.0000	31
30	0.0087	0.0087	114.589	1.0000	30
31	0.0090	0.0090	110.892	1.0000	29
32	0.0093	0.0093	107.426	1.0000	28
33	0.0096	0.0096	104.171	1.0000	27
34	0.0099	0.0099	101.107	1.0000	26
35	0.0102	0.0102	98.2179	0.9999	25
36	0.0105	0.0105	95.4895	0.9999	24
37	0.0108	0.0108	92.9085	0.9999	23
38	0.0111	0.0111	90.4633	0.9999	22
39	0.0113	0.0113	88.1436	0.9999	21
40	0.0116	0.0116	85.9398	0.9999	20
41	0.0119	0.0119	83.8435	0.9999	19
42	0.0122	0.0122	81.8470	0.9999	18
43	0.0125	0.0125	79.9434	0.9999	17
44	0.0128	0.0128	78.1263	0.9999	16
45	0.0131	0.0131	76.3900	0.9999	15
46	0.0134	0.0134	74.7292	0.9999	14
47	0.0137	0.0137	73.1390	0.9999	13
48	0.0140	0.0140	71.6151	0.9999	12
49	0.0143	0.0143	70.1533	0.9999	11
50	0.0145	0.0145	68.7501	0.9999	10
51	0.0148	0.0148	67.4019	0.9999	9
52	0.0151	0.0151	66.1055	0.9999	8
53	0.0154	0.0154	64.8580	0.9999	7
54	0.0157	0.0157	63.6567	0.9999	6
55	0.0160	0.0160	62.4992	0.9999	5
56	0.0163	0.0163	61.3829	0.9999	4
57	0.0166	0.0166	60.3058	0.9999	3
58	0.0169	0.0169	59.2659	0.9999	2
59	0.0172	0.0172	58.2612	0.9999	1
60	0.0175	0.0175	57.2900	0.9998	0
	Cos	Cot	Tan	Sin	′

89°

1°

′	Sin	Tan	Cot	Cos	
0	0.0175	0.0175	57.2900	0.9998	60
1	0.0177	0.0177	56.3506	0.9998	59
2	0.0180	0.0180	55.4415	0.9998	58
3	0.0183	0.0183	54.5613	0.9998	57
4	0.0186	0.0186	53.7086	0.9998	56
5	0.0189	0.0189	52.8821	0.9998	55
6	0 0192	0.0192	52.0807	0.9998	54
7	0.0195	0.0195	51.3032	0.9998	53
8	0.0198	0.0198	50.5485	0.9998	52
9	0.0201	0.0201	49.8157	0.9998	51
10	0.0204	0.0204	49.1039	0.9998	50
11	0.0207	0.0207	48.4121	0.9998	49
12	0.0209	0.0209	47.7395	0.9998	48
13	0.0212	0.0212	47.0853	0.9998	47
14	0.0215	0.0215	46.4489	0.9998	46
15	0.0218	0.0218	45.8294	0.9998	45
16	0.0221	0.0221	45.2261	0.9998	44
17	0.0224	0.0224	44.6386	0.9997	43
18	0.0227	0.0227	44.0661	0.9997	42
19	0.0230	0.0230	43.5081	0.9997	41
20	0.0233	0.0233	42.9641	0.9997	40
21	0.0236	0.0236	42.4335	0.9997	39
22	0.0239	0.0239	41.9158	0.9997	38
23	0.0241	0.0241	41.4106	0.9997	37
24	0.0244	0.0244	40.9174	0.9997	36
25	0.0247	0.0247	40.4358	0.9997	35
26	0.0250	0.0250	39.9655	0.9997	34
27	0.0253	0.0253	39.5059	0.9997	33
28	0.0256	0.0256	39.0568	0.9997	32
29	0.0259	0.0259	38.6177	0.9997	31
30	0.0262	0.0262	38.1885	0.9997	30
31	0.0265	0.0265	37.7686	0.9996	29
32	0.0268	0.0268	37.3579	0.9996	28
33	0.0270	0.0271	36.9560	0.9996	27
34	0.0273	0.0274	36.5627	0.9996	26
35	0.0276	0.0276	36.1776	0.9996	25
36	0.0279	0.0279	35.8006	0.9996	24
37	0.0282	0.0282	35.4313	0.9996	23
38	0.0285	0.0285	35.0695	0.9996	22
39	0.0288	0.0288	34.7151	0.9996	21
40	0.0291	0.0291	34.3678	0.9996	20
41	0.0294	0.0294	34.0273	0.9996	19
42	0.0297	0.0297	33.6935	0.9996	18
43	0.0300	0.0300	33.3662	0.9996	17
44	0.0302	0.0303	33.0452	0.9995	16
45	0.0305	0.0306	32.7303	0.9995	15
46	0.0308	0.0308	32.4213	0.9995	14
47	0.0311	0.0311	32.1181	0.9995	13
48	0.0314	0.0314	31.8205	0.9995	12
49	0.0317	0.0317	31.5284	0.9995	11
50	0.0320	0.0320	31.2416	0.9995	10
51	0.0323	0.0323	30.9599	0.9995	9
52	0.0326	0.0326	30.6833	0.9995	8
53	0.0329	0.0329	30.4116	0.9995	7
54	0.0332	0.0332	30.1446	0.9995	6
55	0.0334	0.0335	29.8823	0.9994	5
56	0.0337	0.0338	29.6245	0.9994	4
57	0.0340	0.0340	29.3711	0.9994	3
58	0.0343	0.0343	29.1220	0.9994	2
59	0.0346	0.0346	28.8771	0.9994	1
60	0.0349	0.0349	28.6363	0.9994	0
	Cos	Cot	Tan	Sin	′

88°

III. Natural Trigonometric Functions

2° ′	Sin	Tan	Cot	Cos		3° ′	Sin	Tan	Cot	Cos	
0	0.0349	0.0349	28.6363	0.9994	60	0	0.0523	0.0524	19.0811	0.9986	60
1	0.0352	0.0352	28.3994	0.9994	59	1	0.0526	0.0527	18.9755	0.9986	59
2	0.0355	0.0355	28.1664	0.9994	58	2	0.0529	0.0530	18.8711	0.9986	58
3	0.0358	0.0358	27.9372	0.9994	57	3	0.0532	0.0533	18.7678	0.9986	57
4	0.0361	0.0361	27.7117	0.9993	56	4	0.0535	0.0536	18.6656	0.9986	56
5	0.0364	0.0364	27.4899	0.9993	55	5	0.0538	0.0539	18.5645	0.9986	55
6	0.0366	0.0367	27.2715	0.9993	54	6	0.0541	0.0542	18.4645	0.9985	54
7	0.0369	0.0370	27.0566	0.9993	53	7	0.0544	0.0544	18.3655	0.9985	53
8	0.0372	0.0373	26.8450	0.9993	52	8	0.0547	0.0547	18.2677	0.9985	52
9	0.0375	0.0375	26.6367	0.9993	51	9	0.0550	0.0550	18.1708	0.9985	51
10	0.0378	0.0378	26.4316	0.9993	50	10	0.0552	0.0553	18.0750	0.9985	50
11	0.0381	0.0381	26.2296	0.9993	49	11	0.0555	0.0556	17.9802	0.9985	49
12	0.0384	0.0384	26.0307	0.9993	48	12	0.0558	0.0559	17.8863	0.9984	48
13	0.0387	0.0387	25.8348	0.9993	47	13	0.0561	0.0562	17.7934	0.9984	47
14	0.0390	0.0390	25.6418	0.9992	46	14	0.0564	0.0565	17.7015	0.9984	46
15	0.0393	0.0393	25.4517	0.9992	45	15	0.0567	0.0568	17.6106	0.9984	45
16	0.0396	0.0396	25.2644	0.9992	44	16	0.0570	0.0571	17.5205	0.9984	44
17	0.0398	0.0399	25.0798	0.9992	43	17	0.0573	0.0574	17.4314	0.9984	43
18	0.0401	0.0402	24.8978	0.9992	42	18	0.0576	0.0577	17.3432	0.9983	42
19	0.0404	0.0405	24.7185	0.9992	41	19	0.0579	0.0580	17.2558	0.9983	41
20	0.0407	0.0407	24.5418	0.9992	40	20	0.0581	0.0582	17.1693	0.9983	40
21	0.0410	0.0410	24.3675	0.9992	39	21	0.0584	0.0585	17.0837	0.9983	39
22	0.0413	0.0413	24.1957	0.9991	38	22	0.0587	0.0588	16.9990	0.9983	38
23	0.0416	0.0416	24.0263	0.9991	37	23	0.0590	0.0591	16.9150	0.9983	37
24	0.0419	0.0419	23.8593	0.9991	36	24	0.0593	0.0594	16.8319	0.9982	36
25	0.0422	0.0422	23.6945	0.9991	35	25	0.0596	0.0597	16.7496	0.9982	35
26	0.0425	0.0425	23.5321	0.9991	34	26	0.0599	0.0600	16.6681	0.9982	34
27	0.0427	0.0428	23.3718	0.9991	33	27	0.0602	0.0603	16.5874	0.9982	33
28	0.0430	0.0431	23.2137	0.9991	32	28	0.0605	0.0606	16.5075	0.9982	32
29	0.0433	0.0434	23.0577	0.9991	31	29	0.0608	0.0609	16.4283	0.9982	31
30	0.0436	0.0437	22.9038	0.9990	30	30	0.0610	0.0612	16.3499	0.9981	30
31	0.0439	0.0440	22.7519	0.9990	29	31	0.0613	0.0615	16.2722	0.9981	29
32	0.0442	0.0442	22.6020	0.9990	28	32	0.0616	0.0617	16.1952	0.9981	28
33	0.0445	0.0445	22.4541	0.9990	27	33	0.0619	0.0620	16.1190	0.9981	27
34	0.0448	0.0448	22.3081	0.9990	26	34	0.0622	0.0623	16.0435	0.9981	26
35	0.0451	0.0451	22.1640	0.9990	25	35	0.0625	0.0626	15.9687	0.9980	25
36	0.0454	0.0454	22.0217	0.9990	24	36	0.0628	0.0629	15.8945	0.9980	24
37	0.0457	0.0457	21.8813	0.9990	23	37	0.0631	0.0632	15.8211	0.9980	23
38	0.0459	0.0460	21.7426	0.9989	22	38	0.0634	0.0635	15.7483	0.9980	22
39	0.0462	0.0463	21.6056	0.9989	21	39	0.0637	0.0638	15.6762	0.9980	21
40	0.0465	0.0466	21.4704	0.9989	20	40	0.0640	0.0641	15.6048	0.9980	20
41	0.0468	0.0469	21.3369	0.9989	19	41	0.0642	0.0644	15.5340	0.9979	19
42	0.0471	0.0472	21.2049	0.9989	18	42	0.0645	0.0647	15.4638	0.9979	18
43	0.0474	0.0475	21.0747	0.9989	17	43	0.0648	0.0650	15.3943	0.9979	17
44	0.0477	0.0477	20.9460	0.9989	16	44	0.0651	0.0653	15.3254	0.9979	16
45	0.0480	0.0480	20.8188	0.9988	15	45	0.0654	0.0655	15.2571	0.9979	15
46	0.0483	0.0483	20.6932	0.9988	14	46	0.0657	0.0658	15.1893	0.9978	14
47	0.0486	0.0486	20.5691	0.9988	13	47	0.0660	0.0661	15.1222	0.9978	13
48	0.0488	0.0489	20.4465	0.9988	12	48	0.0663	0.0664	15.0557	0.9978	12
49	0.0491	0.0492	20.3253	0.9988	11	49	0.0666	0.0667	14.9898	0.9978	11
50	0.0494	0.0495	20.2056	0.9988	10	50	0.0669	0.0670	14.9244	0.9978	10
51	0.0497	0.0498	20.0872	0.9988	9	51	0.0671	0.0673	14.8596	0.9977	9
52	0.0500	0.0501	19.9702	0.9987	8	52	0.0674	0.0676	14.7954	0.9977	8
53	0.0503	0.0504	19.8546	0.9987	7	53	0.0677	0.0679	14.7317	0.9977	7
54	0.0506	0.0507	19.7403	0.9987	6	54	0.0680	0.0682	14.6685	0.9977	6
55	0.0509	0.0509	19.6273	0.9987	5	55	0.0683	0.0685	14.6059	0.9977	5
56	0.0512	0.0512	19.5156	0.9987	4	56	0.0686	0.0688	14.5438	0.9976	4
57	0.0515	0.0515	19.4051	0.9987	3	57	0.0689	0.0690	14.4823	0.9976	3
58	0.0518	0.0518	19.2959	0.9987	2	58	0.0692	0.0693	14.4212	0.9976	2
59	0.0520	0.0521	19.1879	0.9986	1	59	0.0695	0.0696	14.3607	0.9976	1
60	0.0523	0.0524	19.0811	0.9986	0	60	0.0698	0.0699	14.3007	0.9976	0
	Cos	Cot	Tan	Sin	′ 87°		Cos	Cot	Tan	Sin	′ 86°

4°

′	Sin	Tan	Cot	Cos	
0	0.0698	0.0699	14.3007	0.9976	60
1	0.0700	0.0702	14.2411	0.9975	59
2	0.0703	0.0705	14.1821	0.9975	58
3	0.0706	0.0708	14.1235	0.9975	57
4	0.0709	0.0711	14.0655	0.9975	56
5	0.0712	0.0714	14.0079	0.9975	55
6	0.0715	0.0717	13.9507	0.9974	54
7	0.0718	0.0720	13.8940	0.9974	53
8	0.0721	0.0723	13.8378	0.9974	52
9	0.0724	0.0726	13.7821	0.9974	51
10	0.0727	0.0729	13.7267	0.9974	50
11	0.0729	0.0731	13.6719	0.9973	49
12	0.0732	0.0734	13.6174	0.9973	48
13	0.0735	0.0737	13.5634	0.9973	47
14	0.0738	0.0740	13.5098	0.9973	46
15	0.0741	0.0743	13.4566	0.9973	45
16	0.0744	0.0746	13.4039	0.9972	44
17	0.0747	0.0749	13.3515	0.9972	43
18	0.0750	0.0752	13.2996	0.9972	42
19	0.0753	0.0755	13.2480	0.9972	41
20	0.0756	0.0758	13.1969	0.9971	40
21	0.0758	0.0761	13.1461	0.9971	39
22	0.0761	0.0764	13.0958	0.9971	38
23	0.0764	0.0767	13.0458	0.9971	37
24	0.0767	0.0769	12.9962	0.9971	36
25	0.0770	0.0772	12.9469	0.9970	35
26	0.0773	0.0775	12.8981	0.9970	34
27	0.0776	0.0778	12.8496	0.9970	33
28	0.0779	0.0781	12.8014	0.9970	32
29	0.0782	0.0784	12.7536	0.9969	31
30	0.0785	0.0787	12.7062	0.9969	30
31	0.0787	0.0790	12.6591	0.9969	29
32	0.0790	0.0793	12.6124	0.9969	28
33	0.0793	0.0796	12.5660	0.9968	27
34	0.0796	0.0799	12.5199	0.9968	26
35	0.0799	0.0802	12.4742	0.9968	25
36	0.0802	0.0805	12.4288	0.9968	24
37	0.0805	0.0808	12.3838	0.9968	23
38	0.0808	0.0810	12.3390	0.9967	22
39	0.0811	0.0813	12.2946	0.9967	21
40	0.0814	0.0816	12.2505	0.9967	20
41	0.0816	0.0819	12.2067	0.9967	19
42	0.0819	0.0822	12.1632	0.9966	18
43	0.0822	0.0825	12.1201	0.9966	17
44	0.0825	0.0828	12.0772	0.9966	16
45	0.0828	0.0831	12.0346	0.9966	15
46	0.0831	0.0834	11.9923	0.9965	14
47	0.0834	0.0837	11.9504	0.9965	13
48	0.0837	0.0840	11.9087	0.9965	12
49	0.0840	0.0843	11.8673	0.9965	11
50	0.0843	0.0846	11.8262	0.9964	10
51	0.0845	0.0849	11.7853	0.9964	9
52	0.0848	0.0851	11.7448	0.9964	8
53	0.0851	0.0854	11.7045	0.9964	7
54	0.0854	0.0857	11.6645	0.9963	6
55	0.0857	0.0860	11.6248	0.9963	5
56	0.0860	0.0863	11.5853	0.9963	4
57	0.0863	0.0866	11.5461	0.9963	3
58	0.0866	0.0869	11.5072	0.9962	2
59	0.0869	0.0872	11.4685	0.9962	1
60	0.0872	0.0875	11.4301	0.9962	0
	Cos	Cot	Tan	Sin	′

85°

5°

′	Sin	Tan	Cot	Cos	
0	0.0872	0.0875	11.4301	0.9962	60
1	0.0874	0.0878	11.3919	0.9962	59
2	0.0877	0.0881	11.3540	0.9961	58
3	0.0880	0.0884	11.3163	0.9961	57
4	0.0883	0.0887	11.2789	0.9961	56
5	0.0886	0.0890	11.2417	0.9961	55
6	0.0889	0.0892	11.2048	0.9960	54
7	0.0892	0.0895	11.1681	0.9960	53
8	0.0895	0.0898	11.1316	0.9960	52
9	0.0898	0.0901	11.0954	0.9960	51
10	0.0901	0.0904	11.0594	0.9959	50
11	0.0903	0.0907	11.0237	0.9959	49
12	0.0906	0.0910	10.9882	0.9959	48
13	0.0909	0.0913	10.9529	0.9959	47
14	0.0912	0.0916	10.9178	0.9958	46
15	0.0915	0.0919	10.8829	0.9958	45
16	0.0918	0.0922	10.8483	0.9958	44
17	0.0921	0.0925	10.8139	0.9958	43
18	0.0924	0.0928	10.7797	0.9957	42
19	0.0927	0.0931	10.7457	0.9957	41
20	0.0929	0.0934	10.7119	0.9957	40
21	0.0932	0.0936	10.6783	0.9956	39
22	0.0935	0.0939	10.6450	0.9956	38
23	0.0938	0.0942	10.6118	0.9956	37
24	0.0941	0.0945	10.5789	0.9956	36
25	0.0944	0.0948	10.5462	0.9955	35
26	0.0947	0.0951	10.5136	0.9955	34
27	0.0950	0.0954	10.4813	0.9955	33
28	0.0953	0.0957	10.4491	0.9955	32
29	0.0956	0.0960	10.4172	0.9954	31
30	0.0958	0.0963	10.3854	0.9954	30
31	0.0961	0.0966	10.3538	0.9954	29
32	0.0964	0.0969	10.3224	0.9953	28
33	0.0967	0.0972	10.2913	0.9953	27
34	0.0970	0.0975	10.2602	0.9953	26
35	0.0973	0.0978	10.2294	0.9953	25
36	0.0976	0.0981	10.1988	0.9952	24
37	0.0979	0.0983	10.1683	0.9952	23
38	0.0982	0.0986	10.1381	0.9952	22
39	0.0985	0.0989	10.1080	0.9951	21
40	0.0987	0.0992	10.0780	0.9951	20
41	0.0990	0.0995	10.0483	0.9951	19
42	0.0993	0.0998	10.0187	0.9951	18
43	0.0996	0.1001	9.9893	0.9950	17
44	0.0999	0.1004	9.9601	0.9950	16
45	0.1002	0.1007	9.9310	0.9950	15
46	0.1005	0.1010	9.9021	0.9949	14
47	0.1008	0.1013	9.8734	0.9949	13
48	0.1011	0.1016	9.8448	0.9949	12
49	0.1013	0.1019	9.8164	0.9949	11
50	0.1016	0.1022	9.7882	0.9948	10
51	0.1019	0.1025	9.7601	0.9948	9
52	0.1022	0.1028	9.7322	0.9948	8
53	0.1025	0.1030	9.7044	0.9947	7
54	0.1028	0.1033	9.6768	0.9947	6
55	0.1031	0.1036	9.6493	0.9947	5
56	0.1034	0.1039	9.6220	0.9946	4
57	0.1037	0.1042	9.5949	0.9946	3
58	0.1039	0.1045	9.5679	0.9946	2
59	0.1042	0.1048	9.5411	0.9946	1
60	0.1045	0.1051	9.5144	0.9945	0
	Cos	Cot	Tan	Sin	′

84°

III. Natural Trigonometric Functions

6°

′	Sin	Tan	Cot	Cos	
0	0.1045	0.1051	9.5144	0.9945	60
1	0.1048	0.1054	9.4878	0.9945	59
2	0.1051	0.1057	9.4614	0.9945	58
3	0.1054	0.1060	9.4352	0.9944	57
4	0.1057	0.1063	9.4090	0.9944	56
5	0.1060	0.1066	9.3831	0.9944	55
6	0.1063	0.1069	9.3572	0.9943	54
7	0.1066	0.1072	9.3315	0.9943	53
8	0.1068	0.1075	9.3060	0.9943	52
9	0.1071	0.1078	9.2806	0.9942	51
10	0.1074	0.1080	9.2553	0.9942	50
11	0.1077	0.1083	9.2302	0.9942	49
12	0.1080	0.1086	9.2052	0.9942	48
13	0.1083	0.1089	9.1803	0.9941	47
14	0.1086	0.1092	9.1555	0.9941	46
15	0.1089	0.1095	9.1309	0.9941	45
16	0.1092	0.1098	9.1065	0.9940	44
17	0.1094	0.1101	9.0821	0.9940	43
18	0.1097	0.1104	9.0579	0.9940	42
19	0.1100	0.1107	9.0338	0.9939	41
20	0.1103	0.1110	9.0098	0.9939	40
21	0.1106	0.1113	8.9860	0.9939	39
22	0.1109	0.1116	8.9623	0.9938	38
23	0.1112	0.1119	8.9387	0.9938	37
24	0.1115	0.1122	8.9152	0.9938	36
25	0.1118	0.1125	8.8919	0.9937	35
26	0.1120	0.1128	8.8686	0.9937	34
27	0.1123	0.1131	8.8455	0.9937	33
28	0.1126	0.1133	8.8225	0.9936	32
29	0.1129	0.1136	8.7996	0.9936	31
30	0.1132	0.1139	8.7769	0.9936	30
31	0.1135	0.1142	8.7542	0.9935	29
32	0.1138	0.1145	8.7317	0.9935	28
33	0.1141	0.1148	8.7093	0.9935	27
34	0.1144	0.1151	8.6870	0.9934	26
35	0.1146	0.1154	8.6648	0.9934	25
36	0.1149	0.1157	8.6427	0.9934	24
37	0.1152	0.1160	8.6208	0.9933	23
38	0.1155	0.1163	8.5989	0.9933	22
39	0.1158	0.1166	8.5772	0.9933	21
40	0.1161	0.1169	8.5555	0.9932	20
41	0.1164	0.1172	8.5340	0.9932	19
42	0.1167	0.1175	8.5126	0.9932	18
43	0.1170	0.1178	8.4913	0.9931	17
44	0.1172	0.1181	8.4701	0.9931	16
45	0.1175	0.1184	8.4490	0.9931	15
46	0.1178	0.1187	8.4280	0.9930	14
47	0.1181	0.1189	8.4071	0.9930	13
48	0.1184	0.1192	8.3863	0.9930	12
49	0.1187	0.1195	8.3656	0.9929	11
50	0.1190	0.1198	8.3450	0.9929	10
51	0.1193	0.1201	8.3245	0.9929	9
52	0.1196	0.1204	8.3041	0.9928	8
53	0.1198	0.1207	8.2838	0.9928	7
54	0.1201	0.1210	8.2636	0.9928	6
55	0.1204	0.1213	8.2434	0.9927	5
56	0.1207	0.1216	8.2234	0.9927	4
57	0.1210	0.1219	8.2035	0.9927	3
58	0.1213	0.1222	8.1837	0.9926	2
59	0.1216	0.1225	8.1640	0.9926	1
60	0.1219	0.1228	8.1443	0.9925	0
	Cos	Cot	Tan	Sin	′

83°

7°

′	Sin	Tan	Cot	Cos	
0	0.1219	0.1228	8.1443	0.9925	60
1	0.1222	0.1231	8.1248	0.9925	59
2	0.1224	0.1234	8.1054	0.9925	58
3	0.1227	0.1237	8.0860	0.9924	57
4	0.1230	0.1240	8.0667	0.9924	56
5	0.1233	0.1243	8.0476	0.9924	55
6	0.1236	0.1246	8.0285	0.9923	54
7	0.1239	0.1249	8.0095	0.9923	53
8	0.1242	0.1251	7.9906	0.9923	52
9	0.1245	0.1254	7.9718	0.9922	51
10	0.1248	0.1257	7.9530	0.9922	50
11	0.1250	0.1260	7.9344	0.9922	49
12	0.1253	0.1263	7.9158	0.9921	48
13	0.1256	0.1266	7.8973	0.9921	47
14	0.1259	0.1269	7.8789	0.9920	46
15	0.1262	0.1272	7.8606	0.9920	45
16	0.1265	0.1275	7.8424	0.9920	44
17	0.1268	0.1278	7.8243	0.9919	43
18	0.1271	0.1281	7.8062	0.9919	42
19	0.1274	0.1284	7.7882	0.9919	41
20	0.1276	0.1287	7.7704	0.9918	40
21	0.1279	0.1290	7.7525	0.9918	39
22	0.1282	0.1293	7.7348	0.9917	38
23	0.1286	0.1296	7.7171	0.9917	37
24	0.1288	0.1299	7.6996	0.9917	36
25	0.1291	0.1302	7.6821	0.9916	35
26	0.1294	0.1305	7.6647	0.9916	34
27	0.1297	0.1308	7.6473	0.9916	33
28	0.1299	0.1311	7.6301	0.9915	32
29	0.1302	0.1314	7.6129	0.9915	31
30	0.1305	0.1317	7.5958	0.9914	30
31	0.1308	0.1319	7.5787	0.9914	29
32	0.1311	0.1322	7.5618	0.9914	28
33	0.1314	0.1325	7.5449	0.9913	27
34	0.1317	0.1328	7.5281	0.9913	26
35	0.1320	0.1331	7.5113	0.9913	25
36	0.1323	0.1334	7.4947	0.9912	24
37	0.1325	0.1337	7.4781	0.9912	23
38	0.1328	0.1340	7.4615	0.9911	22
39	0.1331	0.1343	7.4451	0.9911	21
40	0.1334	0.1346	7.4287	0.9911	20
41	0.1337	0.1349	7.4124	0.9910	19
42	0.1340	0.1352	7.3962	0.9910	18
43	0.1343	0.1355	7.3800	0.9909	17
44	0.1346	0.1358	7.3639	0.9909	16
45	0.1349	0.1361	7.3479	0.9909	15
46	0.1351	0.1364	7.3319	0.9908	14
47	0.1354	0.1367	7.3160	0.9908	13
48	0.1357	0.1370	7.3002	0.9907	12
49	0.1360	0.1373	7.2844	0.9907	11
50	0.1363	0.1376	7.2687	0.9907	10
51	0.1366	0.1379	7.2531	0.9906	9
52	0.1369	0.1382	7.2375	0.9906	8
53	0.1372	0.1385	7.2220	0.9905	7
54	0.1374	0.1388	7.2066	0.9905	6
55	0.1377	0.1391	7.1912	0.9905	5
56	0.1380	0.1394	7.1759	0.9904	4
57	0.1383	0.1397	7.1607	0.9904	3
58	0.1386	0.1399	7.1455	0.9903	2
59	0.1389	0.1402	7.1304	0.9903	1
60	0.1392	0.1405	7.1154	0.9903	0
	Cos	Cot	Tan	Sin	′

82°

8°

′	Sin	Tan	Cot	Cos	
0	0.1392	0.1405	7.1154	0.9903	60
1	0.1395	0.1408	7.1004	0.9902	59
2	0.1397	0.1411	7.0855	0.9902	58
3	0.1400	0.1414	7.0706	0.9901	57
4	0.1403	0.1417	7.0558	0.9901	56
5	0.1406	0.1420	7.0410	0.9901	55
6	0.1409	0.1423	7.0264	0.9900	54
7	0.1412	0.1426	7.0117	0.9900	53
8	0.1415	0.1429	6.9972	0.9899	52
9	0.1418	0.1432	6.9827	0.9899	51
10	0.1421	0.1435	6.9682	0.9899	50
11	0.1423	0.1438	6.9538	0.9898	49
12	0.1426	0.1441	6.9395	0.9898	48
13	0.1429	0.1444	6.9252	0.9897	47
14	0.1432	0.1447	6.9110	0.9897	46
15	0.1435	0.1450	6.8969	0.9897	45
16	0.1438	0.1453	6.8828	0.9896	44
17	0.1441	0.1456	6.8687	0.9896	43
18	0.1444	0.1459	6.8548	0.9895	42
19	0.1446	0.1462	6.8408	0.9895	41
20	0.1449	0.1465	6.8269	0.9894	40
21	0.1452	0.1468	6.8131	0.9894	39
22	0.1455	0.1471	6.7994	0.9894	38
23	0.1458	0.1474	6.7856	0.9893	37
24	0.1461	0.1477	6.7720	0.9893	36
25	0.1464	0.1480	6.7584	0.9892	35
26	0.1467	0.1483	6.7448	0.9892	34
27	0.1469	0.1486	6.7313	0.9891	33
28	0.1472	0.1489	6.7179	0.9891	32
29	0.1475	0.1492	6.7045	0.9891	31
30	0.1478	0.1495	6.6912	0.9890	30
31	0.1481	0.1497	6.6779	0.9890	29
32	0.1484	0.1500	6.6646	0.9889	28
33	0.1487	0.1503	6.6514	0.9889	27
34	0.1490	0.1506	6.6383	0.9888	26
35	0.1492	0.1509	6.6252	0.9888	25
36	0.1495	0.1512	6.6122	0.9888	24
37	0.1498	0.1515	6.5992	0.9887	23
38	0.1501	0.1518	6.5863	0.9887	22
39	0.1504	0.1521	6.5734	0.9886	21
40	0.1507	0.1524	6.5606	0.9886	20
41	0.1510	0.1527	6.5478	0.9885	19
42	0.1513	0.1530	6.5350	0.9885	18
43	0.1515	0.1533	6.5223	0.9884	17
44	0.1518	0.1536	6.5097	0.9884	16
45	0.1521	0.1539	6.4971	0.9884	15
46	0.1524	0.1542	6.4846	0.9883	14
47	0.1527	0.1545	6.4721	0.9883	13
48	0.1530	0.1548	6.4596	0.9882	12
49	0.1533	0.1551	6.4472	0.9882	11
50	0.1536	0.1554	6.4348	0.9881	10
51	0.1538	0.1557	6.4225	0.9881	9
52	0.1541	0.1560	6.4103	0.9880	8
53	0.1544	0.1563	6.3980	0.9880	7
54	0.1547	0.1566	6.3859	0.9880	6
55	0.1550	0.1569	6.3737	0.9879	5
56	0.1553	0.1572	6.3617	0.9879	4
57	0.1556	0.1575	6.3496	0.9878	3
58	0.1559	0.1578	6.3376	0.9878	2
59	0.1561	0.1581	6.3257	0.9877	1
60	0.1564	0.1584	6.3138	0.9877	0
	Cos	Cot	Tan	Sin	′

81°

9°

′	Sin	Tan	Cot	Cos	
0	0.1564	0.1584	6.3138	0.9877	60
1	0.1567	0.1587	6.3019	0.9876	59
2	0.1570	0.1590	6.2901	0.9876	58
3	0.1573	0.1593	6.2783	0.9876	57
4	0.1576	0.1596	6.2666	0.9875	56
5	0.1579	0.1599	6.2549	0.9875	55
6	0.1582	0.1602	6.2432	0.9874	54
7	0.1584	0.1605	6.2316	0.9874	53
8	0.1587	0.1608	6.2200	0.9873	52
9	0.1590	0.1611	6.2085	0.9873	51
10	0.1593	0.1614	6.1970	0.9872	50
11	0.1596	0.1617	6.1856	0.9872	49
12	0.1599	0.1620	6.1742	0.9871	48
13	0.1602	0.1623	6.1628	0.9871	47
14	0.1605	0.1626	6.1515	0.9870	46
15	0.1607	0.1629	6.1402	0.9870	45
16	0.1610	0.1632	6.1290	0.9869	44
17	0.1613	0.1635	6.1178	0.9869	43
18	0.1616	0.1638	6.1066	0.9869	42
19	0.1619	0.1641	6.0955	0.9868	41
20	0.1622	0.1644	6.0844	0.9868	40
21	0.1625	0.1647	6.0734	0.9867	39
22	0.1628	0.1650	6.0624	0.9867	38
23	0.1630	0.1653	6.0514	0.9866	37
24	0.1633	0.1655	6.0405	0.9866	36
25	0.1636	0.1658	6.0296	0.9865	35
26	0.1639	0.1661	6.0188	0.9865	34
27	0.1642	0.1664	6.0080	0.9864	33
28	0.1645	0.1667	5.9972	0.9864	32
29	0.1648	0.1670	5.9865	0.9863	31
30	0.1650	0.1673	5.9758	0.9863	30
31	0.1653	0.1676	5.9651	0.9862	29
32	0.1656	0.1679	5.9545	0.9862	28
33	0.1659	0.1682	5.9439	0.9861	27
34	0.1662	0.1685	5.9333	0.9861	26
35	0.1665	0.1688	5.9228	0.9860	25
36	0.1668	0.1691	5.9124	0.9860	24
37	0.1671	0.1694	5.9019	0.9859	23
38	0.1673	0.1697	5.8915	0.9859	22
39	0.1676	0.1700	5.8811	0.9859	21
40	0.1679	0.1703	5.8708	0.9858	20
41	0.1682	0.1706	5.8605	0.9858	19
42	0.1685	0.1709	5.8502	0.9857	18
43	0.1688	0.1712	5.8400	0.9857	17
44	0.1691	0.1715	5.8298	0.9856	16
45	0.1693	0.1718	5.8197	0.9856	15
46	0.1696	0.1721	5.8095	0.9855	14
47	0.1699	0.1724	5.7994	0.9855	13
48	0.1702	0.1727	5.7894	0.9854	12
49	0.1705	0.1730	5.7794	0.9854	11
50	0.1708	0.1733	5.7694	0.9853	10
51	0.1711	0.1736	5.7594	0.9853	9
52	0.1714	0.1739	5.7495	0.9852	8
53	0.1716	0.1742	5.7396	0.9852	7
54	0.1719	0.1745	5.7297	0.9851	6
55	0.1722	0.1748	5.7199	0.9851	5
56	0.1725	0.1751	5.7101	0.9850	4
57	0.1728	0.1754	5.7004	0.9850	3
58	0.1731	0.1757	5.6906	0.9849	2
59	0.1734	0.1760	5.6809	0.9849	1
60	0.1736	0.1763	5.6713	0.9848	0
	Cos	Cot	Tan	Sin	′

80°

III. Natural Trigonometric Functions

10°

′	Sin	Tan	Cot	Cos	
0	0.1736	0.1763	5.6713	0.9848	60
1	0.1739	0.1766	5.6617	0.9848	59
2	0.1742	0.1769	5.6521	0.9847	58
3	0.1745	0.1772	5.6425	0.9847	57
4	0.1748	0.1775	5.6329	0.9846	56
5	0.1751	0.1778	5.6234	0.9846	55
6	0.1754	0.1781	5.6140	0.9845	54
7	0.1757	0.1784	5.6045	0.9845	53
8	0.1759	0.1787	5.5951	0.9844	52
9	0.1762	0.1790	5.5857	0.9843	51
10	0.1765	0.1793	5.5764	0.9843	50
11	0.1768	0.1796	5.5671	0.9842	49
12	0.1771	0.1799	5.5578	0.9842	48
13	0.1774	0.1802	5.5485	0.9841	47
14	0.1777	0.1805	5.5393	0.9841	46
15	0.1779	0.1808	5.5301	0.9840	45
16	0.1782	0.1811	5.5209	0.9840	44
17	0.1785	0.1814	5.5118	0.9839	43
18	0.1788	0.1817	5.5026	0.9839	42
19	0.1791	0.1820	5.4936	0.9838	41
20	0.1794	0.1823	5.4845	0.9838	40
21	0.1797	0.1826	5.4755	0.9837	39
22	0.1799	0.1829	5.4665	0.9837	38
23	0.1802	0.1832	5.4575	0.9836	37
24	0.1805	0.1835	5.4486	0.9836	36
25	0.1808	0.1838	5.4397	0.9835	35
26	0.1811	0.1841	5.4308	0.9835	34
27	0.1814	0.1844	5.4219	0.9834	33
28	0.1817	0.1847	5.4131	0.9834	32
29	0.1819	0.1850	5.4043	0.9833	31
30	0.1822	0.1853	5.3955	0.9833	30
31	0.1825	0.1856	5.3868	0.9832	29
32	0.1828	0.1859	5.3781	0.9831	28
33	0.1831	0.1862	5.3694	0.9831	27
34	0.1834	0.1865	5.3607	0.9830	26
35	0.1837	0.1868	5.3521	0.9830	25
36	0.1840	0.1871	5.3435	0.9829	24
37	0.1842	0.1874	5.3349	0.9829	23
38	0.1845	0.1877	5.3263	0.9828	22
39	0.1848	0.1880	5.3178	0.9828	21
40	0.1851	0.1883	5.3093	0.9827	20
41	0.1854	0.1887	5.3008	0.9827	19
42	0.1857	0.1890	5.2924	0.9826	18
43	0.1860	0.1893	5.2839	0.9826	17
44	0.1862	0.1896	5.2755	0.9825	16
45	0.1865	0.1899	5.2672	0.9825	15
46	0.1868	0.1902	5.2588	0.9824	14
47	0.1871	0.1905	5.2505	0.9823	13
48	0.1874	0.1908	5.2422	0.9823	12
49	0.1877	0.1911	5.2339	0.9822	11
50	0.1880	0.1914	5.2257	0.9822	10
51	0.1882	0.1917	5.2174	0.9821	9
52	0.1885	0.1920	5.2092	0.9821	8
53	0.1888	0.1923	5.2011	0.9820	7
54	0.1891	0.1926	5.1929	0.9820	6
55	0.1894	0.1929	5.1848	0.9819	5
56	0.1897	0.1932	5.1767	0.9818	4
57	0.1900	0.1935	5.1686	0.9818	3
58	0.1902	0.1938	5.1606	0.9817	2
59	0.1905	0.1941	5.1526	0.9817	1
60	0.1908	0.1944	5.1446	0.9816	0
	Cos	Cot	Tan	Sin	′

79°

11°

′	Sin	Tan	Cot	Cos	
0	0.1908	0.1944	5.1446	0.9816	60
1	0.1911	0.1947	5.1366	0.9816	59
2	0.1914	0.1950	5.1286	0.9815	58
3	0.1917	0.1953	5.1207	0.9815	57
4	0.1920	0.1956	5.1128	0.9814	56
5	0.1922	0.1959	5.1049	0.9813	55
6	0.1925	0.1962	5.0970	0.9813	54
7	0.1928	0.1965	5.0892	0.9812	53
8	0.1931	0.1968	5.0814	0.9812	52
9	0.1934	0.1971	5.0736	0.9811	51
10	0.1937	0.1974	5.0658	0.9811	50
11	0.1939	0.1977	5.0581	0.9810	49
12	0.1942	0.1980	5.0504	0.9810	48
13	0.1945	0.1983	5.0427	0.9809	47
14	0.1948	0.1986	5.0350	0.9808	46
15	0.1951	0.1989	5.0273	0.9808	45
16	0.1954	0.1992	5.0197	0.9807	44
17	0.1957	0.1995	5.0121	0.9807	43
18	0.1959	0.1998	5.0045	0.9806	42
19	0.1962	0.2001	4.9969	0.9806	41
20	0.1965	0.2004	4.9894	0.9805	40
21	0.1968	0.2007	4.9819	0.9804	39
22	0.1971	0.2010	4.9744	0.9804	38
23	0.1974	0.2013	4.9669	0.9803	37
24	0.1977	0.2016	4.9594	0.9803	36
25	0.1979	0.2019	4.9520	0.9802	35
26	0.1982	0.2022	4.9446	0.9802	34
27	0.1985	0.2025	4.9372	0.9801	33
28	0.1988	0.2028	4.9298	0.9800	32
29	0.1991	0.2031	4.9225	0.9800	31
30	0.1994	0.2035	4.9152	0.9799	30
31	0.1997	0.2038	4.9078	0.9799	29
32	0.1999	0.2041	4.9006	0.9798	28
33	0.2002	0.2044	4.8933	0.9798	27
34	0.2005	0.2047	4.8860	0.9797	26
35	0.2008	0.2050	4.8788	0.9796	25
36	0.2011	0.2053	4.8716	0.9796	24
37	0.2014	0.2056	4.8644	0.9795	23
38	0.2016	0.2059	4.8573	0.9795	22
39	0.2019	0.2062	4.8501	0.9794	21
40	0.2022	0.2065	4.8430	0.9793	20
41	0.2025	0.2068	4.8359	0.9793	19
42	0.2028	0.2071	4.8288	0.9792	18
43	0.2031	0.2074	4.8218	0.9792	17
44	0.2034	0.2077	4.8147	0.9791	16
45	0.2036	0.2080	4.8077	0.9790	15
46	0.2039	0.2083	4.8007	0.9790	14
47	0.2042	0.2086	4.7937	0.9789	13
48	0.2045	0.2089	4.7867	0.9789	12
49	0.2048	0.2092	4.7798	0.9788	11
50	0.2051	0.2095	4.7729	0.9787	10
51	0.2054	0.2098	4.7659	0.9787	9
52	0.2056	0.2101	4.7591	0.9786	8
53	0.2059	0.2104	4.7522	0.9786	7
54	0.2062	0.2107	4.7453	0.9785	6
55	0.2065	0.2110	4.7385	0.9784	5
56	0.2068	0.2113	4.7317	0.9784	4
57	0.2071	0.2116	4.7249	0.9783	3
58	0.2073	0.2119	4.7181	0.9783	2
59	0.2076	0.2123	4.7114	0.9782	1
60	0.2079	0.2126	4.7046	0.9781	0
	Cos	Cot	Tan	Sin	′

78°

12°

′	Sin	Tan	Cot	Cos	
0	0.2079	0.2126	4.7046	0.9781	60
1	0.2082	0.2129	4.6979	0.9781	59
2	0.2085	0.2132	4.6912	0.9780	58
3	0.2088	0.2135	4.6845	0.9780	57
4	0.2090	0.2138	4.6779	0.9779	56
5	0.2093	0.2141	4.6712	0.9778	55
6	0.2096	0.2144	4.6646	0.9778	54
7	0.2099	0.2147	4.6580	0.9777	53
8	0.2102	0.2150	4.6514	0.9777	52
9	0.2105	0.2153	4.6448	0.9776	51
10	0.2108	0.2156	4.6382	0.9775	50
11	0.2110	0.2159	4.6317	0.9775	49
12	0.2113	0.2162	4.6252	0.9774	48
13	0.2116	0.2165	4.6187	0.9774	47
14	0.2119	0.2168	4.6122	0.9773	46
15	0.2122	0.2171	4.6057	0.9772	45
16	0.2125	0.2174	4.5993	0.9772	44
17	0.2127	0.2177	4.5928	0.9771	43
18	0.2130	0.2180	4.5864	0.9770	42
19	0.2133	0.2183	4.5800	0.9770	41
20	0.2136	0.2186	4.5736	0.9769	40
21	0.2139	0.2189	4.5673	0.9769	39
22	0.2142	0.2193	4.5609	0.9768	38
23	0.2145	0.2196	4.5546	0.9767	37
24	0.2147	0.2199	4.5483	0.9767	36
25	0.2150	0.2202	4.5420	0.9766	35
26	0.2153	0.2205	4.5357	0.9765	34
27	0.2156	0.2208	4.5294	0.9765	33
28	0.2159	0.2211	4.5232	0.9764	32
29	0.2162	0.2214	4.5169	0.9764	31
30	0.2164	0.2217	4.5107	0.9763	30
31	0.2167	0.2220	4.5045	0.9762	29
32	0.2170	0.2223	4.4983	0.9762	28
33	0.2173	0.2226	4.4922	0.9761	27
34	0.2176	0.2229	4.4860	0.9760	26
35	0.2179	0.2232	4.4799	0.9760	25
36	0.2181	0.2235	4.4737	0.9759	24
37	0.2184	0.2238	4.4676	0.9759	23
38	0.2187	0.2241	4.4615	0.9758	22
39	0.2190	0.2244	4.4555	0.9757	21
40	0.2193	0.2247	4.4494	0.9757	20
41	0.2196	0.2251	4.4434	0.9756	19
42	0.2198	0.2254	4.4373	0.9755	18
43	0.2201	0.2257	4.4313	0.9755	17
44	0.2204	0.2260	4.4253	0.9754	16
45	0.2207	0.2263	4.4194	0.9753	15
46	0.2210	0.2266	4.4134	0.9753	14
47	0.2213	0.2269	4.4075	0.9752	13
48	0.2215	0.2272	4.4015	0.9751	12
49	0.2218	0.2275	4.3956	0.9751	11
50	0.2221	0.2278	4.3897	0.9750	10
51	0.2224	0.2281	4.3838	0.9750	9
52	0.2227	0.2284	4.3779	0.9749	8
53	0.2230	0.2287	4.3721	0.9748	7
54	0.2233	0.2290	4.3662	0.9748	6
55	0.2235	0.2293	4.3604	0.9747	5
56	0.2238	0.2296	4.3546	0.9746	4
57	0.2241	0.2299	4.3488	0.9746	3
58	0.2244	0.2303	4.3430	0.9745	2
59	0.2247	0.2306	4.3372	0.9744	1
60	0.2250	0.2309	4.3315	0.9744	0
	Cos	Cot	Tan	Sin	′

77°

13°

′	Sin	Tan	Cot	Cos	
0	0.2250	0.2309	4.3315	0.9744	60
1	0.2252	0.2312	4.3257	0.9743	59
2	0.2255	0.2315	4.3200	0.9742	58
3	0.2258	0.2318	4.3143	0.9742	57
4	0.2261	0.2321	4.3086	0.9741	56
5	0.2264	0.2324	4.3029	0.9740	55
6	0.2267	0.2327	4.2972	0.9740	54
7	0.2269	0.2330	4.2916	0.9739	53
8	0.2272	0.2333	4.2859	0.9738	52
9	0.2275	0.2336	4.2803	0.9738	51
10	0.2278	0.2339	4.2747	0.9737	50
11	0.2281	0.2342	4.2691	0.9736	49
12	0.2284	0.2345	4.2635	0.9736	48
13	0.2286	0.2349	4.2580	0.9735	47
14	0.2289	0.2352	4.2524	0.9734	46
15	0.2292	0.2355	4.2468	0.9734	45
16	0.2295	0.2358	4.2413	0.9733	44
17	0.2298	0.2361	4.2358	0.9732	43
18	0.2300	0.2364	4.2303	0.9732	42
19	0.2303	0.2367	4.2248	0.9731	41
20	0.2306	0.2370	4.2193	0.9730	40
21	0.2309	0.2373	4.2139	0.9730	39
22	0.2312	0.2376	4.2084	0.9729	38
23	0.2315	0.2379	4.2030	0.9728	37
24	0.2317	0.2382	4.1976	0.9728	36
25	0.2320	0.2385	4.1922	0.9727	35
26	0.2323	0.2388	4.1868	0.9726	34
27	0.2326	0.2392	4.1814	0.9726	33
28	0.2329	0.2395	4.1760	0.9725	32
29	0.2332	0.2398	4.1706	0.9724	31
30	0.2334	0.2401	4.1653	0.9724	30
31	0.2337	0.2404	4.1600	0.9723	29
32	0.2340	0.2407	4.1547	0.9722	28
33	0.2343	0.2410	4.1493	0.9722	27
34	0.2346	0.2413	4.1441	0.9721	26
35	0.2349	0.2416	4.1388	0.9720	25
36	0.2351	0.2419	4.1335	0.9720	24
37	0.2354	0.2422	4.1282	0.9719	23
38	0.2357	0.2425	4.1230	0.9718	22
39	0.2360	0.2428	4.1178	0.9718	21
40	0.2363	0.2432	4.1126	0.9717	20
41	0.2366	0.2435	4.1074	0.9716	19
42	0.2368	0.2438	4.1022	0.9715	18
43	0.2371	0.2441	4.0970	0.9715	17
44	0.2374	0.2444	4.0918	0.9714	16
45	0.2377	0.2447	4.0867	0.9713	15
46	0.2380	0.2450	4.0815	0.9713	14
47	0.2383	0.2453	4.0764	0.9712	13
48	0.2385	0.2456	4.0713	0.9711	12
49	0.2388	0.2459	4.0662	0.9711	11
50	0.2391	0.2462	4.0611	0.9710	10
51	0.2394	0.2465	4.0560	0.9709	9
52	0.2397	0.2469	4.0509	0.9709	8
53	0.2399	0.2472	4.0459	0.9708	7
54	0.2402	0.2475	4.0408	0.9707	6
55	0.2405	0.2478	4.0358	0.9706	5
56	0.2408	0.2481	4.0308	0.9706	4
57	0.2411	0.2484	4.0257	0.9705	3
58	0.2414	0.2487	4.0207	0.9704	2
59	0.2416	0.2490	4.0158	0.9704	1
60	0.2419	0.2493	4.0108	0.9703	0
	Cos	Cot	Tan	Sin	′

76°

14°

′	Sin	Tan	Cot	Cos	
0	0.2419	0.2493	4.0108	0.9703	60
1	0.2422	0.2496	4.0058	0.9702	59
2	0.2425	0.2499	4.0009	0.9702	58
3	0.2428	0.2503	3.9959	0.9701	57
4	0.2431	0.2506	3.9910	0.9700	56
5	0.2433	0.2509	3.9861	0.9699	55
6	0.2436	0.2512	3.9812	0.9699	54
7	0.2439	0.2515	3.9763	0.9698	53
8	0.2442	0.2518	3.9714	0.9697	52
9	0.2445	0.2521	3.9665	0.9697	51
10	0.2447	0.2524	3.9617	0.9696	50
11	0.2450	0.2527	3.9568	0.9695	49
12	0.2453	0.2530	3.9520	0.9694	48
13	0.2456	0.2533	3.9471	0.9694	47
14	0.2459	0.2537	3.9423	0.9693	46
15	0.2462	0.2540	3.9375	0.9692	45
16	0.2464	0.2543	3.9327	0.9692	44
17	0.2467	0.2546	3.9279	0.9691	43
18	0.2470	0.2549	3.9232	0.9690	42
19	0.2473	0.2552	3.9184	0.9689	41
20	0.2476	0.2555	3.9136	0.9689	40
21	0.2478	0.2558	3.9089	0.9688	39
22	0.2481	0.2561	3.9042	0.9687	38
23	0.2484	0.2564	3.8995	0.9687	37
24	0.2487	0.2568	3.8947	0.9686	36
25	0.2490	0.2571	3.8900	0.9685	35
26	0.2493	0.2574	3.8854	0.9684	34
27	0.2495	0.2577	3.8807	0.9684	33
28	0.2498	0.2580	3.8760	0.9683	32
29	0.2501	0.2583	3.8714	0.9682	31
30	0.2504	0.2586	3.8667	0.9681	30
31	0.2507	0.2589	3.8621	0.9681	29
32	0.2509	0.2592	3.8575	0.9680	28
33	0.2512	0.2595	3.8528	0.9679	27
34	0.2515	0.2599	3.8482	0.9679	26
35	0.2518	0.2602	3.8436	0.9678	25
36	0.2521	0.2605	3.8391	0.9677	24
37	0.2524	0.2608	3.8345	0.9676	23
38	0.2526	0.2611	3.8299	0.9676	22
39	0.2529	0.2614	3.8254	0.9675	21
40	0.2532	0.2617	3.8208	0.9674	20
41	0.2535	0.2620	3.8163	0.9673	19
42	0.2538	0.2623	3.8118	0.9673	18
43	0.2540	0.2627	3.8073	0.9672	17
44	0.2543	0.2630	3.8028	0.9671	16
45	0.2546	0.2633	3.7983	0.9670	15
46	0.2549	0.2636	3.7938	0.9670	14
47	0.2552	0.2639	3.7893	0.9669	13
48	0.2554	0.2642	3.7848	0.9668	12
49	0.2557	0.2645	3.7804	0.9667	11
50	0.2560	0.2648	3.7760	0.9667	10
51	0.2563	0.2651	3.7715	0.9666	9
52	0.2566	0.2655	3.7671	0.9665	8
53	0.2569	0.2658	3.7627	0.9665	7
54	0.2571	0.2661	3.7583	0.9664	6
55	0.2574	0.2664	3.7539	0.9663	5
56	0.2577	0.2667	3.7495	0.9662	4
57	0.2580	0.2670	3.7451	0.9662	3
58	0.2583	0.2673	3.7408	0.9661	2
59	0.2585	0.2676	3.7364	0.9660	1
60	0.2588	0.2679	3.7321	0.9659	0
	Cos	Cot	Tan	Sin	′

75°

15°

′	Sin	Tan	Cot	Cos	
0	0.2588	0.2679	3.7321	0.9659	60
1	0.2591	0.2683	3.7277	0.9659	59
2	0.2594	0.2686	3.7234	0.9658	58
3	0.2597	0.2689	3.7191	0.9657	57
4	0.2599	0.2692	3.7148	0.9656	56
5	0.2602	0.2695	3.7105	0.9655	55
6	0.2605	0.2698	3.7062	0.9655	54
7	0.2608	0.2701	3.7019	0.9654	53
8	0.2611	0.2704	3.6976	0.9653	52
9	0.2613	0.2708	3.6933	0.9652	51
10	0.2616	0.2711	3.6891	0.9652	50
11	0.2619	0.2714	3.6848	0.9651	49
12	0.2622	0.2717	3.6806	0.9650	48
13	0.2625	0.2720	3.6764	0.9649	47
14	0.2628	0.2723	3.6722	0.9649	46
15	0.2630	0.2726	3.6680	0.9648	45
16	0.2633	0.2729	3.6638	0.9647	44
17	0.2636	0.2733	3.6596	0.9646	43
18	0.2639	0.2736	3.6554	0.9646	42
19	0.2642	0.2739	3.6512	0.9645	41
20	0.2644	0.2742	3.6470	0.9644	40
21	0.2647	0.2745	3.6429	0.9643	39
22	0.2650	0.2748	3.6387	0.9642	38
23	0.2653	0.2751	3.6346	0.9642	37
24	0.2656	0.2754	3.6305	0.9641	36
25	0.2658	0.2758	3.6264	0.9640	35
26	0.2661	0.2761	3.6222	0.9639	34
27	0.2664	0.2764	3.6181	0.9639	33
28	0.2667	0.2767	3.6140	0.9638	32
29	0.2670	0.2770	3.6100	0.9637	31
30	0.2672	0.2773	3.6059	0.9636	30
31	0.2675	0.2776	3.6018	0.9636	29
32	0.2678	0.2780	3.5978	0.9635	28
33	0.2681	0.2783	3.5937	0.9634	27
34	0.2684	0.2786	3.5897	0.9633	26
35	0.2686	0.2789	3.5856	0.9632	25
36	0.2689	0.2792	3.5816	0.9632	24
37	0.2692	0.2795	3.5776	0.9631	23
38	0.2695	0.2798	3.5736	0.9630	22
39	0.2698	0.2801	3.5696	0.9629	21
40	0.2700	0.2805	3.5656	0.9628	20
41	0.2703	0.2808	3.5616	0.9628	19
42	0.2706	0.2811	3.5576	0.9627	18
43	0.2709	0.2814	3.5536	0.9626	17
44	0.2712	0.2817	3.5497	0.9625	16
45	0.2714	0.2820	3.5457	0.9625	15
46	0.2717	0.2823	3.5418	0.9624	14
47	0.2720	0.2827	3.5379	0.9623	13
48	0.2723	0.2830	3.5339	0.9622	12
49	0.2726	0.2833	3.5300	0.9621	11
50	0.2728	0.2836	3.5261	0.9621	10
51	0.2731	0.2839	3.5222	0.9620	9
52	0.2734	0.2842	3.5183	0.9619	8
53	0.2737	0.2845	3.5144	0.9618	7
54	0.2740	0.2849	3.5105	0.9617	6
55	0.2742	0.2852	3.5067	0.9617	5
56	0.2745	0.2855	3.5028	0.9616	4
57	0.2748	0.2858	3.4989	0.9615	3
58	0.2751	0.2861	3.4951	0.9614	2
59	0.2754	0.2864	3.4912	0.9613	1
60	0.2756	0.2867	3.4874	0.9613	0
	Cos	Cot	Tan	Sin	′

74°

16°

′	Sin	Tan	Cot	Cos	
0	0.2756	0.2867	3.4874	0.9613	60
1	0.2759	0.2871	3.4836	0.9612	59
2	0.2762	0.2874	3.4798	0.9611	58
3	0.2765	0.2877	3.4760	0.9610	57
4	0.2768	0.2880	3.4722	0.9609	56
5	0.2770	0.2883	3.4684	0.9609	55
6	0.2773	0.2886	3.4646	0.9608	54
7	0.2776	0.2890	3.4608	0.9607	53
8	0.2779	0.2893	3.4570	0.9606	52
9	0.2782	0.2896	3.4533	0.9605	51
10	0.2784	0.2899	3.4495	0.9605	50
11	0.2787	0.2902	3.4458	0.9604	49
12	0.2790	0.2905	3.4420	0.9603	48
13	0.2793	0.2908	3.4383	0.9602	47
14	0.2795	0.2912	3.4346	0.9601	46
15	0.2798	0.2915	3.4308	0.9600	45
16	0.2801	0.2918	3.4271	0.9600	44
17	0.2804	0.2921	3.4234	0.9599	43
18	0.2807	0.2924	3.4197	0.9598	42
19	0.2809	0.2927	3.4160	0.9597	41
20	0.2812	0.2931	3.4124	0.9596	40
21	0.2815	0.2934	3.4087	0.9596	39
22	0.2818	0.2937	3.4050	0.9595	38
23	0.2821	0.2940	3.4014	0.9594	37
24	0.2823	0.2943	3.3977	0.9593	36
25	0.2826	0.2946	3.3941	0.9592	35
26	0.2829	0.2949	3.3904	0.9591	34
27	0.2832	0.2953	3.3868	0.9591	33
28	0.2835	0.2956	3.3832	0.9590	32
29	0.2837	0.2959	3.3796	0.9589	31
30	0.2840	0.2962	3.3759	0.9588	30
31	0.2843	0.2965	3.3723	0.9587	29
32	0.2846	0.2968	3.3687	0.9587	28
33	0.2849	0.2972	3.3652	0.9586	27
34	0.2851	0.2975	3.3616	0.9585	26
35	0.2854	0.2978	3.3580	0.9584	25
36	0.2857	0.2981	3.3544	0.9583	24
37	0.2860	0.2984	3.3509	0.9582	23
38	0.2862	0.2987	3.3473	0.9582	22
39	0.2865	0.2991	3.3438	0.9581	21
40	0.2868	0.2994	3.3402	0.9580	20
41	0.2871	0.2997	3.3367	0.9579	19
42	0.2874	0.3000	3.3332	0.9578	18
43	0.2876	0.3003	3.3297	0.9577	17
44	0.2879	0.3006	3.3261	0.9577	16
45	0.2882	0.3010	3.3226	0.9576	15
46	0.2885	0.3013	3.3191	0.9575	14
47	0.2888	0.3016	3.3156	0.9574	13
48	0.2890	0.3019	3.3122	0.9573	12
49	0.2893	0.3022	3.3087	0.9572	11
50	0.2896	0.3026	3.3052	0.9572	10
51	0.2899	0.3029	3.3017	0.9571	9
52	0.2901	0.3032	3.2983	0.9570	8
53	0.2904	0.3035	3.2948	0.9560	7
54	0.2907	0.3038	3.2914	0.9568	6
55	0.2910	0.3041	3.2879	0.9567	5
56	0.2913	0.3045	3.2845	0.9566	4
57	0.2915	0.3048	3.2811	0.9566	3
58	0.2918	0.3051	3.2777	0.9565	2
59	0.2921	0.3054	3.2743	0.9564	1
60	0.2924	0.3057	3.2709	0.9563	0
	Cos	Cot	Tan	Sin	′

73°

17°

′	Sin	Tan	Cot	Cos	
0	0.2924	0.3057	3.2709	0.9563	60
1	0.2926	0.3060	3.2675	0.9562	59
2	0.2929	0.3064	3.2641	0.9561	58
3	0.2932	0.3067	3.2607	0.9560	57
4	0.2935	0.3070	3.2573	0.9560	56
5	0.2938	0.3073	3.2539	0.9559	55
6	0.2940	0.3076	3.2506	0.9558	54
7	0.2943	0.3080	3.2472	0.9557	53
8	0.2946	0.3083	3.2438	0.9556	52
9	0.2949	0.3086	3.2405	0.9555	51
10	0.2952	0.3089	3.2371	0.9555	50
11	0.2954	0.3092	3.2338	0.9554	49
12	0.2957	0.3096	3.2305	0.9553	48
13	0.2960	0.3099	3.2272	0.9552	47
14	0.2963	0.3102	3.2238	0.9551	46
15	0.2965	0.3105	3.2205	0.9550	45
16	0.2968	0.3108	3.2172	0.9549	44
17	0.2971	0.3111	3.2139	0.9548	43
18	0.2974	0.3115	3.2106	0.9548	42
19	0.2977	0.3118	3.2073	0.9547	41
20	0.2979	0.3121	3.2041	0.9546	40
21	0.2982	0.3124	3.2008	0.9545	39
22	0.2985	0.3127	3.1975	0.9544	38
23	0.2988	0.3131	3.1943	0.9543	37
24	0.2990	0.3134	3.1910	0.9542	36
25	0.2993	0.3137	3.1878	0.9542	35
26	0.2996	0.3140	3.1845	0.9541	34
27	0.2999	0.3143	3.1813	0.9540	33
28	0.3002	0.3147	3.1780	0.9539	32
29	0.3004	0.3150	3.1748	0.9538	31
30	0.3007	0.3153	3.1716	0.9537	30
31	0.3010	0.3156	3.1684	0.9536	29
32	0.3013	0.3159	3.1652	0.9535	28
33	0.3015	0.3163	3.1620	0.9535	27
34	0.3018	0.3166	3.1588	0.9534	26
35	0.3021	0.3169	3.1556	0.9533	25
36	0.3024	0.3172	3.1524	0.9532	24
37	0.3026	0.3175	3.1492	0.9531	23
38	0.3029	0.3179	3.1460	0.9530	22
39	0.3032	0.3182	3.1429	0.9529	21
40	0.3035	0.3185	3.1397	0.9528	20
41	0.3038	0.3188	3.1366	0.9527	19
42	0.3040	0.3191	3.1334	0.9527	18
43	0.3043	0.3195	3.1303	0.9526	17
44	0.3046	0.3198	3.1271	0.9525	16
45	0.3049	0.3201	3.1240	0.9524	15
46	0.3051	0.3204	3.1209	0.9523	14
47	0.3054	0.3207	3.1178	0.9522	13
48	0.3057	0.3211	3.1146	0.9521	12
49	0.3060	0.3214	3.1115	0.9520	11
50	0.3062	0.3217	3.1084	0.9520	10
51	0.3065	0.3220	3.1053	0.9519	9
52	0.3068	0.3223	3.1022	0.9518	8
53	0.3071	0.3227	3.0991	0.9517	7
54	0.3074	0.3230	3.0961	0.9516	6
55	0.3076	0.3233	3.0930	0.9515	5
56	0.3079	0.3236	3.0899	0.9514	4
57	0.3082	0.3240	3.0868	0.9513	3
58	0.3085	0.3243	3.0838	0.9512	2
59	0.3087	0.3246	3.0807	0.9511	1
60	0.3090	0.3249	3.0777	0.9511	0
	Cos	Cot	Tan	Sin	′

72°

III. Natural Trigonometric Functions

18°

′	Sin	Tan	Cot	Cos	
0	0.3090	0.3249	3.0777	0.9511	60
1	0.3093	0.3252	3.0746	0.9510	59
2	0.3096	0.3256	3.0716	0.9509	58
3	0.3098	0.3259	3.0686	0.9508	57
4	0.3101	0.3262	3.0655	0.9507	56
5	0.3104	0.3265	3.0625	0.9506	55
6	0.3107	0.3269	3.0595	0.9505	54
7	0.3110	0.3272	3.0565	0.9504	53
8	0.3112	0.3275	3.0535	0.9503	52
9	0.3115	0.3278	3.0505	0.9502	51
10	0.3118	0.3281	3.0475	0.9502	50
11	0.3121	0.3285	3.0445	0.9501	49
12	0.3123	0.3288	3.0415	0.9500	48
13	0.3126	0.3291	3.0385	0.9499	47
14	0.3129	0.3294	3.0356	0.9498	46
15	0.3132	0.3298	3.0326	0.9497	45
16	0.3134	0.3301	3.0296	0.9496	44
17	0.3137	0.3304	3.0267	0.9495	43
18	0.3140	0.3307	3.0237	0.9494	42
19	0.3143	0.3310	3.0208	0.9493	41
20	0.3145	0.3314	3.0178	0.9492	40
21	0.3148	0.3317	3.0149	0.9492	39
22	0.3151	0.3320	3.0120	0.9491	38
23	0.3154	0.3323	3.0090	0.9490	37
24	0.3156	0.3327	3.0061	0.9489	36
25	0.3159	0.3330	3.0032	0.9488	35
26	0.3162	0.3333	3.0003	0.9487	34
27	0.3165	0.3336	2.9974	0.9486	33
28	0.3168	0.3339	2.9945	0.9485	32
29	0.3170	0.3343	2.9916	0.9484	31
30	0.3173	0.3346	2.9887	0.9483	30
31	0.3176	0.3349	2.9858	0.9482	29
32	0.3179	0.3352	2.9829	0.9481	28
33	0.3181	0.3356	2.9800	0.9480	27
34	0.3184	0.3359	2.9772	0.9480	26
35	0.3187	0.3362	2.9743	0.9479	25
36	0.3190	0.3365	2.9714	0.9478	24
37	0.3192	0.3369	2.9686	0.9477	23
38	0.3195	0.3372	2.9657	0.9476	22
39	0.3198	0.3375	2.9629	0.9475	21
40	0.3201	0.3378	2.9600	0.9474	20
41	0.3203	0.3382	2.9572	0.9473	19
42	0.3206	0.3385	2.9544	0.9472	18
43	0.3209	0.3388	2.9515	0.9471	17
44	0.3212	0.3391	2.9487	0.9470	16
45	0.3214	0.3395	2.9459	0.9469	15
46	0.3217	0.3398	2.9431	0.9468	14
47	0.3220	0.3401	2.9403	0.9467	13
48	0.3223	0.3404	2.9375	0.9466	12
49	0.3225	0.3408	2.9347	0.9466	11
50	0.3228	0.3411	2.9319	0.9465	10
51	0.3231	0.3414	2.9291	0.9464	9
52	0.3234	0.3417	2.9263	0.9463	8
53	0.3236	0.3421	2.9235	0.9462	7
54	0.3239	0.3424	2.9208	0.9461	6
55	0.3242	0.3427	2.9180	0.9460	5
56	0.3245	0.3430	2.9152	0.9459	4
57	0.3247	0.3434	2.9125	0.9458	3
58	0.3250	0.3437	2.9097	0.9457	2
59	0.3253	0.3440	2.9070	0.9456	1
60	0.3256	0.3443	2.9042	0.9455	0
	Cos	Cot	Tan	Sin	′

71°

19°

′	Sin	Tan	Cot	Cos	
0	0.3256	0.3443	2.9042	0.9455	60
1	0.3258	0.3447	2.9015	0.9454	59
2	0.3261	0.3450	2.8987	0.9453	58
3	0.3264	0.3453	2.8960	0.9452	57
4	0.3267	0.3456	2.8933	0.9451	56
5	0.3269	0.3460	2.8905	0.9450	55
6	0.3272	0.3463	2.8878	0.9449	54
7	0.3275	0.3466	2.8851	0.9449	53
8	0.3278	0.3469	2.8824	0.9448	52
9	0.3280	0.3473	2.8797	0.9447	51
10	0.3283	0.3476	2.8770	0.9446	50
11	0.3286	0.3479	2.8743	0.9445	49
12	0.3289	0.3482	2.8716	0.9444	48
13	0.3291	0.3486	2.8689	0.9443	47
14	0.3294	0.3489	2.8662	0.9442	46
15	0.3297	0.3492	2.8636	0.9441	45
16	0.3300	0.3495	2.8609	0.9440	44
17	0.3302	0.3499	2.8582	0.9439	43
18	0.3305	0.3502	2.8556	0.9438	42
19	0.3308	0.3505	2.8529	0.9437	41
20	0.3311	0.3508	2.8502	0.9436	40
21	0.3313	0.3512	2.8476	0.9435	39
22	0.3316	0.3515	2.8449	0.9434	38
23	0.3319	0.3518	2.8423	0.9433	37
24	0.3322	0.3522	2.8397	0.9432	36
25	0.3324	0.3525	2.8370	0.9431	35
26	0.3327	0.3528	2.8344	0.9430	34
27	0.3330	0.3531	2.8318	0.9429	33
28	0.3333	0.3535	2.8291	0.9428	32
29	0.3335	0.3538	2.8265	0.9427	31
30	0.3338	0.3541	2.8239	0.9426	30
31	0.3341	0.3544	2.8213	0.9425	29
32	0.3344	0.3548	2.8187	0.9424	28
33	0.3346	0.3551	2.8161	0.9423	27
34	0.3349	0.3554	2.8135	0.9423	26
35	0.3352	0.3558	2.8109	0.9422	25
36	0.3355	0.3561	2.8083	0.9421	24
37	0.3357	0.3564	2.8057	0.9420	23
38	0.3360	0.3567	2.8032	0.9419	22
39	0.3363	0.3571	2.8006	0.9418	21
40	0.3365	0.3574	2.7980	0.9417	20
41	0.3368	0.3577	2.7955	0.9416	19
42	0.3371	0.3581	2.7929	0.9415	18
43	0.3374	0.3584	2.7903	0.9414	17
44	0.3376	0.3587	2.7878	0.9413	16
45	0.3379	0.3590	2.7852	0.9412	15
46	0.3382	0.3594	2.7827	0.9411	14
47	0.3385	0.3597	2.7801	0.9410	13
48	0.3387	0.3600	2.7776	0.9409	12
49	0.3390	0.3604	2.7751	0.9408	11
50	0.3393	0.3607	2.7725	0.9407	10
51	0.3396	0.3610	2.7700	0.9406	9
52	0.3398	0.3613	2.7675	0.9405	8
53	0.3401	0.3617	2.7650	0.9404	7
54	0.3404	0.3620	2.7625	0.9403	6
55	0.3407	0.3623	2.7600	0.9402	5
56	0.3409	0.3627	2.7575	0.9401	4
57	0.3412	0.3630	2.7550	0.9400	3
58	0.3415	0.3633	2.7525	0.9399	2
59	0.3417	0.3636	2.7500	0.9398	1
60	0.3420	0.3640	2.7475	0.9397	0
	Cos	Cot	Tan	Sin	′

70°

20°

′	Sin	Tan	Cot	Cos	
0	0.3420	0.3640	2.7475	0.9397	60
1	0.3423	0.3643	2.7450	0.9396	59
2	0.3426	0.3646	2.7425	0.9395	58
3	0.3428	0.3650	2.7400	0.9394	57
4	0.3431	0.3653	2.7376	0.9393	56
5	0.3434	0.3656	2.7351	0.9392	55
6	0.3437	0.3659	2.7326	0.9391	54
7	0.3439	0.3663	2.7302	0.9390	53
8	0.3442	0.3666	2.7277	0.9389	52
9	0.3445	0.3669	2.7253	0.9388	51
10	0.3448	0.3673	2.7228	0.9387	50
11	0.3450	0.3676	2.7204	0.9386	49
12	0.3453	0.3679	2.7179	0.9385	48
13	0.3456	0.3683	2.7155	0.9384	47
14	0.3458	0.3686	2.7130	0.9383	46
15	0.3461	0.3689	2.7106	0.9382	45
16	0.3464	0.3693	2.7082	0.9381	44
17	0.3467	0.3696	2.7058	0.9380	43
18	0.3469	0.3699	2.7034	0.9379	42
19	0.3472	0.3702	2.7009	0.9378	41
20	0.3475	0.3706	2.6985	0.9377	40
21	0.3478	0.3709	2.6961	0.9376	39
22	0.3480	0.3712	2.6937	0.9375	38
23	0.3483	0.3716	2.6913	0.9374	37
24	0.3486	0.3719	2.6889	0.9373	36
25	0.3488	0.3722	2.6865	0.9372	35
26	0.3491	0.3726	2.6841	0.9371	34
27	0.3494	0.3729	2.6818	0.9370	33
28	0.3497	0.3732	2.6794	0.9369	32
29	0.3499	0.3736	2.6770	0.9368	31
30	0.3502	0.3739	2.6746	0.9367	30
31	0.3505	0.3742	2.6723	0.9366	29
32	0.3508	0.3745	2.6699	0.9365	28
33	0.3510	0.3749	2.6675	0.9364	27
34	0.3513	0.3752	2.6652	0.9363	26
35	0.3516	0.3755	2.6628	0.9362	25
36	0.3518	0.3759	2.6605	0.9361	24
37	0.3521	0.3762	2.6581	0.9360	23
38	0.3524	0.3765	2.6558	0.9359	22
39	0.3527	0.3769	2.6534	0.9358	21
40	0.3529	0.3772	2.6511	0.9356	20
41	0.3532	0.3775	2.6488	0.9355	19
42	0.3535	0.3779	2.6464	0.9354	18
43	0.3537	0.3782	2.6441	0.9353	17
44	0.3540	0.3785	2.6418	0.9352	16
45	0.3543	0.3789	2.6395	0.9351	15
46	0.3546	0.3792	2.6371	0.9350	14
47	0.3548	0.3795	2.6348	0.9349	13
48	0.3551	0.3799	2.6325	0.9348	12
49	0.3554	0.3802	2.6302	0.9347	11
50	0.3557	0.3805	2.6279	0.9346	10
51	0.3559	0.3809	2.6256	0.9345	9
52	0.3562	0.3812	2.6233	0.9344	8
53	0.3565	0.3815	2.6210	0.9343	7
54	0.3567	0.3819	2.6187	0.9342	6
55	0.3570	0.3822	2.6165	0.9341	5
56	0.3573	0.3825	2.6142	0.9340	4
57	0.3576	0.3829	2.6119	0.9339	3
58	0.3578	0.3832	2.6096	0.9338	2
59	0.3581	0.3835	2.6074	0.9337	1
60	0.3584	0.3839	2.6051	0.9336	0
	Cos	Cot	Tan	Sin	′

69°

21°

′	Sin	Tan	Cot	Cos	
0	0.3584	0.3839	2.6051	0.9336	60
1	0.3586	0.3842	2.6028	0.9335	59
2	0.3589	0.3845	2.6006	0.9334	58
3	0.3592	0.3849	2.5983	0.9333	57
4	0.3595	0.3852	2.5961	0.9332	56
5	0.3597	0.3855	2.5938	0.9331	55
6	0.3600	0.3859	2.5916	0.9330	54
7	0.3603	0.3862	2.5893	0.9328	53
8	0.3605	0.3865	2.5871	0.9327	52
9	0.3608	0.3869	2.5848	0.9326	51
10	0.3611	0.3872	2.5826	0.9325	50
11	0.3614	0.3875	2.5804	0.9324	49
12	0.3616	0.3879	2.5782	0.9323	48
13	0.3619	0.3882	2.5759	0.9322	47
14	0.3622	0.3885	2.5737	0.9321	46
15	0.3624	0.3889	2.5715	0.9320	45
16	0.3627	0.3892	2.5693	0.9319	44
17	0.3630	0.3895	2.5671	0.9318	43
18	0.3633	0.3899	2.5649	0.9317	42
19	0.3635	0.3902	2.5627	0.9316	41
20	0.3638	0.3906	2.5605	0.9315	40
21	0.3641	0.3909	2.5583	0.9314	39
22	0.3643	0.3912	2.5561	0.9313	38
23	0.3646	0.3916	2.5539	0.9312	37
24	0.3649	0.3919	2.5517	0.9311	36
25	0.3651	0.3922	2.5495	0.9309	35
26	0.3654	0.3926	2.5473	0.9308	34
27	0.3657	0.3929	2.5452	0.9307	33
28	0.3660	0.3932	2.5430	0.9306	32
29	0.3662	0.3936	2.5408	0.9305	31
30	0.3665	0.3939	2.5386	0.9304	30
31	0.3668	0.3942	2.5365	0.9303	29
32	0.3670	0.3946	2.5343	0.9302	28
33	0.3673	0.3949	2.5322	0.9301	27
34	0.3676	0.3953	2.5300	0.9300	26
35	0.3679	0.3956	2.5279	0.9299	25
36	0.3681	0.3959	2.5257	0.9298	24
37	0.3684	0.3963	2.5236	0.9297	23
38	0.3687	0.3966	2.5214	0.9296	22
39	0.3689	0.3969	2.5193	0.9295	21
40	0.3692	0.3973	2.5172	0.9293	20
41	0.3695	0.3976	2.5150	0.9292	19
42	0.3697	0.3979	2.5129	0.9291	18
43	0.3700	0.3983	2.5108	0.9290	17
44	0.3703	0.3986	2.5086	0.9289	16
45	0.3706	0.3990	2.5065	0.9288	15
46	0.3708	0.3993	2.5044	0.9287	14
47	0.3711	0.3996	2.5023	0.9286	13
48	0.3714	0.4000	2.5002	0.9285	12
49	0.3716	0.4003	2.4981	0.9284	11
50	0.3719	0.4006	2.4960	0.9283	10
51	0.3722	0.4010	2.4939	0.9282	9
52	0.3724	0.4013	2.4918	0.9281	8
53	0.3727	0.4017	2.4897	0.9279	7
54	0.3730	0.4020	2.4876	0.9278	6
55	0.3733	0.4023	2.4855	0.9277	5
56	0.3735	0.4027	2.4834	0.9276	4
57	0.3738	0.4030	2.4813	0.9275	3
58	0.3741	0.4033	2.4792	0.9274	2
59	0.3743	0.4037	2.4772	0.9273	1
60	0.3746	0.4040	2.4751	0.9272	0
	Cos	Cot	Tan	Sin	′

68°

III. Natural Trigonometric Functions

22°

′	Sin	Tan	Cot	Cos	
0	0.3746	0.4040	2.4751	0.9272	60
1	0.3749	0.4044	2.4730	0.9271	59
2	0.3751	0.4047	2.4709	0.9270	58
3	0.3754	0.4050	2.4689	0.9269	57
4	0.3757	0.4054	2.4668	0.9267	56
5	0.3760	0.4057	2.4648	0.9266	55
6	0.3762	0.4061	2.4627	0.9265	54
7	0.3765	0.4064	2.4606	0.9264	53
8	0.3768	0.4067	2.4586	0.9263	52
9	0.3770	0.4071	2.4566	0.9262	51
10	0.3773	0.4074	2.4545	0.9261	50
11	0.3776	0.4078	2.4525	0.9260	49
12	0.3778	0.4081	2.4504	0.9259	48
13	0.3781	0.4084	2.4484	0.9258	47
14	0.3784	0.4088	2.4464	0.9257	46
15	0.3786	0.4091	2.4443	0.9255	45
16	0.3789	0.4095	2.4423	0.9254	44
17	0.3792	0.4098	2.4403	0.9253	43
18	0.3795	0.4101	2.4383	0.9252	42
19	0.3797	0.4105	2.4362	0.9251	41
20	0.3800	0.4108	2.4342	0.9250	40
21	0.3803	0.4111	2.4322	0.9249	39
22	0.3805	0.4115	2.4302	0.9248	38
23	0.3808	0.4118	2.4282	0.9247	37
24	0.3811	0.4122	2.4262	0.9245	36
25	0.3813	0.4125	2.4242	0.9244	35
26	0.3816	0.4129	2.4222	0.9243	34
27	0.3819	0.4132	2.4202	0.9242	33
28	0.3821	0.4135	2.4182	0.9241	32
29	0.3824	0.4139	2.4162	0.9240	31
30	0.3827	0.4142	2.4142	0.9239	30
31	0.3830	0.4146	2.4122	0.9238	29
32	0.3832	0.4149	2.4102	0.9237	28
33	0.3835	0.4152	2.4083	0.9235	27
34	0.3838	0.4156	2.4063	0.9234	26
35	0.3840	0.4159	2.4043	0.9233	25
36	0.3843	0.4163	2.4023	0.9232	24
37	0.3846	0.4166	2.4004	0.9231	23
38	0.3848	0.4169	2.3984	0.9230	22
39	0.3851	0.4173	2.3964	0.9229	21
40	0.3854	0.4176	2.3945	0.9228	20
41	0.3856	0.4180	2.3925	0.9227	19
42	0.3859	0.4183	2.3906	0.9225	18
43	0.3862	0.4187	2.3886	0.9224	17
44	0.3864	0.4190	2.3867	0.9223	16
45	0.3867	0.4193	2.3847	0.9222	15
46	0.3870	0.4197	2.3828	0.9221	14
47	0.3872	0.4200	2.3808	0.9220	13
48	0.3875	0.4204	2.3789	0.9219	12
49	0.3878	0.4207	2.3770	0.9218	11
50	0.3881	0.4210	2.3750	0.9216	10
51	0.3883	0.4214	2.3731	0.9215	9
52	0.3886	0.4217	2.3712	0.9214	8
53	0.3889	0.4221	2.3693	0.9213	7
54	0.3891	0.4224	2.3673	0.9212	6
55	0.3894	0.4228	2.3654	0.9211	5
56	0.3897	0.4231	2.3635	0.9210	4
57	0.3899	0.4234	2.3616	0.9208	3
58	0.3902	0.4238	2.3597	0.9207	2
59	0.3905	0.4241	2.3578	0.9206	1
60	0.3907	0.4245	2.3559	0.9205	0
	Cos	Cot	Tan	Sin	′

67°

23°

′	Sin	Tan	Cot	Cos	
0	0.3907	0.4245	2.3559	0.9205	60
1	0.3910	0.4248	2.3539	0.9204	59
2	0.3913	0.4252	2.3520	0.9203	58
3	0.3915	0.4255	2.3501	0.9202	57
4	0.3918	0.4258	2.3483	0.9200	56
5	0.3921	0.4262	2.3464	0.9199	55
6	0.3923	0.4265	2.3445	0.9198	54
7	0.3926	0.4269	2.3426	0.9197	53
8	0.3929	0.4272	2.3407	0.9196	52
9	0.3931	0.4276	2.3388	0.9195	51
10	0.3934	0.4279	2.3369	0.9194	50
11	0.3937	0.4283	2.3351	0.9192	49
12	0.3939	0.4286	2.3332	0.9191	48
13	0.3942	0.4289	2.3313	0.9190	47
14	0.3945	0.4293	2.3294	0.9189	46
15	0.3947	0.4296	2.3276	0.9188	45
16	0.3950	0.4300	2.3257	0.9187	44
17	0.3953	0.4303	2.3238	0.9186	43
18	0.3955	0.4307	2.3220	0.9184	42
19	0.3958	0.4310	2.3201	0.9183	41
20	0.3961	0.4314	2.3183	0.9182	40
21	0.3963	0.4317	2.3164	0.9181	39
22	0.3966	0.4320	2.3146	0.9180	38
23	0.3969	0.4324	2.3127	0.9179	37
24	0.3971	0.4327	2.3109	0.9178	36
25	0.3974	0.4331	2.3090	0.9176	35
26	0.3977	0.4334	2.3072	0.9175	34
27	0.3979	0.4338	2.3053	0.9174	33
28	0.3982	0.4341	2.3035	0.9173	32
29	0.3985	0.4345	2.3017	0.9172	31
30	0.3987	0.4348	2.2998	0.9171	30
31	0.3990	0.4352	2.2980	0.9169	29
32	0.3993	0.4355	2.2962	0.9168	28
33	0.3995	0.4359	2.2944	0.9167	27
34	0.3998	0.4362	2.2925	0.9166	26
35	0.4001	0.4365	2.2907	0.9165	25
36	0.4003	0.4369	2.2889	0.9164	24
37	0.4006	0.4372	2.2871	0.9162	23
38	0.4009	0.4376	2.2853	0.9161	22
39	0.4011	0.4379	2.2835	0.9160	21
40	0.4014	0.4383	2.2817	0.9159	20
41	0.4017	0.4386	2.2799	0.9158	19
42	0.4019	0.4390	2.2781	0.9157	18
43	0.4022	0.4393	2.2763	0.9155	17
44	0.4025	0.4397	2.2745	0.9154	16
45	0.4027	0.4400	2.2727	0.9153	15
46	0.4030	0.4404	2.2709	0.9152	14
47	0.4033	0.4407	2.2691	0.9151	13
48	0.4035	0.4411	2.2673	0.9150	12
49	0.4038	0.4414	2.2655	0.9148	11
50	0.4041	0.4417	2.2637	0.9147	10
51	0.4043	0.4421	2.2620	0.9146	9
52	0.4046	0.4424	2.2602	0.9145	8
53	0.4049	0.4428	2.2584	0.9144	7
54	0.4051	0.4431	2.2566	0.9143	6
55	0.4054	0.4435	2.2549	0.9141	5
56	0.4057	0.4438	2.2531	0.9140	4
57	0.4059	0.4442	2.2513	0.9139	3
58	0.4062	0.4445	2.2496	0.9138	2
59	0.4065	0.4449	2.2478	0.9137	1
60	0.4067	0.4452	2.2460	0.9135	0
	Cos	Cot	Tan	Sin	′

66°

24°

′	Sin	Tan	Cot	Cos	
0	0.4067	0.4452	2.2460	0.9135	60
1	0.4070	0.4456	2.2443	0.9134	59
2	0.4073	0.4459	2.2425	0.9133	58
3	0.4075	0.4463	2.2408	0.9132	57
4	0.4078	0.4466	2.2390	0.9131	56
5	0.4081	0.4470	2.2373	0.9130	55
6	0.4083	0.4473	2.2355	0.9128	54
7	0.4086	0.4477	2.2338	0.9127	53
8	0.4089	0.4480	2.2320	0.9126	52
9	0.4091	0.4484	2.2303	0.9125	51
10	0.4094	0.4487	2.2286	0.9124	50
11	0.4097	0.4491	2.2268	0.9122	49
12	0.4099	0.4494	2.2251	0.9121	48
13	0.4102	0.4498	2.2234	0.9120	47
14	0.4105	0.4501	2.2216	0.9119	46
15	0.4107	0.4505	2.2199	0.9118	45
16	0.4110	0.4508	2.2182	0.9116	44
17	0.4112	0.4512	2.2165	0.9115	43
18	0.4115	0.4515	2.2148	0.9114	42
19	0.4118	0.4519	2.2130	0.9113	41
20	0.4120	0.4522	2.2113	0.9112	40
21	0.4123	0.4526	2.2096	0.9110	39
22	0.4126	0.4529	2.2079	0.9109	38
23	0.4128	0.4533	2.2062	0.9108	37
24	0.4131	0.4536	2.2045	0.9107	36
25	0.4134	0.4540	2.2028	0.9106	35
26	0.4136	0.4543	2.2011	0.9104	34
27	0.4139	0.4547	2.1994	0.9103	33
28	0.4142	0.4550	2.1977	0.9102	32
29	0.4144	0.4554	2.1960	0.9101	31
30	0.4147	0.4557	2.1943	0.9100	30
31	0.4150	0.4561	2.1926	0.9098	29
32	0.4152	0.4564	2.1909	0.9097	28
33	0.4155	0.4568	2.1892	0.9096	27
34	0.4158	0.4571	2.1876	0.9095	26
35	0.4160	0.4575	2.1859	0.9094	25
36	0.4163	0.4578	2.1842	0.9092	24
37	0.4165	0.4582	2.1825	0.9091	23
38	0.4168	0.4585	2.1808	0.9090	22
39	0.4171	0.4589	2.1792	0.9089	21
40	0.4173	0.4592	2.1775	0.9088	20
41	0.4176	0.4596	2.1758	0.9086	19
42	0.4179	0.4599	2.1742	0.9085	18
43	0.4181	0.4603	2.1725	0.9084	17
44	0.4184	0.4607	2.1708	0.9083	16
45	0.4187	0.4610	2.1692	0.9081	15
46	0.4189	0.4614	2.1675	0.9080	14
47	0.4192	0.4617	2.1659	0.9079	13
48	0.4195	0.4621	2.1642	0.9078	12
49	0.4197	0.4624	2.1625	0.9077	11
50	0.4200	0.4628	2.1609	0.9075	10
51	0.4202	0.4631	2.1592	0.9074	9
52	0.4205	0.4635	2.1576	0.9073	8
53	0.4208	0.4638	2.1560	0.9072	7
54	0.4210	0.4642	2.1543	0.9070	6
55	0.4213	0.4645	2.1527	0.9069	5
56	0.4216	0.4649	2.1510	0.9068	4
57	0.4218	0.4652	2.1494	0.9067	3
58	0.4221	0.4656	2.1478	0.9066	2
59	0.4224	0.4660	2.1461	0.9064	1
60	0.4226	0.4663	2.1445	0.9063	0
	Cos	Cot	Tan	Sin	′

65°

25°

′	Sin	Tan	Cot	Cos	
0	0.4226	0.4663	2.1445	0.9063	60
1	0.4229	0.4667	2.1429	0.9062	59
2	0.4231	0.4670	2.1413	0.9061	58
3	0.4234	0.4674	2.1396	0.9059	57
4	0.4237	0.4677	2.1380	0.9058	56
5	0.4239	0.4681	2.1364	0.9057	55
6	0.4242	0.4684	2.1348	0.9056	54
7	0.4245	0.4688	2.1332	0.9054	53
8	0.4247	0.4691	2.1315	0.9053	52
9	0.4250	0.4695	2.1299	0.9052	51
10	0.4253	0.4699	2.1283	0.9051	50
11	0.4255	0.4702	2.1267	0.9050	49
12	0.4258	0.4706	2.1251	0.9048	48
13	0.4260	0.4709	2.1235	0.9047	47
14	0.4263	0.4713	2.1219	0.9046	46
15	0.4266	0.4716	2.1203	0.9045	45
16	0.4268	0.4720	2.1187	0.9043	44
17	0.4271	0.4723	2.1171	0.9042	43
18	0.4274	0.4727	2.1155	0.9041	42
19	0.4276	0.4731	2.1139	0.9040	41
20	0.4279	0.4734	2.1123	0.9038	40
21	0.4281	0.4738	2.1107	0.9037	39
22	0.4284	0.4741	2.1092	0.9036	38
23	0.4287	0.4745	2.1076	0.9035	37
24	0.4289	0.4748	2.1060	0.9033	36
25	0.4292	0.4752	2.1044	0.9032	35
26	0.4295	0.4755	2.1028	0.9031	34
27	0.4297	0.4759	2.1013	0.9030	33
28	0.4300	0.4763	2.0997	0.9028	32
29	0.4302	0.4766	2.0981	0.9027	31
30	0.4305	0.4770	2.0965	0.9026	30
31	0.4308	0.4773	2.0950	0.9025	29
32	0.4310	0.4777	2.0934	0.9023	28
33	0.4313	0.4780	2.0918	0.9022	27
34	0.4316	0.4784	2.0903	0.9021	26
35	0.4318	0.4788	2.0887	0.9020	25
36	0.4321	0.4791	2.0872	0.9018	24
37	0.4323	0.4795	2.0859	0.9017	23
38	0.4326	0.4798	2.0840	0.9016	22
39	0.4329	0.4802	2.0825	0.9015	21
40	0.4331	0.4806	2.0809	0.9013	20
41	0.4334	0.4809	2.0794	0.9012	19
42	0.4337	0.4813	2.0778	0.9011	18
43	0.4339	0.4816	2.0763	0.9010	17
44	0.4342	0.4820	2.0748	0.9008	16
45	0.4344	0.4823	2.0732	0.9007	15
46	0.4347	0.4827	2.0717	0.9006	14
47	0.4350	0.4831	2.0701	0.9004	13
48	0.4352	0.4834	2.0686	0.9003	12
49	0.4355	0.4838	2.0671	0.9002	11
50	0.4358	0.4841	2.0655	0.9001	10
51	0.4360	0.4845	2.0640	0.8999	9
52	0.4363	0.4849	2.0625	0.8998	8
53	0.4365	0.4852	2.0609	0.8997	7
54	0.4368	0.4856	2.0594	0.8996	6
55	0.4371	0.4859	2.0579	0.8994	5
56	0.4373	0.4863	2.0564	0.8993	4
57	0.4376	0.4867	2.0549	0.8992	3
58	0.4378	0.4870	2.0533	0.8990	2
59	0.4381	0.4874	2.0518	0.8989	1
60	0.4384	0.4877	2.0503	0.8988	0
	Cos	Cot	Tan	Sin	′

64°

III. Natural Trigonometric Functions

26°

′	Sin	Tan	Cot	Cos	
0	0.4384	0.4877	2.0503	0.8988	60
1	0.4386	0.4881	2.0488	0.8987	59
2	0.4389	0.4885	2.0473	0.8985	58
3	0.4392	0.4888	2.0458	0.8984	57
4	0.4394	0.4892	2.0443	0.8983	56
5	0.4397	0.4895	2.0428	0.8982	55
6	0.4399	0.4899	2.0413	0.8980	54
7	0.4402	0.4903	2.0398	0.8979	53
8	0.4405	0.4906	2.0383	0.8978	52
9	0.4407	0.4910	2.0368	0.8976	51
10	0.4410	0.4913	2.0353	0.8975	50
11	0.4412	0.4917	2.0338	0.8974	49
12	0.4415	0.4921	2.0323	0.8973	48
13	0.4418	0.4924	2.0308	0.8971	47
14	0.4420	0.4928	2.0293	0.8970	46
15	0.4423	0.4931	2.0278	0.8969	45
16	0.4425	0.4935	2.0263	0.8967	44
17	0.4428	0.4939	2.0248	0.8966	43
18	0.4431	0.4942	2.0233	0.8965	42
19	0.4433	0.4946	2.0219	0.8964	41
20	0.4436	0.4950	2.0204	0.8962	40
21	0.4439	0.4953	2.0189	0.8961	39
22	0.4441	0.4957	2.0174	0.8960	38
23	0.4444	0.4960	2.0160	0.8958	37
24	0.4446	0.4964	2.0145	0.8957	36
25	0.4449	0.4968	2.0130	0.8956	35
26	0.4452	0.4971	2.0115	0.8955	34
27	0.4454	0.4975	2.0101	0.8953	33
28	0.4457	0.4979	2.0086	0.8952	32
29	0.4459	0.4982	2.0072	0.8951	31
30	0.4462	0.4986	2.0057	0.8949	30
31	0.4465	0.4989	2.0042	0.8948	29
32	0.4467	0.4993	2.0028	0.8947	28
33	0.4470	0.4997	2.0013	0.8945	27
34	0.4472	0.5000	1.9999	0.8944	26
35	0.4475	0.5004	1.9984	0.8943	25
36	0.4478	0.5008	1.9970	0.8942	24
37	0.4480	0.5011	1.9955	0.8940	23
38	0.4483	0.5015	1.9941	0.8939	22
39	0.4485	0.5019	1.9926	0.8938	21
40	0.4488	0.5022	1.9912	0.8936	20
41	0.4491	0.5026	1.9897	0.8935	19
42	0.4493	0.5029	1.9883	0.8934	18
43	0.4496	0.5033	1.9868	0.8932	17
44	0.4498	0.5037	1.9854	0.8931	16
45	0.4501	0.5040	1.9840	0.8930	15
46	0.4504	0.5044	1.9825	0.8928	14
47	0.4506	0.5048	1.9811	0.8927	13
48	0.4509	0.5051	1.9797	0.8926	12
49	0.4511	0.5055	1.9782	0.8925	11
50	0.4514	0.5059	1.9768	0.8923	10
51	0.4517	0.5062	1.9754	0.8922	9
52	0.4519	0.5066	1.9740	0.8921	8
53	0.4522	0.5070	1.9725	0.8919	7
54	0.4524	0.5073	1.9711	0.8918	6
55	0.4527	0.5077	1.9697	0.8917	5
56	0.4530	0.5081	1.9683	0.8915	4
57	0.4532	0.5084	1.9669	0.8914	3
58	0.4545	0.5088	1.9654	0.8913	2
59	0.4537	0.5092	1.9640	0.8911	1
60	0.4540	0.5095	1.9626	0.8910	0
	Cos	Cot	Tan	Sin	′

63°

27°

′	Sin	Tan	Cot	Cos	
0	0.4540	0.5095	1.9626	0.8910	60
1	0.4542	0.5099	1.9612	0.8909	59
2	0.4545	0.5103	1.9598	0.8907	58
3	0.4548	0.5106	1.9584	0.8906	57
4	0.4550	0.5110	1.9570	0.8905	56
5	0.4553	0.5114	1.9556	0.8903	55
6	0.4555	0.5117	1.9542	0.8902	54
7	0.4558	0.5121	1.9528	0.8901	53
8	0.4561	0.5125	1.9514	0.8899	52
9	0.4563	0.5128	1.9500	0.8898	51
10	0.4566	0.5132	1.9486	0.8897	50
11	0.4568	0.5136	1.9472	0.8895	49
12	0.4571	0.5139	1.9458	0.8894	48
13	0.4574	0.5143	1.9444	0.8893	47
14	0.4576	0.5147	1.9430	0.8892	46
15	0.4579	0.5150	1.9416	0.8890	45
16	0.4581	0.5154	1.9402	0.8889	44
17	0.4584	0.5158	1.9388	0.8888	43
18	0.4586	0.5161	1.9375	0.8886	42
19	0.4589	0.5165	1.9361	0.8885	41
20	0.4592	0.5169	1.9347	0.8884	40
21	0.4594	0.5172	1.9333	0.8882	39
22	0.4597	0.5176	1.9319	0.8881	38
23	0.4599	0.5180	1.9306	0.8879	37
24	0.4602	0.5184	1.9292	0.8878	36
25	0.4605	0.5187	1.9278	0.8877	35
26	0.4607	0.5191	1.9265	0.8875	34
27	0.4610	0.5195	1.9251	0.8874	33
28	0.4612	0.5198	1.9237	0.8873	32
29	0.4615	0.5202	1.9223	0.8871	31
30	0.4617	0.5206	1.9210	0.8870	30
31	0.4620	0.5209	1.9196	0.8869	29
32	0.4623	0.5213	1.9183	0.8867	28
33	0.4625	0.5217	1.9169	0.8866	27
34	0.4628	0.5220	1.9155	0.8865	26
35	0.4630	0.5224	1.9142	0.8863	25
36	0.4633	0.5228	1.9128	0.8862	24
37	0.4636	0.5232	1.9115	0.8861	23
38	0.4638	0.5235	1.9101	0.8859	22
39	0.4641	0.5239	1.9088	0.8858	21
40	0.4643	0.5243	1.9074	0.8857	20
41	0.4646	0.5246	1.9061	0.8855	19
42	0.4648	0.5250	1.9047	0.8854	18
43	0.4651	0.5254	1.9034	0.8853	17
44	0.4654	0.5258	1.9020	0.8851	16
45	0.4656	0.5261	1.9007	0.8850	15
46	0.4659	0.5265	1.8993	0.8849	14
47	0.4661	0.5269	1.8980	0.8847	13
48	0.4664	0.5272	1.8967	0.8846	12
49	0.4666	0.5276	1.8953	0.8844	11
50	0.4669	0.5280	1.8940	0.8843	10
51	0.4672	0.5284	1.8927	0.8842	9
52	0.4674	0.5287	1.8913	0.8840	8
53	0.4677	0.5291	1.8900	0.8839	7
54	0.4679	0.5295	1.8887	0.8838	6
55	0.4682	0.5298	1.8873	0.8836	5
56	0.4684	0.5302	1.8860	0.8835	4
57	0.4687	0.5306	1.8847	0.8834	3
58	0.4690	0.5310	1.8834	0.8832	2
59	0.4692	0.5313	1.8820	0.8831	1
60	0.4695	0.5317	1.8807	0.8829	0
	Cos	Cot	Tan	Sin	′

62°

28°

′	Sin	Tan	Cot	Cos	
0	0.4695	0.5317	1.8807	0.8829	60
1	0.4697	0.5321	1.8794	0.8828	59
2	0.4700	0.5325	1.8781	0.8827	58
3	0.4702	0.5328	1.8768	0.8825	57
4	0.4705	0.5332	1.8755	0.8824	56
5	0.4708	0.5336	1.8741	0.8823	55
6	0.4710	0.5340	1.8728	0.8821	54
7	0.4713	0.5343	1.8715	0.8820	53
8	0.4715	0.5347	1.8702	0.8819	52
9	0.4718	0.5351	1.8689	0.8817	51
10	0.4720	0.5354	1.8676	0.8816	50
11	0.4723	0.5358	1.8663	0.8814	49
12	0.4726	0.5362	1.8650	0.8813	48
13	0.4728	0.5366	1.8637	0.8812	47
14	0.4731	0.5369	1.8624	0.8810	46
15	0.4733	0.5373	1.8611	0.8809	45
16	0.4736	0.5377	1.8598	0.8808	44
17	0.4738	0.5381	1.8585	0.8806	43
18	0.4741	0.5384	1.8572	0.8805	42
19	0.4743	0.5388	1.8559	0.8803	41
20	0.4746	0.5392	1.8546	0.8802	40
21	0.4749	0.5396	1.8533	0.8801	39
22	0.4751	0.5399	1.8520	0.8799	38
23	0.4754	0.5403	1.8507	0.8798	37
24	0.4756	0.5407	1.8495	0.8796	36
25	0.4759	0.5411	1.8482	0.8795	35
26	0.4761	0.5415	1.8469	0.8794	34
27	0.4764	0.5418	1.8456	0.8792	33
28	0.4766	0.5422	1.8443	0.8791	32
29	0.4769	0.5426	1.8430	0.8790	31
30	0.4772	0.5430	1.8418	0.8788	30
31	0.4774	0.5433	1.8405	0.8787	29
32	0.4777	0.5437	1.8392	0.8785	28
33	0.4779	0.5441	1.8379	0.8784	27
34	0.4782	0.5445	1.8367	0.8783	26
35	0.4784	0.5448	1.8354	0.8781	25
36	0.4787	0.5452	1.8341	0.8780	24
37	0.4789	0.5456	1.8329	0.8778	23
38	0.4792	0.5460	1.8316	0.8777	22
39	0.4795	0.5464	1.8303	0.8776	21
40	0.4797	0.5467	1.8291	0.8774	20
41	0.4800	0.5471	1.8278	0.8773	19
42	0.4802	0.5475	1.8265	0.8771	18
43	0.4805	0.5479	1.8253	0.8770	17
44	0.4807	0.5482	1.8240	0.8769	16
45	0.4810	0.5486	1.8228	0.8767	15
46	0.4812	0.5490	1.8215	0.8766	14
47	0.4815	0.5494	1.8202	0.8764	13
48	0.4818	0.5498	1.8190	0.8763	12
49	0.4820	0.5501	1.8177	0.8762	11
50	0.4823	0.5505	1.8165	0.8760	10
51	0.4825	0.5509	1.8152	0.8759	9
52	0.4828	0.5513	1.8140	0.8757	8
53	0.4830	0.5517	1.8127	0.8756	7
54	0.4833	0.5520	1.8115	0.8755	6
55	0.4835	0.5524	1.8103	0.8753	5
56	0.4838	0.5528	1.8090	0.8752	4
57	0.4840	0.5532	1.8078	0.8750	3
58	0.4843	0.5535	1.8065	0.8749	2
59	0.4846	0.5539	1.8053	0.8748	1
60	0.4848	0.5543	1.8040	0.8746	0
	Cos	Cot	Tan	Sin	′

61°

29°

′	Sin	Tan	Cot	Cos	
0	0.4848	0.5543	1.8040	0.8746	60
1	0.4851	0.5547	1.8028	0.8745	59
2	0.4853	0.5551	1.8016	0.8743	58
3	0.4856	0.5555	1.8003	0.8742	57
4	0.4858	0.5558	1.7991	0.8741	56
5	0.4861	0.5562	1.7979	0.8739	55
6	0.4863	0.5566	1.7966	0.8738	54
7	0.4866	0.5570	1.7954	0.8736	53
8	0.4868	0.5574	1.7942	0.8735	52
9	0.4871	0.5577	1.7930	0.8733	51
10	0.4874	0.5581	1.7917	0.8732	50
11	0.4876	0.5585	1.7905	0.8731	49
12	0.4879	0.5589	1.7893	0.8729	48
13	0.4881	0.5593	1.7881	0.8728	47
14	0.4884	0.5596	1.7868	0.8726	46
15	0.4886	0.5600	1.7856	0.8725	45
16	0.4889	0.5604	1.7844	0.8724	44
17	0.4891	0.5608	1.7832	0.8722	43
18	0.4894	0.5612	1.7820	0.8721	42
19	0.4896	0.5616	1.7808	0.8719	41
20	0.4899	0.5619	1.7796	0.8718	40
21	0.4901	0.5623	1.7783	0.8716	39
22	0.4904	0.5627	1.7771	0.8715	38
23	0.4907	0.5631	1.7759	0.8714	37
24	0.4909	0.5635	1.7747	0.8712	36
25	0.4912	0.5639	1.7735	0.8711	35
26	0.4914	0.5642	1.7723	0.8709	34
27	0.4917	0.5646	1.7711	0.8708	33
28	0.4919	0.5650	1.7699	0.8706	32
29	0.4922	0.5654	1.7687	0.8705	31
30	0.4924	0.5658	1.7675	0.8704	30
31	0.4927	0.5662	1.7663	0.8702	29
32	0.4929	0.5665	1.7651	0.8701	28
33	0.4932	0.5669	1.7639	0.8699	27
34	0.4934	0.5673	1.7627	0.8698	26
35	0.4937	0.5677	1.7615	0.8696	25
36	0.4939	0.5681	1.7603	0.8695	24
37	0.4942	0.5685	1.7591	0.8694	23
38	0.4944	0.5688	1.7579	0.8692	22
39	0.4947	0.5692	1.7567	0.8691	21
40	0.4950	0.5696	1.7556	0.8689	20
41	0.4952	0.5700	1.7544	0.8688	19
42	0.4955	0.5704	1.7532	0.8686	18
43	0.4957	0.5708	1.7520	0.8685	17
44	0.4960	0.5712	1.7508	0.8683	16
45	0.4962	0.5715	1.7496	0.8682	15
46	0.4965	0.5719	1.7485	0.8681	14
47	0.4967	0.5723	1.7473	0.8679	13
48	0.4970	0.5727	1.7461	0.8678	12
49	0.4972	0.5731	1.7449	0.8676	11
50	0.4975	0.5735	1.7437	0.8675	10
51	0.4977	0.5739	1.7426	0.8673	9
52	0.4980	0.5743	1.7414	0.8672	8
53	0.4982	0.5746	1.7402	0.8670	7
54	0.4985	0.5750	1.7391	0.8669	6
55	0.4987	0.5754	1.7379	0.8668	5
56	0.4990	0.5758	1.7367	0.8666	4
57	0.4992	0.5762	1.7355	0.8665	3
58	0.4995	0.5766	1.7344	0.8663	2
59	0.4997	0.5770	1.7332	0.8662	1
60	0.5000	0.5774	1.7321	0.8660	0
	Cos	Cot	Tan	Sin	′

60°

III. Natural Trigonometric Functions

30°

′	Sin	Tan	Cot	Cos	
0	0.5000	0.5774	1.7321	0.8660	60
1	0.5003	0.5777	1.7309	0.8659	59
2	0.5005	0.5781	1.7297	0.8657	58
3	0.5008	0.5785	1.7286	0.8656	57
4	0.5010	0.5789	1.7274	0.8654	56
5	0.5013	0.5793	1.7262	0.8653	55
6	0.5015	0.5797	1.7251	0.8652	54
7	0.5018	0.5801	1.7239	0.8650	53
8	0.5020	0.5805	1.7228	0.8649	52
9	0.5023	0.5808	1.7216	0.8647	51
10	0.5025	0.5812	1.7205	0.8646	50
11	0.5028	0.5816	1.7193	0.8644	49
12	0.5030	0.5820	1.7182	0.8643	48
13	0.5033	0.5824	1.7170	0.8641	47
14	0.5035	0.5828	1.7159	0.8640	46
15	0.5038	0.5832	1.7147	0.8638	45
16	0.5040	0.5836	1.7136	0.8637	44
17	0.5043	0.5840	1.7124	0.8635	43
18	0.5045	0.5844	1.7113	0.8634	42
19	0.5048	0.5847	1.7102	0.8632	41
20	0.5050	0.5851	1.7090	0.8631	40
21	0.5053	0.5855	1.7079	0.8630	39
22	0.5055	0.5859	1.7067	0.8628	38
23	0.5058	0.5863	1.7056	0.8627	37
24	0.5060	0.5867	1.7045	0.8625	36
25	0.5063	0.5871	1.7033	0.8624	35
26	0.5065	0.5875	1.7022	0.8622	34
27	0.5068	0.5879	1.7011	0.8621	33
28	0.5070	0.5883	1.6999	0.8619	32
29	0.5073	0.5887	1.6988	0.8618	31
30	0.5075	0.5890	1.6977	0.8616	30
31	0.5078	0.5894	1.6965	0.8615	29
32	0.5080	0.5898	1.6954	0.8613	28
33	0.5083	0.5902	1.6943	0.8612	27
34	0.5085	0.5906	1.6932	0.8610	26
35	0.5088	0.5910	1.6920	0.8609	25
36	0.5090	0.5914	1.6909	0.8607	24
37	0.5093	0.5918	1.6898	0.8606	23
38	0.5095	0.5922	1.6887	0.8604	22
39	0.5098	0.5926	1.6875	0.8603	21
40	0.5100	0.5930	1.6864	0.8601	20
41	0.5103	0.5934	1.6853	0.8600	19
42	0.5105	0.5938	1.6842	0.8599	18
43	0.5108	0.5942	1.6831	0.8597	17
44	0.5110	0.5945	1.6820	0.8596	16
45	0.5113	0.5949	1.6808	0.8594	15
46	0.5115	0.5953	1.6797	0.8593	14
47	0.5118	0.5957	1.6786	0.8591	13
48	0.5120	0.5961	1.6775	0.8590	12
49	0.5123	0.5965	1.6764	0.8588	11
50	0.5125	0.5969	1.6753	0.8587	10
51	0.5128	0.5973	1.6742	0.8585	9
52	0.5130	0.5977	1.6731	0.8584	8
53	0.5133	0.5981	1.6720	0.8582	7
54	0.5135	0.5985	1.6709	0.8581	6
55	0.5138	0.5989	1.6698	0.8579	5
56	0.5140	0.5993	1.6687	0.8578	4
57	0.5143	0.5997	1.6676	0.8576	3
58	0.5145	0.6001	1.6665	0.8575	2
59	0.5148	0.6005	1.6654	0.8573	1
60	0.5150	0.6009	1.6643	0.8572	0
	Cos	Cot	Tan	Sin	′

59°

31°

′	Sin	Tan	Cot	Cos	
0	0.5150	0.6009	1.6643	0.8572	60
1	0.5153	0.6013	1.6632	0.8570	59
2	0.5155	0.6017	1.6621	0.8569	58
3	0.5158	0.6020	1.6610	0.8567	57
4	0.5160	0.6024	1.6599	0.8566	56
5	0.5163	0.6028	1.6588	0.8564	55
6	0.5165	0.6032	1.6577	0.8563	54
7	0.5168	0.6036	1.6566	0.8561	53
8	0.5170	0.6040	1.6555	0.8560	52
9	0.5173	0.6044	1.6545	0.8558	51
10	0.5175	0.6048	1.6534	0.8557	50
11	0.5178	0.6052	1.6523	0.8555	49
12	0.5180	0.6056	1.6512	0.8554	48
13	0.5183	0.6060	1.6501	0.8552	47
14	0.5185	0.6064	1.6490	0.8551	46
15	0.5188	0.6068	1.6479	0.8549	45
16	0.5190	0.6072	1.6469	0.8548	44
17	0.5193	0.6076	1.6458	0.8546	43
18	0.5195	0.6080	1.6447	0.8545	42
19	0.5198	0.6084	1.6436	0.8543	41
20	0.5200	0.6088	1.6426	0.8542	40
21	0.5203	0.6092	1.6415	0.8540	39
22	0.5205	0.6096	1.6404	0.8539	38
23	0.5208	0.6100	1.6393	0.8537	37
24	0.5210	0.6104	1.6383	0.8536	36
25	0.5213	0.6108	1.6372	0.8534	35
26	0.5215	0.6112	1.6361	0.8532	34
27	0.5218	0.6116	1.6351	0.8531	33
28	0.5220	0.6120	1.6340	0.8529	32
29	0.5223	0.6124	1.6329	0.8528	31
30	0.5225	0.6128	1.6319	0.8526	30
31	0.5227	0.6132	1.6308	0.8525	29
32	0.5230	0.6136	1.6297	0.8523	28
33	0.5232	0.6140	1.6287	0.8522	27
34	0.5235	0.6144	1.6276	0.8520	26
35	0.5237	0.6148	1.6265	0.8519	25
36	0.5240	0.6152	1.6255	0.8517	24
37	0.5242	0.6156	1.6244	0.8516	23
38	0.5245	0.6160	1.6234	0.8514	22
39	0.5247	0.6164	1.6223	0.8513	21
40	0.5250	0.6168	1.6212	0.8511	20
41	0.5252	0.6172	1.6202	0.8510	19
42	0.5255	0.6176	1.6191	0.8508	18
43	0.5257	0.6180	1.6181	0.8507	17
44	0.5260	0.6184	1.6170	0.8505	16
45	0.5262	0.6188	1.6160	0.8504	15
46	0.5265	0.6192	1.6149	0.8502	14
47	0.5267	0.6196	1.6139	0.8500	13
48	0.5270	0.6200	1.6128	0.8499	12
49	0.5272	0.6204	1.6118	0.8497	11
50	0.5275	0.6208	1.6107	0.8496	10
51	0.5277	0.6212	1.6097	0.8494	9
52	0.5279	0.6216	1.6087	0.8493	8
53	0.5282	0.6220	1.6076	0.8491	7
54	0.5284	0.6224	1.6066	0.8490	6
55	0.5287	0.6228	1.6055	0.8488	5
56	0.5289	0.6233	1.6045	0.8487	4
57	0.5292	0.6237	1.6034	0.8485	3
58	0.5294	0.6241	1.6024	0.8484	2
59	0.5297	0.6245	1.6014	0.8482	1
60	0.5299	0.6249	1.6003	0.8480	0
	Cos	Cot	Tan	Sin	′

58°

32°

′	Sin	Tan	Cot	Cos	
0	0.5299	0.6249	1.6003	0.8480	60
1	0.5302	0.6253	1.5993	0.8479	59
2	0.5304	0.6257	1.5983	0.8477	58
3	0.5307	0.6261	1.5972	0.8476	57
4	0.5309	0.6265	1.5062	0.8474	56
5	0.5312	0.6269	1.5952	0.8473	55
6	0.5314	0.6273	1.5941	0.8471	54
7	0.5316	0.6277	1.5931	0.8470	53
8	0.5319	0.6281	1.5921	0.8468	52
9	0.5321	0.6285	1.5911	0.8467	51
10	0.5324	0.6289	1.5900	0.8465	50
11	0.5326	0.6293	1.5890	0.8463	49
12	0.5329	0.6297	1.5880	0.8462	48
13	0.5331	0.6301	1.5869	0.8460	47
14	0.5334	0.6305	1.5859	0.8459	46
15	0.5336	0.6310	1.5849	0.8457	45
16	0.5339	0.6314	1.5839	0.8456	44
17	0.5341	0.6318	1.5829	0.8454	43
18	0.5344	0.6322	1.5818	0.8453	42
19	0.5346	0.6326	1.5808	0.8451	41
20	0.5348	0.6330	1.5798	0.8450	40
21	0.5351	0.6334	1.5788	0.8448	39
22	0.5353	0.6338	1.5778	0.8446	38
23	0.5356	0.6342	1.5768	0.8445	37
24	0.5358	0.6346	1.5757	0.8443	36
25	0.5361	0.6350	1.5747	0.8442	35
26	0.5363	0.6354	1.5737	0.8440	34
27	0.5366	0.6358	1.5727	0.8439	33
28	0.5368	0.6363	1.5717	0.8437	32
29	0.5371	0.6367	1.5707	0.8435	31
30	0.5373	0.6371	1.5697	0.8434	30
31	0.5375	0.6375	1.5687	0.8432	29
32	0.5378	0.6379	1.5677	0.8431	28
33	0.5380	0.6383	1.5667	0.8429	27
34	0.5383	0.6387	1.5657	0.8428	26
35	0.5385	0.6391	1.5647	0.8426	25
36	0.5388	0.6395	1.5637	0.8425	24
37	0.5390	0.6399	1.5627	0.8423	23
38	0.5393	0.6403	1.5617	0.8421	22
39	0.5395	0.6408	1.5607	0.8420	21
40	0.5398	0.6412	1.5597	0.8418	20
41	0.5400	0.6416	1.5587	0.8417	19
42	0.5402	0.6420	1.5577	0.8415	18
43	0.5405	0.6424	1.5567	0.8414	17
44	0.5407	0.6428	1.5557	0.8412	16
45	0.5410	0.6432	1.5547	0.8410	15
46	0.5412	0.6436	1.5537	0.8409	14
47	0.5415	0.6440	1.5527	0.8407	13
48	0.5417	0.6445	1.5517	0.8406	12
49	0.5420	0.6449	1.5507	0.8404	11
50	0.5422	0.6453	1.5497	0.8403	10
51	0.5424	0.6457	1.5487	0.8401	9
52	0.5427	0.6461	1.5477	0.8399	8
53	0.5429	0.6465	1.5468	0.8398	7
54	0.5432	0.6469	1.5458	0.8396	6
55	0.5434	0.6473	1.5448	0.8395	5
56	0.5437	0.6478	1.5438	0.8393	4
57	0.5439	0.6482	1.5428	0.8391	3
58	0.5442	0.6486	1.5418	0.8390	2
59	0.5444	0.6490	1.5408	0.8388	1
60	0.5446	0.6494	1.5399	0.8387	0
	Cos	Cot	Tan	Sin	′

57°

33°

′	Sin	Tan	Cot	Cos	
0	0.5446	0.6494	1.5399	0.8387	60
1	0.5449	0.6498	1.5389	0.8385	59
2	0.5451	0.6502	1.5379	0.8384	58
3	0.5454	0.6506	1.5369	0.8382	57
4	0.5456	0.6511	1.5359	0.8380	56
5	0.5459	0.6515	1.5350	0.8379	55
6	0.5461	0.6519	1.5340	0.8377	54
7	0.5463	0.6523	1.5330	0.8376	53
8	0.5466	0.6527	1.5320	0.8374	52
9	0.5468	0.6531	1.5311	0.8372	51
10	0.5471	0.6536	1.5301	0.8371	50
11	0.5473	0.6540	1.5291	0.8369	49
12	0.5476	0.6544	1.5282	0.8368	48
13	0.5478	0.6548	1.5272	0.8366	47
14	0.5480	0.6552	1.5262	0.8364	46
15	0.5483	0.6556	1.5253	0.8363	45
16	0.5485	0.6560	1.5243	0.8361	44
17	0.5488	0.6565	1.5233	0.8360	43
18	0.5490	0.6569	1.5224	0.8358	42
19	0.5493	0.6573	1.5214	0.8356	41
20	0.5495	0.6577	1.5204	0.8355	40
21	0.5498	0.6581	1.5195	0.8353	39
22	0.5500	0.6585	1.5185	0.8352	38
23	0.5502	0.6590	1.5175	0.8350	37
24	0.5505	0.6594	1.5166	0.8348	36
25	0.5507	0.6598	1.5156	0.8347	35
26	0.5510	0.6602	1.5147	0.8345	34
27	0.5512	0.6606	1.5137	0.8344	33
28	0.5515	0.6610	1.5127	0.8342	32
29	0.5517	0.6615	1.5118	0.8340	31
30	0.5519	0.6619	1.5108	0.8339	30
31	0.5522	0.6623	1.5099	0.8337	29
32	0.5524	0.6627	1.5089	0.8336	28
33	0.5527	0.6631	1.5080	0.8334	27
34	0.5529	0.6636	1.5070	0.8332	26
35	0.5531	0.6640	1.5061	0.8331	25
36	0.5534	0.6644	1.5051	0.8329	24
37	0.5536	0.6648	1.5042	0.8328	23
38	0.5539	0.6652	1.5032	0.8326	22
39	0.5541	0.6657	1.5023	0.8324	21
40	0.5544	0.6661	1.5013	0.8323	20
41	0.5546	0.6665	1.5004	0.8321	19
42	0.5548	0.6669	1.4994	0.8320	18
43	0.5551	0.6673	1.4985	0.8318	17
44	0.5553	0.6678	1.4975	0.8316	16
45	0.5556	0.6682	1.4966	0.8315	15
46	0.5558	0.6686	1.4957	0.8313	14
47	0.5561	0.6690	1.4947	0.8311	13
48	0.5563	0.6694	1.4938	0.8310	12
49	0.5565	0.6699	1.4928	0.8308	11
50	0.5568	0.6703	1.4919	0.8307	10
51	0.5570	0.6707	1.4910	0.8305	9
52	0.5573	0.6711	1.4900	0.8303	8
53	0.5575	0.6715	1.4891	0.8302	7
54	0.5577	0.6720	1.4882	0.8300	6
55	0.5580	0.6724	1.4872	0.8298	5
56	0.5582	0.6728	1.4863	0.8297	4
57	0.5585	0.6732	1.4854	0.8295	3
58	0.5587	0.6737	1.4844	0.8294	2
59	0.5590	0.6741	1.4835	0.8292	1
60	0.5592	0.6745	1.4826	0.8290	0
	Cos	Cot	Tan	Sin	′

56°

34°

′	Sin	Tan	Cot	Cos	
0	0.5592	0.6745	1.4826	0.8290	60
1	0.5594	0.6749	1.4816	0.8289	59
2	0.5597	0.6754	1.4807	0.8287	58
3	0.5599	0.6758	1.4798	0.8285	57
4	0.5602	0.6762	1.4788	0.8284	56
5	0.5604	0.6766	1.4779	0.8282	55
6	0.5606	0.6771	1.4770	0.8281	54
7	0.5609	0.6775	1.4761	0.8279	53
8	0.5611	0.6779	1.4751	0.8277	52
9	0.5614	0.6783	1.4742	0.8276	51
10	0.5616	0.6787	1.4733	0.8274	50
11	0.5618	0.6792	1.4724	0.8272	49
12	0.5621	0.6796	1.4715	0.8271	48
13	0.5623	0.6800	1.4705	0.8269	47
14	0.5626	0.6805	1.4696	0.8268	46
15	0.5628	0.6809	1.4687	0.8266	45
16	0.5630	0.6813	1.4678	0.8264	44
17	0.5633	0.6817	1.4669	0.8263	43
18	0.5635	0.6822	1.4659	0.8261	42
19	0.5638	0.6826	1.4650	0.8259	41
20	0.5640	0.6830	1.4641	0.8258	40
21	0.5642	0.6834	1.4632	0.8256	39
22	0.5645	0.6839	1.4623	0.8254	38
23	0.5647	0.6843	1.4614	0.8253	37
24	0.5650	0.6847	1.4605	0.8251	36
25	0.5652	0.6851	1.4596	0.8249	35
26	0.5654	0.6856	1.4586	0.8248	34
27	0.5657	0.6860	1.4577	0.8246	33
28	0.5659	0.6864	1.4568	0.8245	32
29	0.5662	0.6869	1.4559	0.8243	31
30	0.5664	0.6873	1.4550	0.8241	30
31	0.5666	0.6877	1.4541	0.8240	29
32	0.5669	0.6881	1.4532	0.8238	28
33	0.5671	0.6886	1.4523	0.8236	27
34	0.5674	0.6890	1.4514	0.8235	26
35	0.5676	0.6894	1.4505	0.8233	25
36	0.5678	0.6899	1.4496	0.8231	24
37	0.5681	0.6903	1.4487	0.8230	23
38	0.5683	0.6907	1.4478	0.8228	22
39	0.5686	0.6911	1.4469	0.8226	21
40	0.5688	0.6916	1.4460	0.8225	20
41	0.5690	0.6920	1.4451	0.8223	19
42	0.5693	0.6924	1.4442	0.8221	18
43	0.5695	0.6929	1.4433	0.8220	17
44	0.5698	0.6933	1.4424	0.8218	16
45	0.5700	0.6937	1.4415	0.8216	15
46	0.5702	0.6942	1.4406	0.8215	14
47	0.5705	0.6946	1.4397	0.8213	13
48	0.5707	0.6950	1.4388	0.8211	12
49	0.5710	0.6954	1.4379	0.8210	11
50	0.5712	0.6959	1.4370	0.8208	10
51	0.5714	0.6963	1.4361	0.8207	9
52	0.5717	0.6967	1.4352	0.8205	8
53	0.5719	0.6972	1.4344	0.8203	7
54	0.5721	0.6976	1.4335	0.8202	6
55	0.5724	0.6980	1.4326	0.8200	5
56	0.5726	0.6985	1.4317	0.8198	4
57	0.5729	0.6989	1.4308	0.8197	3
58	0.5731	0.6993	1.4299	0.8195	2
59	0.5733	0.6998	1.4290	0.8193	1
60	0.5736	0.7002	1.4281	0.8192	0
	Cos	Cot	Tan	Sin	′

55°

35°

′	Sin	Tan	Cot	Cos	
0	0.5736	0.7002	1.4281	0.8192	60
1	0.5738	0.7006	1.4273	0.8190	59
2	0.5741	0.7011	1.4264	0.8188	58
3	0.5743	0.7015	1.4255	0.8187	57
4	0.5745	0.7019	1.4246	0.8185	56
5	0.5748	0.7024	1.4237	0.8183	55
6	0.5750	0.7028	1.4229	0.8181	54
7	0.5752	0.7032	1.4220	0.8180	53
8	0.5755	0.7037	1.4211	0.8178	52
9	0.5757	0.7041	1.4202	0.8176	51
10	0.5760	0.7046	1.4193	0.8175	50
11	0.5762	0.7050	1.4185	0.8173	49
12	0.5764	0.7054	1.4176	0.8171	48
13	0.5767	0.7059	1.4167	0.8170	47
14	0.5769	0.7063	1.4158	0.8168	46
15	0.5771	0.7067	1.4150	0.8166	45
16	0.5774	0.7072	1.4141	0.8165	44
17	0.5776	0.7076	1.4132	0.8163	43
18	0.5779	0.7080	1.4124	0.8161	42
19	0.5781	0.7085	1.4115	0.8160	41
20	0.5783	0.7089	1.4106	0.8158	40
21	0.5786	0.7094	1.4097	0.8156	39
22	0.5788	0.7098	1.4089	0.8155	38
23	0.5790	0.7102	1.4080	0.8153	37
24	0.5793	0.7107	1.4071	0.8151	36
25	0.5795	0.7111	1.4063	0.8150	35
26	0.5798	0.7115	1.4054	0.8148	34
27	0.5800	0.7120	1.4045	0.8146	33
28	0.5802	0.7124	1.4037	0.8145	32
29	0.5805	0.7129	1.4028	0.8143	31
30	0.5807	0.7133	1.4019	0.8141	30
31	0.5809	0.7137	1.4011	0.8139	29
32	0.5812	0.7142	1.4002	0.8138	28
33	0.5814	0.7146	1.3994	0.8136	27
34	0.5816	0.7151	1.3985	0.8134	26
35	0.5819	0.7155	1.3976	0.8133	25
36	0.5821	0.7159	1.3968	0.8131	24
37	0.5824	0.7164	1.3959	0.8129	23
38	0.5826	0.7168	1.3951	0.8128	22
39	0.5828	0.7173	1.3942	0.8126	21
40	0.5831	0.7177	1.3934	0.8124	20
41	0.5833	0.7181	1.3925	0.8123	19
42	0.5835	0.7186	1.3916	0.8121	18
43	0.5838	0.7190	1.3908	0.8119	17
44	0.5840	0.7195	1.3899	0.8117	16
45	0.5842	0.7199	1.3891	0.8116	15
46	0.5845	0.7203	1.3882	0.8114	14
47	0.5847	0.7208	1.3874	0.8112	13
48	0.5850	0.7212	1.3865	0.8111	12
49	0.5852	0.7217	1.3857	0.8109	11
50	0.5854	0.7221	1.3848	0.8107	10
51	0.5857	0.7226	1.3840	0.8106	9
52	0.5859	0.7230	1.3831	0.8104	8
53	0.5861	0.7234	1.3823	0.8102	7
54	0.5864	0.7239	1.3814	0.8100	6
55	0.5866	0.7243	1.3806	0.8099	5
56	0.5868	0.7248	1.3798	0.8097	4
57	0.5871	0.7252	1.3789	0.8095	3
58	0.5873	0.7257	1.3781	0.8094	2
59	0.5875	0.7261	1.3772	0.8092	1
60	0.5878	0.7265	1.3764	0.8090	0
	Cos	Cot	Tan	Sin	′

54°

36°

′	Sin	Tan	Cot	Cos	
0	0.5878	0.7265	1.3764	0.8090	60
1	0.5880	0.7270	1.3755	0.8088	59
2	0.5883	0.7274	1.3747	0.8087	58
3	0.5885	0.7279	1.3739	0.8085	57
4	0.5887	0.7283	1.3730	0.8083	56
5	0.5890	0.7288	1.3722	0.8082	55
6	0.5892	0.7292	1.3713	0.8080	54
7	0.5894	0.7297	1.3705	0.8078	53
8	0.5897	0.7301	1.3697	0.8076	52
9	0.5899	0.7306	1.3688	0.8075	51
10	0.5901	0.7310	1.3680	0.8073	50
11	0.5904	0.7314	1.3672	0.8071	49
12	0.5906	0.7319	1.3663	0.8070	48
13	0.5908	0.7323	1.3655	0.8068	47
14	0.5911	0.7328	1.3647	0.8066	46
15	0.5913	0.7332	1.3638	0.8064	45
16	0.5915	0.7337	1.3630	0.8063	44
17	0.5918	0.7341	1.3622	0.8061	43
18	0.5920	0.7346	1.3613	0.8059	42
19	0.5922	0.7350	1.3605	0.8058	41
20	0.5925	0.7355	1.3597	0.8056	40
21	0.5927	0.7359	1.3588	0.8054	39
22	0.5930	0.7364	1.3580	0.8052	38
23	0.5932	0.7368	1.3572	0.8051	37
24	0.5934	0.7373	1.3564	0.8049	36
25	0.5937	0.7377	1.3555	0.8047	35
26	0.5939	0.7382	1.3547	0.8045	34
27	0.5941	0.7386	1.3539	0.8044	33
28	0.5944	0.7391	1.3531	0.8042	32
29	0.5946	0.7395	1.3522	0.8040	31
30	0.5948	0.7400	1.3514	0.8039	30
31	0.5951	0.7404	1.3506	0.8037	29
32	0.5953	0.7409	1.3498	0.8035	28
33	0.5955	0.7413	1.3490	0.8033	27
34	0.5958	0.7418	1.3481	0.8032	26
35	0.5960	0.7422	1.3473	0.8030	25
36	0.5962	0.7427	1.3465	0.8028	24
37	0.5965	0.7431	1.3457	0.8026	23
38	0.5967	0.7436	1.3449	0.8025	22
39	0.5969	0.7440	1.3440	0.8023	21
40	0.5972	0.7445	1.3432	0.8021	20
41	0.5974	0.7449	1.3424	0.8019	19
42	0.5976	0.7454	1.3416	0.8018	18
43	0.5979	0.7458	1.3408	0.8016	17
44	0.5981	0.7463	1.3400	0.8014	16
45	0.5983	0.7467	1.3392	0.8013	15
46	0.5986	0.7472	1.3384	0.8011	14
47	0.5988	0.7476	1.3375	0.8009	13
48	0.5990	0.7481	1.3367	0.8007	12
49	0.5993	0.7485	1.3359	0.8006	11
50	0.5995	0.7490	1.3351	0.8004	10
51	0.5997	0.7495	1.3343	0.8002	9
52	0.6000	0.7499	1.3335	0.8000	8
53	0.6002	0.7504	1.3327	0.7999	7
54	0.6004	0.7508	1.3319	0.7997	6
55	0.6007	0.7513	1.3311	0.7995	5
56	0.6009	0.7517	1.3303	0.7993	4
57	0.6011	0.7522	1.3295	0.7992	3
58	0.6014	0.7526	1.3287	0.7990	2
59	0.6016	0.7531	1.3278	0.7988	1
60	0.6018	0.7536	1.3270	0.7986	0
	Cos	Cot	Tan	Sin	′

53°

37°

′	Sin	Tan	Cot	Cos	
0	0.6018	0.7536	1.3270	0.7986	60
1	0.6020	0.7540	1.3262	0.7985	59
2	0.6023	0.7545	1.3254	0.7983	58
3	0.6025	0.7549	1.3246	0.7981	57
4	0.6027	0.7554	1.3238	0.7979	56
5	0.6030	0.7558	1.3230	0.7978	55
6	0.6032	0.7563	1.3222	0.7976	54
7	0.6034	0.7568	1.3214	0.7974	53
8	0.6037	0.7572	1.3206	0.7972	52
9	0.6039	0.7577	1.3198	0.7971	51
10	0.6041	0.7581	1.3190	0.7969	50
11	0.6044	0.7586	1.3182	0.7967	49
12	0.6046	0.7590	1.3175	0.7965	48
13	0.6048	0.7595	1.3167	0.7964	47
14	0.6051	0.7600	1.3159	0.7962	46
15	0.6053	0.7604	1.3151	0.7960	45
16	0.6055	0.7609	1.3143	0.7958	44
17	0.6058	0.7613	1.3135	0.7956	43
18	0.6060	0.7618	1.3127	0.7955	42
19	0.6062	0.7623	1.3119	0.7953	41
20	0.6065	0.7627	1.3111	0.7951	40
21	0.6067	0.7632	1.3103	0.7949	39
22	0.6069	0.7636	1.3095	0.7948	38
23	0.6071	0.7641	1.3087	0.7946	37
24	0.6074	0.7646	1.3079	0.7944	36
25	0.6076	0.7650	1.3072	0.7942	35
26	0.6078	0.7655	1.3064	0.7941	34
27	0.6081	0.7659	1.3056	0.7939	33
28	0.6083	0.7664	1.3048	0.7937	32
29	0.6085	0.7669	1.3040	0.7935	31
30	0.6088	0.7673	1.3032	0.7934	30
31	0.6090	0.7678	1.3024	0.7932	29
32	0.6092	0.7683	1.3017	0.7930	28
33	0.6095	0.7687	1.3009	0.7928	27
34	0.6097	0.7692	1.3001	0.7926	26
35	0.6099	0.7696	1.2993	0.7925	25
36	0.6101	0.7701	1.2985	0.7923	24
37	0.6104	0.7706	1.2977	0.7921	23
38	0.6106	0.7710	1.2970	0.7919	22
39	0.6108	0.7715	1.2962	0.7918	21
40	0.6111	0.7720	1.2954	0.7916	20
41	0.6113	0.7724	1.2946	0.7914	19
42	0.6115	0.7729	1.2938	0.7912	18
43	0.6118	0.7734	1.2931	0.7910	17
44	0.6120	0.7738	1.2923	0.7909	16
45	0.6122	0.7743	1.2915	0.7907	15
46	0.6124	0.7747	1.2907	0.7905	14
47	0.6127	0.7752	1.2900	0.7903	13
48	0.6129	0.7757	1.2892	0.7902	12
49	0.6131	0.7761	1.2884	0.7900	11
50	0.6134	0.7766	1.2876	0.7898	10
51	0.6136	0.7771	1.2869	0.7896	9
52	0.6138	0.7775	1.2861	0.7894	8
53	0.6141	0.7780	1.2853	0.7893	7
54	0.6143	0.7785	1.2846	0.7891	6
55	0.6145	0.7789	1.2838	0.7889	5
56	0.6147	0.7794	1.2830	0.7887	4
57	0.6150	0.7799	1.2822	0.7885	3
58	0.6152	0.7803	1.2815	0.7884	2
59	0.6154	0.7808	1.2807	0.7882	1
60	0.6157	0.7813	1.2799	0.7880	0
	Cos	Cot	Tan	Sin	′

52°

III. Natural Trigonometric Functions

38°

′	Sin	Tan	Cot	Cos	
0	0.6157	0.7813	1.2799	0.7880	60
1	0.6159	0.7818	1.2792	0.7878	59
2	0.6161	0.7822	1.2784	0.7877	58
3	0.6163	0.7827	1.2776	0.7875	57
4	0.6166	0.7832	1.2769	0.7873	56
5	0.6168	0.7836	1.2761	0.7871	55
6	0.6170	0.7841	1.2753	0.7869	54
7	0.6173	0.7846	1.2746	0.7868	53
8	0.6175	0.7850	1.2738	0.7866	52
9	0.6177	0.7855	1.2731	0.7864	51
10	0.6180	0.7860	1.2723	0.7862	50
11	0.6182	0.7865	1.2715	0.7860	49
12	0.6184	0.7869	1.2708	0.7859	48
13	0.6186	0.7874	1.2700	0.7857	47
14	0.6189	0.7879	1.2693	0.7855	46
15	0.6191	0.7883	1.2685	0.7853	45
16	0.6193	0.7888	1.2677	0.7851	44
17	0.6196	0.7893	1.2670	0.7850	43
18	0.6198	0.7898	1.2662	0.7848	42
19	0.6200	0.7902	1.2655	0.7846	41
20	0.6202	0.7907	1.2647	0.7844	40
21	0.6205	0.7912	1.2640	0.7842	39
22	0.6207	0.7916	1.2632	0.7841	38
23	0.6209	0.7921	1.2624	0.7839	37
24	0.6211	0.7926	1.2617	0.7837	36
25	0.6214	0.7931	1.2609	0.7835	35
26	0.6216	0.7935	1.2602	0.7833	34
27	0.6218	0.7940	1.2594	0.7832	33
28	0.6221	0.7945	1.2587	0.7830	32
29	0.6223	0.7950	1.2579	0.7828	31
30	0.6225	0.7954	1.2572	0.7826	30
31	0.6227	0.7959	1.2564	0.7824	29
32	0.6230	0.7964	1.2557	0.7822	28
33	0.6232	0.7969	1.2549	0.7821	27
34	0.6234	0.7973	1.2542	0.7819	26
35	0.6237	0.7978	1.2534	0.7817	25
36	0.6239	0.7983	1.2527	0.7815	24
37	0.6241	0.7988	1.2519	0.7813	23
38	0.6243	0.7992	1.2512	0.7812	22
39	0.6246	0.7997	1.2504	0.7810	21
40	0.6248	0.8002	1.2497	0.7808	20
41	0.6250	0.8007	1.2489	0.7806	19
42	0.6252	0.8012	1.2482	0.7804	18
43	0.6255	0.8016	1.2475	0.7802	17
44	0.6257	0.8021	1.2467	0.7801	16
45	0.6259	0.8026	1.2460	0.7799	15
46	0.6262	0.8031	1.2452	0.7797	14
47	0.6264	0.8035	1.2445	0.7795	13
48	0.6266	0.8040	1.2437	0.7793	12
49	0.6268	0.8045	1.2430	0.7792	11
50	0.6271	0.8050	1.2423	0.7790	10
51	0.6273	0.8055	1.2415	0.7788	9
52	0.6275	0.8059	1.2408	0.7786	8
53	0.6277	0.8064	1.2401	0.7784	7
54	0.6280	0.8069	1.2393	0.7782	6
55	0.6282	0.8074	1.2386	0.7781	5
56	0.6284	0.8079	1.2378	0.7779	4
57	0.6286	0.8083	1.2371	0.7777	3
58	0.6289	0.8088	1.2364	0.7775	2
59	0.6291	0.8093	1.2356	0.7773	1
60	0.6293	0.8098	1.2349	0.7771	0
	Cos	Cot	Tan	Sin	′

51°

39°

′	Sin	Tan	Cot	Cos	
0	0.6293	0.8098	1.2349	0.7771	60
1	0.6295	0.8103	1.2342	0.7770	59
2	0.6298	0.8107	1.2334	0.7768	58
3	0.6300	0.8112	1.2327	0.7766	57
4	0.6302	0.8117	1.2320	0.7764	56
5	0.6305	0.8122	1.2312	0.7762	55
6	0.6307	0.8127	1.2305	0.7760	54
7	0.6309	0.8132	1.2298	0.7759	53
8	0.6311	0.8136	1.2290	0.7757	52
9	0.6314	0.8141	1.2283	0.7755	51
10	0.6316	0.8146	1.2276	0.7753	50
11	0.6318	0.8151	1.2268	0.7751	49
12	0.6320	0.8156	1.2261	0.7749	48
13	0.6323	0.8161	1.2254	0.7748	47
14	0.6325	0.8165	1.2247	0.7746	46
15	0.6327	0.8170	1.2239	0.7744	45
16	0.6329	0.8175	1.2232	0.7742	44
17	0.6332	0.8180	1.2225	0.7740	43
18	0.6334	0.8185	1.2218	0.7738	42
19	0.6336	0.8190	1.2210	0.7737	41
20	0.6338	0.8195	1.2203	0.7735	40
21	0.6341	0.8199	1.2196	0.7733	39
22	0.6343	0.8204	1.2189	0.7731	38
23	0.6345	0.8209	1.2181	0.7729	37
24	0.6347	0.8214	1.2174	0.7727	36
25	0.6350	0.8219	1.2167	0.7725	35
26	0.6352	0.8224	1.2160	0.7724	34
27	0.6354	0.8229	1.2153	0.7722	33
28	0.6356	0.8234	1.2145	0.7720	32
29	0.6359	0.8238	1.2138	0.7718	31
30	0.6361	0.8243	1.2131	0.7716	30
31	0.6363	0.8248	1.2124	0.7714	29
32	0.6365	0.8253	1.2117	0.7713	28
33	0.6368	0.8258	1.2109	0.7711	27
34	0.6370	0.8263	1.2102	0.7709	26
35	0.6372	0.8268	1.2095	0.7707	25
36	0.6374	0.8273	1.2088	0.7705	24
37	0.6376	0.8278	1.2081	0.7703	23
38	0.6379	0.8283	1.2074	0.7701	22
39	0.6381	0.8287	1.2066	0.7700	21
40	0.6383	0.8292	1.2059	0.7698	20
41	0.6385	0.8297	1.2052	0.7696	19
42	0.6388	0.8302	1.2045	0.7694	18
43	0.6390	0.8307	1.2038	0.7692	17
44	0.6392	0.8312	1.2031	0.7690	16
45	0.6394	0.8317	1.2024	0.7688	15
46	0.6397	0.8322	1.2017	0.7687	14
47	0.6399	0.8327	1.2009	0.7685	13
48	0.6401	0.8332	1.2002	0.7683	12
49	0.6403	0.8337	1.1995	0.7681	11
50	0.6406	0.8342	1.1988	0.7679	10
51	0.6408	0.8346	1.1981	0.7677	9
52	0.6410	0.8351	1.1974	0.7675	8
53	0.6412	0.8356	1.1967	0.7674	7
54	0.6414	0.8361	1.1960	0.7672	6
55	0.6417	0.8366	1.1953	0.7670	5
56	0.6419	0.8371	1.1946	0.7668	4
57	0.6421	0.8376	1.1939	0.7666	3
58	0.6423	0.8381	1.1932	0.7664	2
59	0.6426	0.8386	1.1925	0.7662	1
60	0.6428	0.8391	1.1918	0.7660	0
	Cos	Cot	Tan	Sin	′

50°

40°

′	Sin	Tan	Cot	Cos	
0	0.6428	0.8391	1.1918	0.7660	60
1	0.6430	0.8396	1.1910	0.7659	59
2	0.6432	0.8401	1.1903	0.7657	58
3	0.6435	0.8406	1.1896	0.7655	57
4	0.6437	0.8411	1.1889	0.7653	56
5	0.6439	0.8416	1.1882	0.7651	55
6	0.6441	0.8421	1.1875	0.7649	54
7	0.6443	0.8426	1.1868	0.7647	53
8	0.6446	0.8431	1.1861	0.7645	52
9	0.6448	0.8436	1.1854	0.7644	51
10	0.6450	0.8441	1.1847	0.7642	50
11	0.6452	0.8446	1.1840	0.7640	49
12	0.6455	0.8451	1.1833	0.7638	48
13	0.6457	0.8456	1.1826	0.7636	47
14	0.6459	0.8461	1.1819	0.7634	46
15	0.6461	0.8466	1.1812	0.7632	45
16	0.6463	0.8471	1.1806	0.7630	44
17	0.6466	0.8476	1.1799	0.7629	43
18	0.6468	0.8481	1.1792	0.7627	42
19	0.6470	0.8486	1.1785	0.7625	41
20	0.6472	0.8491	1.1778	0.7623	40
21	0.6475	0.8496	1.1771	0.7621	39
22	0.6477	0.8501	1.1764	0.7619	38
23	0.6479	0.8506	1.1757	0.7617	37
24	0.6481	0.8511	1.1750	0.7615	36
25	0.6483	0.8516	1.1743	0.7613	35
26	0.6486	0.8521	1.1736	0.7612	34
27	0.6488	0.8526	1.1729	0.7610	33
28	0.6490	0.8531	1.1722	0.7608	32
29	0.6492	0.8536	1.1715	0.7606	31
30	0.6494	0.8541	1.1708	0.7604	30
31	0.6497	0.8546	1.1702	0.7602	29
32	0.6499	0.8551	1.1695	0.7600	28
33	0.6501	0.8556	1.1688	0.7598	27
34	0.6503	0.8561	1.1681	0.7596	26
35	0.6506	0.8566	1.1674	0.7595	25
36	0.6508	0.8571	1.1667	0.7593	24
37	0.6510	0.8576	1.1660	0.7591	23
38	0.6512	0.8581	1.1653	0.7589	22
39	0.6514	0.8586	1.1647	0.7587	21
40	0.6517	0.8591	1.1640	0.7585	20
41	0.6519	0.8596	1.1633	0.7583	19
42	0.6521	0.8601	1.1626	0.7581	18
43	0.6523	0.8606	1.1619	0.7579	17
44	0.6525	0.8611	1.1612	0.7578	16
45	0.6528	0.8617	1.1606	0.7576	15
46	0.6530	0.8622	1.1599	0.7574	14
47	0.6532	0.8627	1.1592	0.7572	13
48	0.6534	0.8632	1.1585	0.7570	12
49	0.6536	0.8637	1.1578	0.7568	11
50	0.6539	0.8642	1.1571	0.7566	10
51	0.6541	0.8647	1.1565	0.7564	9
52	0.6543	0.8652	1.1558	0.7562	8
53	0.6545	0.8657	1.1551	0.7560	7
54	0.6547	0.8662	1.1544	0.7559	6
55	0.6550	0.8667	1.1538	0.7557	5
56	0.6552	0.8672	1.1531	0.7555	4
57	0.6554	0.8678	1.1524	0.7553	3
58	0.6556	0.8683	1.1517	0.7551	2
59	0.6558	0.8688	1.1510	0.7549	1
60	0.6561	0.8693	1.1504	0.7547	0
	Cos	Cot	Tan	Sin	′

49°

41°

′	Sin	Tan	Cot	Cos	
0	0.6561	0.8693	1.1504	0.7547	60
1	0.6563	0.8698	1.1497	0.7545	59
2	0.6565	0.8703	1.1490	0.7543	58
3	0.6567	0.8708	1.1483	0.7541	57
4	0.6569	0.8713	1.1477	0.7539	56
5	0.6572	0.8718	1.1470	0.7538	55
6	0.6574	0.8724	1.1463	0.7536	54
7	0.6576	0.8729	1.1456	0.7534	53
8	0.6578	0.8734	1.1450	0.7532	52
9	0.6580	0.8739	1.1443	0.7530	51
10	0.6583	0.8744	1.1436	0.7528	50
11	0.6585	0.8749	1.1430	0.7526	49
12	0.6587	0.8754	1.1423	0.7524	48
13	0.6589	0.8759	1.1416	0.7522	47
14	0.6591	0.8765	1.1410	0.7520	46
15	0.6593	0.8770	1.1403	0.7518	45
16	0.6596	0.8775	1.1396	0.7516	44
17	0.6598	0.8780	1.1389	0.7515	43
18	0.6600	0.8785	1.1383	0.7513	42
19	0.6602	0.8790	1.1376	0.7511	41
20	0.6604	0.8796	1.1369	0.7509	40
21	0.6607	0.8801	1.1363	0.7507	39
22	0.6609	0.8806	1.1356	0.7505	38
23	0.6611	0.8811	1.1349	0.7503	37
24	0.6613	0.8816	1.1343	0.7501	36
25	0.6615	0.8821	1.1336	0.7499	35
26	0.6617	0.8827	1.1329	0.7497	34
27	0.6620	0.8832	1.1323	0.7495	33
28	0.6622	0.8837	1.1316	0.7493	32
29	0.6624	0.8842	1.1310	0.7491	31
30	0.6626	0.8847	1.1303	0.7490	30
31	0.6628	0.8852	1.1296	0.7488	29
32	0.6631	0.8858	1.1290	0.7486	28
33	0.6633	0.8863	1.1283	0.7484	27
34	0.6635	0.8868	1.1276	0.7482	26
35	0.6637	0.8873	1.1270	0.7480	25
36	0.6639	0.8878	1.1263	0.7478	24
37	0.6641	0.8884	1.1257	0.7476	23
38	0.6644	0.8889	1.1250	0.7474	22
39	0.6646	0.8894	1.1243	0.7472	21
40	0.6648	0.8899	1.1237	0.7470	20
41	0.6650	0.8904	1.1230	0.7468	19
42	0.6652	0.8910	1.1224	0.7466	18
43	0.6654	0.8915	1.1217	0.7464	17
44	0.6657	0.8920	1.1211	0.7463	16
45	0.6659	0.8925	1.1204	0.7461	15
46	0.6661	0.8931	1.1197	0.7459	14
47	0.6663	0.8936	1.1191	0.7457	13
48	0.6665	0.8941	1.1184	0.7455	12
49	0.6667	0.8946	1.1178	0.7453	11
50	0.6670	0.8952	1.1171	0.7451	10
51	0.6672	0.8957	1.1165	0.7449	9
52	0.6674	0.8962	1.1158	0.7447	8
53	0.6676	0.8967	1.1152	0.7445	7
54	0.6678	0.8972	1.1145	0.7443	6
55	0.6680	0.8978	1.1139	0.7441	5
56	0.6683	0.8983	1.1132	0.7439	4
57	0.6685	0.8988	1.1126	0.7437	3
58	0.6687	0.8994	1.1119	0.7435	2
59	0.6689	0.8999	1.1113	0.7433	1
60	0.6691	0.9004	1.1106	0.7431	0
	Cos	Cot	Tan	Sin	′

48°

III. Natural Trigonometric Functions

42°

′	Sin	Tan	Cot	Cos	
0	0.6691	0.9004	1.1106	0.7431	60
1	0.6693	0.9009	1.1100	0.7430	59
2	0.6696	0.9015	1.1093	0.7428	58
3	0.6698	0.9020	1.1087	0.7426	57
4	0.6700	0.9025	1.1080	0.7424	56
5	0.6702	0.9030	1.1074	0.7422	55
6	0.6704	0.9036	1.1067	0.7420	54
7	0.6706	0.9041	1.1061	0.7418	53
8	0.6709	0.9046	1.1054	0.7416	52
9	0.6711	0.9052	1.1048	0.7414	51
10	0.6713	0.9057	1.1041	0.7412	50
11	0.6715	0.9062	1.1035	0.7410	49
12	0.6717	0.9067	1.1028	0.7408	48
13	0.6719	0.9073	1.1022	0.7406	47
14	0.6722	0.9078	1.1016	0.7404	46
15	0.6724	0.9083	1.1009	0.7402	45
16	0.6726	0.9089	1.1003	0.7400	44
17	0.6728	0.9094	1.0996	0.7398	43
18	0.6730	0.9099	1.0990	0.7396	42
19	0.6732	0.9105	1.0983	0.7394	41
20	0.6734	0.9110	1.0977	0.7392	40
21	0.6737	0.9115	1.0971	0.7390	39
22	0.6739	0.9121	1.0964	0.7388	38
23	0.6741	0.9126	1.0958	0.7387	37
24	0.6743	0.9131	1.0951	0.7385	36
25	0.6745	0.9137	1.0945	0.7383	35
26	0.6747	0.9142	1.0939	0.7381	34
27	0.6749	0.9147	1.0932	0.7379	33
28	0.6752	0.9153	1.0926	0.7377	32
29	0.6754	0.9158	1.0919	0.7375	31
30	0.6756	0.9163	1.0913	0.7373	30
31	0.6758	0.9169	1.0907	0.7371	29
32	0.6760	0.9174	1.0900	0.7369	28
33	0.6762	0.9179	1.0894	0.7367	27
34	0.6764	0.9185	1.0888	0.7365	26
35	0.6767	0.9190	1.0881	0.7363	25
36	0.6769	0.9195	1.0875	0.7361	24
37	0.6771	0.9201	1.0869	0.7359	23
38	0.6773	0.9206	1.0862	0.7357	22
39	0.6775	0.9212	1.0856	0.7355	21
40	0.6777	0.9217	1.0850	0.7353	20
41	0.6779	0.9222	1.0843	0.7351	19
42	0.6782	0.9228	1.0837	0.7349	18
43	0.6784	0.9233	1.0831	0.7347	17
44	0.6786	0.9239	1.0824	0.7345	16
45	0.6788	0.9244	1.0818	0.7343	15
46	0.6790	0.9249	1.0812	0.7341	14
47	0.6792	0.9255	1.0805	0.7339	13
48	0.6794	0.9260	1.0799	0.7337	12
49	0.6797	0.9266	1.0793	0.7335	11
50	0.6799	0.9271	1.0786	0.7333	10
51	0.6801	0.9276	1.0780	0.7331	9
52	0.6803	0.9282	1.0774	0.7329	8
53	0.6805	0.9287	1.0768	0.7327	7
54	0.6807	0.9293	1.0761	0.7325	6
55	0.6809	0.9298	1.0755	0.7323	5
56	0.6811	0.9303	1.0749	0.7321	4
57	0.6814	0.9309	1.0742	0.7319	3
58	0.6816	0.9314	1.0736	0.7318	2
59	0.6818	0.9320	1.0730	0.7316	1
60	0.6820	0.9325	1.0724	0.7314	0
	Cos	Cot	Tan	Sin	′
					47°

43°

′	Sin	Tan	Cot	Cos	
0	0.6820	0.9325	1.0724	0.7314	60
1	0.6822	0.9331	1.0717	0.7312	59
2	0.6824	0.9336	1.0711	0.7310	58
3	0.6826	0.9341	1.0705	0.7308	57
4	0.6828	0.9347	1.0699	0.7306	56
5	0.6831	0.9352	1.0692	0.7304	55
6	0.6833	0.9358	1.0686	0.7302	54
7	0.6835	0.9363	1.0680	0.7300	53
8	0.6837	0.9369	1.0674	0.7298	52
9	0.6839	0.9374	1.0668	0.7296	51
10	0.6841	0.9380	1.0661	0.7294	50
11	0.6843	0.9385	1.0655	0.7292	49
12	0.6845	0.9391	1.0649	0.7290	48
13	0.6848	0.9396	1.0643	0.7288	47
14	0.6850	0.9402	1.0637	0.7286	46
15	0.6852	0.9407	1.0630	0.7284	45
16	0.6854	0.9413	1.0624	0.7282	44
17	0.6856	0.9418	1.0618	0.7280	43
18	0.6858	0.9424	1.0612	0.7278	42
19	0.6860	0.9429	1.0606	0.7276	41
20	0.6862	0.9435	1.0599	0.7274	40
21	0.6865	0.9440	1.0593	0.7272	39
22	0.6867	0.9446	1.0587	0.7270	38
23	0.6869	0.9451	1.0581	0.7268	37
24	0.6871	0.9457	1.0575	0.7266	36
25	0.6873	0.9462	1.0569	0.7264	35
26	0.6875	0.9468	1.0562	0.7262	34
27	0.6877	0.9473	1.0556	0.7260	33
28	0.6879	0.9479	1.0550	0.7258	32
29	0.6881	0.9484	1.0544	0.7256	31
30	0.6884	0.9490	1.0538	0.7254	30
31	0.6886	0.9495	1.0532	0.7252	29
32	0.6888	0.9501	1.0526	0.7250	28
33	0.6890	0.9506	1.0519	0.7248	27
34	0.6892	0.9512	1.0513	0.7246	26
35	0.6894	0.9517	1.0507	0.7244	25
36	0.6896	0.9523	1.0501	0.7242	24
37	0.6898	0.9528	1.0495	0.7240	23
38	0.6900	0.9534	1.0489	0.7238	22
39	0.6903	0.9540	1.0483	0.7236	21
40	0.6905	0.9545	1.0477	0.7234	20
41	0.6907	0.9551	1.0470	0.7232	19
42	0.6909	0.9556	1.0464	0.7230	18
43	0.6911	0.9562	1.0458	0.7228	17
44	0.6913	0.9567	1.0452	0.7226	16
45	0.6915	0.9573	1.0446	0.7224	15
46	0.6917	0.9578	1.0440	0.7222	14
47	0.6919	0.9584	1.0434	0.7220	13
48	0.6921	0.9590	1.0428	0.7218	12
49	0.6924	0.9595	1.0422	0.7216	11
50	0.6926	0.9601	1.0416	0.7214	10
51	0.6928	0.9606	1.0410	0.7212	9
52	0.6930	0.9612	1.0404	0.7210	8
53	0.6932	0.9618	1.0398	0.7208	7
54	0.6934	0.9623	1.0392	0.7206	6
55	0.6936	0.9629	1.0385	0.7203	5
56	0.6938	0.9634	1.0379	0.7201	4
57	0.6940	0.9640	1.0373	0.7199	3
58	0.6942	0.9646	1.0367	0.7197	2
59	0.6944	0.9651	1.0361	0.7195	1
60	0.6947	0.9657	1.0355	0.7193	0
	Cos	Cot	Tan	Sin	′
					46°

44°

′	Sin	Tan	Cot	Cos	
0	0.6947	0.9657	1.0355	0.7193	60
1	0.6949	0.9663	1.0349	0.7191	59
2	0.6951	0.9668	1.0343	0.7189	58
3	0.6953	0.9674	1.0337	0.7187	57
4	0.6955	0.9679	1.0331	0.7185	56
5	0.6957	0.9685	1.0325	0.7183	55
6	0.6959	0.9691	1.0319	0.7181	54
7	0.6961	0.9696	1.0313	0.7179	53
8	0.6963	0.9702	1.0307	0.7177	52
9	0.6965	0.9708	1.0301	0.7175	51
10	0.6967	0.9713	1.0295	0.7173	50
11	0.6970	0.9719	1.0289	0.7171	49
12	0.6972	0.9725	1.0283	0.7169	48
13	0.6974	0.9730	1.0277	0.7167	47
14	0.6976	0.9736	1.0271	0.7165	46
15	0.6978	0.9742	1.0265	0.7163	45
16	0.6980	0.9747	1.0259	0.7161	44
17	0.6982	0.9753	1.0253	0.7159	43
18	0.6984	0.9759	1.0247	0.7157	42
19	0.6986	0.9764	1.0241	0.7155	41
20	0.6988	0.9770	1.0235	0.7153	40
21	0.6990	0.9776	1.0230	0.7151	39
22	0.6992	0.9781	1.0224	0.7149	38
23	0.6995	0.9787	1.0218	0.7147	37
24	0.6997	0.9793	1.0212	0.7145	36
25	0.6999	0.9798	1.0206	0.7143	35
26	0.7001	0.9804	1.0200	0.7141	34
27	0.7003	0.9810	1.0194	0.7139	33
28	0.7005	0.9816	1.0188	0.7137	32
29	0.7007	0.9821	1.0182	0.7135	31
30	0.7009	0.9827	1.0176	0.7133	30
31	0.7011	0.9833	1.0170	0.7130	29
32	0.7013	0.9838	1.0164	0.7128	28
33	0.7015	0.9844	1.0158	0.7126	27
34	0.7017	0.9850	1.0152	0.7124	26
35	0.7019	0.9856	1.0147	0.7122	25
36	0.7022	0.9861	1.0141	0.7120	24
37	0.7024	0.9867	1.0135	0.7118	23
38	0.7026	0.9873	1.0129	0.7116	22
39	0.7028	0.9879	1.0123	0.7114	21
40	0.7030	0.9884	1.0117	0.7112	20
41	0.7032	0.9890	1.0111	0.7110	19
42	0.7034	0.9896	1.0105	0.7108	18
43	0.7036	0.9902	1.0099	0.7106	17
44	0.7038	0.9907	1.0094	0.7104	16
45	0.7040	0.9913	1.0088	0.7102	15
46	0.7042	0.9919	1.0082	0.7100	14
47	0.7044	0.9925	1.0076	0.7098	13
48	0.7046	0.9930	1.0070	0.7096	12
49	0.7048	0.9936	1.0064	0.7094	11
50	0.7050	0.9942	1.0058	0.7092	10
51	0.7053	0.9948	1.0052	0.7090	9
52	0.7055	0.9954	1.0047	0.7088	8
53	0.7057	0.9959	1.0041	0.7085	7
54	0.7059	0.9965	1.0035	0.7083	6
55	0.7061	0.9971	1.0029	0.7081	5
56	0.7063	0.9977	1.0023	0.7079	4
57	0.7065	0.9983	1.0017	0.7077	3
58	0.7067	0.9988	1.0012	0.7075	2
59	0.7069	0.9994	1.0006	0.7073	1
60	0.7071	1.0000	1.0000	0.7071	0
	Cos	Cot	Tan	Sin	′

45°

IV. Squares and Square Roots

N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$
1.00	1.000	1.000	3.162	1.60	2.560	1.265	4.000	2.20	4.840	1.483	4.690
1.01	1.020	1.005	3.178	1.61	2.592	1.269	4.012	2.21	4.884	1.487	4.701
1.02	1.040	1.010	3.194	1.62	2.624	1.273	4.025	2.22	4.928	1.490	4.712
1.03	1.061	1.015	3.209	1.63	2.657	1.277	4.037	2.23	4.973	1.493	4.722
1.04	1.082	1.020	3.225	1.64	2.690	1.281	4.050	2.24	5.018	1.497	4.733
1.05	1.102	1.025	3.240	1.65	2.722	1.285	4.062	2.25	5.062	1.500	4.743
1.06	1.124	1.030	3.256	1.66	2.756	1.288	4.074	2.26	5.108	1.503	4.754
1.07	1.145	1.034	3.271	1.67	2.789	1.292	4.087	2.27	5.153	1.507	4.764
1.08	1.166	1.039	3.286	1.68	2.822	1.296	4.099	2.28	5.198	1.510	4.775
1.09	1.188	1.044	3.302	1.69	2.856	1.300	4.111	2.29	5.244	1.513	4.785
1.10	1.210	1.049	3.317	1.70	2.890	1.304	4.123	2.30	5.290	1.517	4.796
1.11	1.232	1.054	3.332	1.71	2.924	1.308	4.135	2.31	5.336	1.520	4.806
1.12	1.254	1.058	3.347	1.72	2.958	1.311	4.147	2.32	5.382	1.523	4.817
1.13	1.277	1.063	3.362	1.73	2.993	1.315	4.159	2.33	5.429	1.526	4.827
1.14	1.300	1.068	3.376	1.74	3.208	1.319	4.171	2.34	5.476	1.530	4.837
1.15	1.322	1.072	3.391	1.75	3.062	1.323	4.183	2.35	5.522	1.533	4.848
1.16	1.346	1.077	3.406	1.76	3.098	1.327	4.195	2.36	5.570	1.536	4.858
1.17	1.369	1.082	3.421	1.77	3.133	1.330	4.207	2.37	5.617	1.539	4.868
1.18	1.392	1.086	3.435	1.78	3.168	1.334	4.219	2.38	5.664	1.543	4.879
1.19	1.416	1.091	3.450	1.79	3.204	1.338	4.231	2.39	5.712	1.546	4.889
1.20	1.440	1.095	3.464	1.80	3.240	1.342	4.243	2.40	5.760	1.549	4.899
1.21	1.464	1.100	3.479	1.81	3.276	1.345	4.254	2.41	5.808	1.552	4.909
1.22	1.488	1.105	3.493	1.82	3.312	1.349	4.266	2.42	5.856	1.556	4.919
1.23	1.513	1.109	3.507	1.83	3.349	1.353	4.278	2.43	5.905	1.559	4.930
1.24	1.538	1.114	3.521	1.84	3.386	1.356	4.290	2.44	5.954	1.562	4.940
1.25	1.562	1.118	3.536	1.85	3.422	1.360	4.301	2.45	6.002	1.565	4.950
1.26	1.588	1.122	3.550	1.86	3.460	1.364	4.313	2.46	6.052	1.568	4.960
1.27	1.613	1.127	3.564	1.87	3.497	1.367	4.324	2.47	6.101	1.572	4.970
1.28	1.638	1.131	3.578	1.88	3.534	1.371	4.336	2.48	6.150	1.575	4.980
1.29	1.664	1.136	3.592	1.89	3.572	1.375	4.347	2.49	6.200	1.578	4.990
1.30	1.690	1.140	3.606	1.90	3.610	1.378	4.359	2.50	6.250	1.581	5.000
1.31	1.716	1.145	3.619	1.91	3.648	1.382	4.370	2.51	6.300	1.584	5.010
1.32	1.742	1.149	3.633	1.92	3.686	1.386	4.382	2.52	6.350	1.587	5.020
1.33	1.769	1.153	3.647	1.93	3.725	1.389	4.393	2.53	6.401	1.591	5.030
1.34	1.796	1.158	3.661	1.94	3.764	1.393	4.405	2.54	6.452	1.594	5.040
1.35	1.822	1.162	3.674	1.95	3.802	1.396	4.416	2.55	6.502	1.597	5.050
1.36	1.850	1.166	3.688	1.96	3.842	1.400	4.427	2.56	6.554	1.600	5.060
1.37	1.877	1.170	3.701	1.97	3.881	1.404	4.438	2.57	6.605	1.603	5.070
1.38	1.904	1.175	3.715	1.98	3.920	1.407	4.450	2.58	6.656	1.606	5.079
1.39	1.932	1.179	3.728	1.99	3.960	1.411	4.461	2.59	6.708	1.609	5.089
1.40	1.960	1.183	3.742	2.00	4.000	1.414	4.472	2.60	6.760	1.612	5.099
1.41	1.988	1.187	3.755	2.01	4.040	1.418	4.483	2.61	6.812	1.616	5.109
1.42	2.016	1.192	3.768	2.02	4.080	1.421	4.494	2.62	6.864	1.619	5.119
1.43	2.045	1.196	3.782	2.03	4.121	1.425	4.506	2.63	6.917	1.622	5.128
1.44	2.074	1.200	3.795	2.04	4.162	1.428	4.517	2.64	6.970	1.625	5.138
1.45	2.102	1.204	3.808	2.05	4.202	1.432	4.528	2.65	7.022	1.628	5.148
1.46	2.132	1.208	3.821	2.06	4.244	1.435	4.539	2.66	7.076	1.631	5.158
1.47	2.161	1.212	3.834	2.07	4.285	1.439	4.550	2.67	7.129	1.634	5.167
1.48	2.190	1.217	3.847	2.08	4.326	1.442	4.561	2.68	7.182	1.637	5.177
1.49	2.220	1.221	3.860	2.09	4.368	1.446	4.572	2.69	7.236	1.640	5.187
1.50	2.250	1.225	3.873	2.10	4.410	1.449	4.583	2.70	7.290	1.643	5.196
1.51	2.280	1.229	3.886	2.11	4.452	1.453	4.593	2.71	7.344	1.646	5.206
1.52	2.310	1.233	3.899	2.12	4.494	1.456	4.604	2.72	7.398	1.649	5.215
1.53	2.341	1.237	3.912	2.13	4.537	1.459	4.615	2.73	7.453	1.652	5.225
1.54	2.372	1.241	3.924	2.14	4.580	1.463	4.626	2.74	7.508	1.655	5.235
1.55	2.402	1.245	3.937	2.15	4.622	1.466	4.637	2.75	7.562	1.658	5.244
1.56	2.434	1.249	3.950	2.16	4.666	1.470	4.648	2.76	7.618	1.661	5.254
1.57	2.465	1.253	3.962	2.17	4.709	1.473	4.658	2.77	7.673	1.664	5.263
1.58	2.496	1.257	3.975	2.18	4.752	1.476	4.669	2.78	7.728	1.667	5.273
1.59	2.528	1.261	3.987	2.19	4.796	1.480	4.680	2.79	7.784	1.670	5.282
1.60	2.560	1.265	4.000	2.20	4.840	1.483	4.690	2.80	7.840	1.673	5.292
N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$

N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$
2.80	7.840	1.673	5.292	3.40	11.56	1.844	5.831	4.00	16.00	2.000	6.325
2.81	7.896	1.676	5.301	3.41	11.63	1.847	5.840	4.01	16.08	2.002	6.332
2.82	7.952	1.679	5.310	3.42	11.70	1.849	5.848	4.02	16.16	2.005	6.340
2.83	8.009	1.682	5.320	3.43	11.76	1.852	5.857	4.03	16.24	2.007	6.348
2.84	8.066	1.685	5.329	3.44	11.83	1.855	5.865	4.04	16.32	2.010	6.356
2.85	8.122	1.688	5.339	3.45	11.90	1.857	5.874	4.05	16.40	2.012	6.364
2.86	8.180	1.691	5.348	3.46	11.97	1.860	5.882	4.06	16.48	2.015	6.372
2.87	8.237	1.694	5.357	3.47	12.04	1.863	5.891	4.07	16.56	2.017	6.380
2.88	8.294	1.697	5.367	3.48	12.11	1.865	5.899	4.08	16.65	2.020	6.387
2.89	8.352	1.700	5.376	3.49	12.18	1.868	5.908	4.09	16.73	2.022	6.395
2.90	8.410	1.703	5.385	3.50	12.25	1.871	5.916	4.10	16.81	2.025	6.403
2.91	8.468	1.706	5.394	3.51	12.32	1.873	5.925	4.11	16.89	2.027	6.411
2.92	8.526	1.709	5.404	3.52	12.39	1.876	5.933	4.12	16.97	2.030	6.419
2.93	8.585	1.712	5.413	3.53	12.46	1.879	5.941	4.13	17.06	2.032	6.427
2.94	8.644	1.715	5.422	3.54	12.53	1.881	5.950	4.14	17.14	2.035	6.434
2.95	8.702	1.718	5.431	3.55	12.60	1.884	5.958	4.15	17.22	2.037	6.442
2.96	8.762	1.720	5.441	3.56	12.67	1.887	5.967	4.16	17.31	2.040	6.450
2.97	8.821	1.723	5.450	3.57	12.74	1.889	5.975	4.17	17.39	2.042	6.458
2.98	8.880	1.726	5.459	3.58	12.82	1.892	5.983	4.18	17.47	2.045	6.465
2.99	8.940	1.729	5.468	3.59	12.89	1.895	5.992	4.19	17.56	2.047	6.473
3.00	9.000	1.732	5.477	3.60	12.96	1.897	6.000	4.20	17.64	2.049	6.481
3.01	9.060	1.735	5.486	3.61	13.03	1.900	6.008	4.21	17.72	2.052	6.488
3.02	9.120	1.738	5.495	3.62	13.10	1.903	6.017	4.22	17.81	2.054	6.496
3.03	9.181	1.741	5.505	3.63	13.18	1.905	6.025	4.23	17.89	2.057	6.504
3.04	9.242	1.744	5.514	3.64	13.25	1.908	6.033	4.24	17.98	2.059	6.512
3.05	9.302	1.746	5.523	3.65	13.32	1.910	6.042	4.25	18.06	2.062	6.519
3.06	9.364	1.749	5.532	3.66	13.40	1.913	6.050	4.26	18.15	2.064	6.527
3.07	9.425	1.752	5.541	3.67	13.47	1.916	6.058	4.27	18.23	2.066	6.535
3.08	9.486	1.755	5.550	3.68	13.54	1.918	6.066	4.28	18.32	2.069	6.542
3.09	9.548	1.758	5.559	3.69	13.62	1.921	6.075	4.29	18.40	2.071	6.550
3.10	9.610	1.761	5.568	3.70	13.69	1.924	6.083	4.30	18.49	2.074	6.557
3.11	9.672	1.764	5.577	3.71	13.76	1.926	6.091	4.31	18.58	2.076	6.565
3.12	9.734	1.766	5.586	3.72	13.84	1.929	6.099	4.32	18.66	2.078	6.573
3.13	9.797	1.769	5.595	3.73	13.91	1.931	6.107	4.33	18.75	2.081	6.580
3.14	9.860	1.772	5.604	3.74	13.99	1.934	6.116	4.34	18.84	2.083	6.588
3.15	9.922	1.775	5.612	3.75	14.06	1.936	6.124	4.35	18.92	2.086	6.595
3.16	9.986	1.778	5.621	3.76	14.14	1.939	6.132	4.36	19.01	2.088	6.603
3.17	10.05	1.780	5.630	3.77	14.21	1.942	6.140	4.37	19.10	2.090	6.611
3.18	10.11	1.783	5.639	3.78	14.29	1.944	6.148	4.38	19.18	2.093	6.618
3.19	10.18	1.786	5.648	3.79	14.36	1.947	6.156	4.39	19.27	2.095	6.626
3.20	10.24	1.789	5.657	3.80	14.44	1.949	6.164	4.40	19.36	2.098	6.633
3.21	10.30	1.792	5.666	3.81	14.52	1.952	6.173	4.41	19.45	2.100	6.641
3.22	10.37	1.794	5.675	3.82	14.59	1.954	6.181	4.42	19.54	2.102	6.648
3.23	10.43	1.797	5.683	3.83	14.67	1.957	6.189	4.43	19.62	2.105	6.656
3.24	10.50	1.800	5.692	3.84	14.75	1.960	6.197	4.44	19.71	2.107	6.663
3.25	10.56	1.803	5.701	3.85	14.82	1.962	6.205	4.45	19.80	2.110	6.671
3.26	10.63	1.806	5.710	3.86	14.90	1.965	6.213	4.46	19.89	2.112	6.678
3.27	10.69	1.808	5.718	3.87	14.98	1.967	6.221	4.47	19.98	2.114	6.686
3.28	10.76	1.811	5.727	3.88	15.05	1.970	6.229	4.48	20.07	2.117	6.693
3.29	10.82	1.814	5.736	3.89	15.13	1.972	6.237	4.49	20.16	2.119	6.701
3.30	10.89	1.817	5.745	3.90	15.21	1.975	6.245	4.50	20.25	2.121	6.708
3.31	10.96	1.819	5.753	3.91	15.29	1.977	6.253	4.51	20.34	2.124	6.716
3.32	11.02	1.822	5.762	3.92	15.37	1.980	6.261	4.52	20.43	2.126	6.723
3.33	11.09	1.825	5.771	3.93	15.44	1.982	6.269	4.53	20.52	2.128	6.731
3.34	11.16	1.828	5.779	3.94	15.52	1.985	6.277	4.54	20.61	2.131	6.738
3.35	11.22	1.830	5.788	3.95	15.60	1.987	6.285	4.55	20.70	2.133	6.745
3.36	11.29	1.833	5.797	3.96	15.68	1.990	6.293	4.56	20.79	2.135	5.753
3.37	11.36	1.836	5.805	3.97	15.76	1.992	6.301	4.57	20.88	2.138	6.760
3.38	11.42	1.838	5.814	3.98	15.84	1.995	6.309	4.58	20.98	2.140	6.768
3.39	11.49	1.841	5.822	3.99	15.92	1.997	6.317	4.59	21.07	2.142	6.775
3.40	11.56	1.844	5.831	4.00	16.00	2.000	6.325	4.60	21.16	2.145	6.782
N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$

IV. Squares and Square Roots

N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$
4.60	21.16	2.145	6.782	5.20	27.04	2.280	7.211	5.80	33.64	2.408	7.616
4.61	21.25	2.147	6.790	5.21	27.14	2.283	7.218	5.81	33.76	2.410	7.622
4.62	21.34	2.149	6.797	5.22	27.25	2.285	7.225	5.82	33.87	2.412	7.629
4.63	21.44	2.152	6.804	5.23	27.35	2.287	7.232	5.83	33.99	2.415	7.635
4.64	21.53	2.154	6.812	5.24	27.46	2.289	7.239	5.84	34.11	2.417	7.642
4.65	21.62	2.156	6.819	5.25	27.56	2.291	7.246	5.85	34.22	2.419	7.649
4.66	21.72	2.159	6.826	5.26	27.67	2.293	7.253	5.86	34.34	2.421	7.655
4.67	21.81	2.161	6.834	5.27	27.77	2.296	7.259	5.87	34.46	2.423	7.662
4.68	21.90	2.163	6.841	5.28	27.88	2.298	7.266	5.88	34.57	2.425	7.668
4.69	22.00	2.166	6.848	5.29	27.98	2.300	7.273	5.89	34.69	2.427	7.675
4.70	22.09	2.168	6.856	5.30	28.09	2.302	7.280	5.90	34.81	2.429	7.681
4.71	22.18	2.170	6.863	5.31	28.20	2.304	7.287	5.91	34.93	2.431	7.688
4.72	22.28	2.173	6.870	5.32	28.30	2.307	7.294	5.92	35.05	2.433	7.694
4.73	22.37	2.175	6.877	5.33	28.41	2.309	7.301	5.93	35.16	2.435	7.701
4.74	22.47	2.177	6.885	5.34	28.52	2.311	7.308	5.94	35.28	2.437	7.707
4.75	22.56	2.179	6.892	5.35	28.62	2.313	7.314	5.95	35.40	2.439	7.714
4.76	22.66	2.182	6.899	5.36	28.73	2.315	7.321	5.96	35.52	2.441	7.720
4.77	22.75	2.184	6.907	5.37	28.84	2.317	7.328	5.97	35.64	2.443	7.727
4.78	22.85	2.186	6.914	5.38	28.94	2.319	7.335	5.98	35.76	2.445	7.733
4.79	22.94	2.189	6.921	5.39	29.05	2.322	7.342	5.99	35.88	2.447	7.740
4.80	23.04	2.191	6.928	5.40	29.16	2.324	7.348	6.00	36.00	2.449	7.746
4.81	23.14	2.193	6.935	5.41	29.27	2.326	7.355	6.01	36.12	2.452	7.752
4.82	23.23	2.195	6.943	5.42	29.38	2.328	7.362	6.02	36.24	2.454	7.759
4.83	23.33	2.198	6.950	5.43	29.48	2.330	7.369	6.03	36.36	2.456	7.765
4.84	23.43	2.200	6.957	5.44	29.59	2.332	7.376	6.04	36.48	2.458	7.772
4.85	23.52	2.202	6.964	5.45	29.70	2.335	7.382	6.05	36.60	2.460	7.778
4.86	23.62	2.205	6.971	5.46	29.81	2.337	7.389	6.06	36.72	2.462	7.785
4.87	23.72	2.207	6.979	5.47	29.92	2.339	7.396	6.07	36.84	2.464	7.791
4.88	23.81	2.209	6.986	5.48	30.03	2.341	7.403	6.08	36.97	2.466	7.797
4.89	23.91	2.211	6.993	5.49	30.14	2.343	7.409	6.09	37.09	2.468	7.804
4.90	24.01	2.214	7.000	5.50	30.25	2.345	7.416	6.10	37.21	2.470	7.810
4.91	24.11	2.216	7.007	5.51	30.36	2.347	7.423	6.11	37.33	2.472	7.817
4.92	24.21	2.218	7.014	5.52	30.47	2.349	7.430	6.12	37.45	2.474	7.823
4.93	24.30	2.220	7.021	5.53	30.58	2.352	7.436	6.13	37.58	2.476	7.829
4.94	24.40	2.223	7.029	5.54	30.69	2.354	7.443	6.14	37.70	2.478	7.836
4.95	24.50	2.225	7.036	5.55	30.80	2.356	7.450	6.15	37.82	2.480	7.842
4.96	24.60	2.227	7.043	5.56	30.91	2.358	7.457	6.16	37.95	2.482	7.849
4.97	24.70	2.229	7.050	5.57	31.02	2.360	7.463	6.17	38.07	2.484	7.855
4.98	24.80	2.232	7.057	5.58	31.14	2.362	7.470	6.18	38.19	2.486	7.861
4.99	24.90	2.234	7.064	5.59	31.25	2.364	7.477	6.19	38.32	2.488	7.868
5.00	25.00	2.236	7.071	5.60	31.36	2.366	7.483	6.20	38.44	2.490	7.874
5.01	25.10	2.238	7.078	5.61	31.47	2.369	7.490	6.21	38.56	2.492	7.880
5.02	25.20	2.241	7.085	5.62	31.58	2.371	7.497	6.22	38.69	2.494	7.887
5.03	25.30	2.243	7.092	5.63	31.70	2.373	7.503	6.23	38.81	2.496	7.893
5.04	25.40	2.245	7.099	5.64	31.81	2.375	7.510	6.24	38.94	2.498	7.899
5.05	25.50	2.247	7.106	5.65	31.92	2.377	7.517	6.25	39.06	2.500	7.906
5.06	25.60	2.249	7.113	5.66	32.04	2.379	7.523	6.26	39.19	2.502	7.912
5.07	25.70	2.252	7.120	5.67	32.15	2.381	7.530	6.27	39.31	2.504	7.918
5.08	25.81	2.254	7.127	5.68	32.26	2.383	7.537	6.28	39.44	2.506	7.925
5.09	25.91	2.256	7.134	5.69	32.38	2.385	7.543	6.29	39.56	2.508	7.931
5.10	26.01	2.258	7.141	5.70	32.49	2.387	7.550	6.30	39.69	2.510	7.937
5.11	26.11	2.261	7.148	5.71	32.60	2.390	7.556	6.31	39.82	2.512	7.944
5.12	26.21	2.263	7.155	5.72	32.72	2.392	6.563	6.32	39.94	2.514	7.950
5.13	26.32	2.265	7.162	5.73	32.83	2.394	7.570	6.33	40.07	2.516	7.956
5.14	26.42	2.267	7.169	5.74	32.95	2.396	7.576	6.34	40.20	2.518	7.962
5.15	26.52	2.269	7.176	5.75	33.06	2.398	7.583	6.35	40.32	2.520	7.969
5.16	26.63	2.272	7.183	5.76	33.18	2.400	7.589	6.36	40.45	2.522	7.975
5.17	26.73	2.274	7.190	5.77	33.29	2.402	7.596	6.37	40.58	2.524	7.981
5.18	26.83	2.276	7.197	5.78	33.41	2.404	7.603	6.38	40.70	2.526	7.987
5.19	26.94	2.278	7.204	5.79	33.52	2.406	7.609	6.39	40.83	2.528	7.994
5.20	27.04	2.280	7.211	5.80	33.64	2.408	7.616	6.40	40.96	2.530	8.000
N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$

IV. Squares and Square Roots

N	N²	√N	√10N	N	N²	√N	√10N	N	N²	√N	√10N
6.40	40.96	2.530	8.000	7.00	49.00	2.646	8.367	7.60	57.76	2.757	8.718
6.41	41.09	2.532	8.006	7.01	49.14	2.648	8.373	7.61	57.91	2.759	8.724
6.42	41.22	2.534	8.012	7.02	49.28	2.650	8.379	7.62	58.06	2.760	8.729
6.43	41.34	2.536	8.019	7.03	49.42	2.651	8.385	7.63	58.22	2.762	8.735
6.44	41.47	2.538	8.025	7.04	49.56	2.653	8.390	7.64	58.37	2.764	8.741
6.45	41.60	2.540	8.031	7.05	49.70	2.655	8.396	7.65	58.52	2.766	8.746
6.46	41.73	2.542	8.037	7.06	49.84	2.657	8.402	7.66	58.68	2.768	8.752
6.47	41.86	2.544	8.044	7.07	49.98	2.659	8.408	7.67	58.83	2.769	8.758
6.48	41.99	2.546	8.050	7.08	50.13	2.661	8.414	7.68	58.98	2.771	8.764
6.49	42.12	2.548	8.056	7.09	50.27	2.663	8.420	7.69	59.14	2.773	8.769
6.50	42.25	2.550	8.062	7.10	50.41	2.665	8.426	7.70	59.29	2.775	8.775
6.51	42.38	2.551	8.068	7.11	50.55	2.666	8.432	7.71	59.44	2.777	8.781
6.52	42.51	2.553	8.075	7.12	50.69	2.668	8.438	7.72	59.60	2.778	8.786
6.53	42.64	2.555	8.081	7.13	50.84	2.670	8.444	7.73	59.75	2.780	8.792
6.54	42.77	2.557	8.087	7.14	50.98	2.672	8.450	7.74	59.91	2.782	8.798
6.55	42.90	2.559	8.093	7.15	51.12	2.674	8.456	7.75	60.06	2.784	8.803
6.56	43.03	2.561	8.099	7.16	51.27	2.676	8.462	7.76	60.22	2.786	8.809
6.57	43.16	2.563	8.106	7.17	51.41	2.678	8.468	7.77	60.37	2.787	8.815
6.58	43.30	2.565	8.112	7.18	51.55	2.680	8.473	7.78	60.53	2.789	8.820
6.59	43.43	2.567	8.118	7.19	51.70	2.681	8.479	7.79	60.68	2.791	8.826
6.60	43.56	2.569	8.124	7.20	51.84	2.683	8.485	7.80	60.84	2.793	8.832
6.61	43.69	2.571	8.130	7.21	51.98	2.685	8.491	7.81	61.00	2.795	8.837
6.62	43.82	2.573	8.136	7.22	52.13	2.687	8.497	7.82	61.15	2.796	8.843
6.63	43.96	2.575	8.142	7.23	52.27	2.689	8.503	7.83	61.31	2.798	8.849
6.64	44.09	2.577	8.149	7.24	52.42	2.691	8.509	7.84	61.47	2.800	8.854
6.65	44.22	2.579	8.155	7.25	52.56	2.693	8.515	7.85	61.62	2.802	8.860
6.66	44.36	2.581	8.161	7.26	52.71	2.694	8.521	7.86	61.78	2.804	8.866
6.67	44.49	2.583	8.167	7.27	52.85	2.696	8.526	7.87	61.94	2.805	8.871
6.68	44.62	2.585	8.173	7.28	53.00	2.698	8.532	7.88	62.09	2.807	8.877
6.69	44.76	2.587	8.179	7.29	53.14	2.700	8.538	7.89	62.25	2.809	8.883
6.70	44.89	2.588	8.185	7.30	53.29	2.702	8.544	7.90	62.41	2.811	8.888
6.71	45.02	2.590	8.191	7.31	53.44	2.704	8.550	7.91	62.57	2.812	8.894
6.72	45.16	2.592	8.198	7.32	53.58	2.706	8.556	7.92	62.73	2.814	8.899
6.73	45.29	2.594	8.204	7.33	53.73	2.707	8.562	7.93	62.88	2.816	8.905
6.74	45.43	2.596	8.210	7.34	53.88	2.709	8.567	7.94	63.04	2.818	8.911
6.75	45.56	2.598	8.216	7.35	54.02	2.711	8.573	7.95	63.20	2.820	8.916
6.76	45.70	2.600	8.222	7.36	54.17	2.713	8.579	7.96	63.36	2.821	8.922
6.77	45.83	2.602	8.228	7.37	54.32	2.715	8.585	7.97	63.52	2.823	8.927
6.78	45.97	2.604	8.234	7.38	54.46	2.717	8.591	7.98	63.68	2.825	8.933
6.79	46.10	2.606	8.240	7.39	54.61	2.718	8.597	7.99	63.84	2.827	8.939
6.80	46.24	2.608	8.246	7.40	54.76	7.720	8.602	8.00	64.00	2.828	8.944
6.81	46.38	2.610	8.252	7.41	54.91	2.722	8.608	8.01	64.16	2.830	8.950
6.82	46.51	2.612	8.258	7.42	55.06	2.724	8.614	8.02	64.32	2.832	8.955
6.83	46.65	2.613	8.264	7.43	55.20	2.726	8.620	8.03	64.48	2.834	8.961
6.84	46.79	2.615	8.270	7.44	55.35	2.728	8.626	8.04	64.64	2.835	8.967
6.85	46.92	2.617	8.276	7.45	55.50	2.729	8.631	8.05	64.80	2.837	8.972
6.86	47.06	2.619	8.283	7.46	55.65	2.731	8.637	8.06	64.96	2.839	8.978
6.87	47.20	2.621	8.289	7.47	55.80	2.733	8.643	8.07	65.12	2.841	8.983
6.88	47.33	2.623	8.295	7.48	55.95	2.735	8.649	8.08	65.29	2.843	8.989
6.89	47.47	2.625	8.301	7.49	56.10	2.737	8.654	8.09	65.45	2.844	8.994
6.90	47.61	2.627	8.307	7.50	56.25	2.739	8.660	8.10	65.61	2.846	9.000
6.91	47.75	2.629	8.313	7.51	56.40	2.740	8.666	8.11	65.77	2.848	9.006
6.92	47.89	2.631	8.319	7.52	56.55	2.742	8.672	8.12	65.93	2.850	9.011
6.93	48.02	2.632	8.325	7.53	56.70	2.744	8.678	8.13	66.10	2.851	9.017
6.94	48.16	2.634	8.331	7.54	56.85	2.746	8.683	8.14	66.26	2.853	9.022
6.95	48.30	2.636	8.337	7.55	57.00	2.748	8.689	8.15	66.42	2.855	9.028
6.96	48.44	2.638	8.343	7.56	57.15	2.750	8.695	8.16	66.59	2.856	9.033
6.97	48.58	2.640	8.349	7.57	57.30	2.751	8.701	8.17	66.75	2.858	9.039
6.98	48.72	2.642	8.355	7.58	57.46	2.753	8.706	8.18	66.91	2.860	9.044
6.99	48.86	2.644	8.361	7.59	57.61	2.755	8.712	8.19	67.08	2.862	9.050
7.00	49.00	2.646	8.367	7.60	57.76	2.757	8.718	8.20	67.24	2.864	9.055
N	N²	√N	√10N	N	N²	√N	√10N	N	N²	√N	√10N

IV. Squares and Square Roots

N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$
8.20	67.24	2.864	9.055	8.80	77.44	2.966	9.381	9.40	88.36	3.066	9.695
8.21	67.40	2.865	9.061	8.81	77.62	2.968	9.386	9.41	88.55	3.068	9.701
8.22	67.57	2.867	9.066	8.82	77.79	2.970	9.391	9.42	88.74	3.069	9.706
8.23	67.73	2.869	9.072	8.83	77.97	2.972	9.397	9.43	88.92	3.071	9.711
8.24	67.90	2.871	9.077	8.84	78.15	2.973	9.402	9.44	89.11	3.072	9.716
8.25	68.06	2.872	9.083	8.85	78.32	2.975	9.407	9.45	89.30	3.074	9.721
8.26	68.23	2.874	9.088	8.86	78.50	2.977	9.413	9.46	89.49	3.076	9.726
8.27	68.39	2.876	0.094	8.87	78.68	2.978	9.418	9.47	89.68	3.077	9.731
8.28	68.56	2.877	9.099	8.88	78.85	2.980	9.423	9.48	89.87	3.079	9.737
8.29	68.72	2.879	9.105	8.89	79.03	2.982	9.429	9.49	90.06	3.081	9.742
8.30	68.89	2.881	9.110	8.90	79.21	2.983	9.434	9.50	90.25	3.082	9.747
8.31	69.06	2.883	9.116	8.91	79.39	2.985	9.439	9.51	90.44	3.084	9.752
8.32	69.22	2.884	9.121	8.92	79.57	2.987	9.445	9.52	90.63	3.085	9.757
8.33	69.39	2.886	9.127	8.93	79.74	2.988	9.450	9.53	90.82	3.087	9.762
8.34	69.56	2.888	9.132	8.94	79.92	2.990	9.455	9.54	91.01	3.089	9.767
8.35	69.72	2.890	9.138	8.95	80.10	2.992	9.460	9.55	91.20	3.090	9.772
8.36	69.89	2.891	9.143	8.96	80.28	2.993	9.466	9.56	91.39	3.092	9.778
8.37	70.06	2.893	9.149	8.97	80.46	2.995	9.471	9.57	91.58	3.094	9.783
8.38	70.22	2.895	9.154	8.98	80.64	2.997	9.476	9.58	91.78	3.095	9.788
8.39	70.39	2.897	9.160	8.99	80.82	2.998	9.482	9.59	91.97	3.097	9.793
8.40	70.56	2.898	9.165	9.00	81.00	3.000	9.487	9.60	92.16	3.098	9.798
8.41	70.73	2.900	9.171	9.01	81.18	3.002	9.492	9.61	92.35	3.100	9.803
8.42	70.90	2.902	9.176	9.02	81.36	3.003	9.497	9.62	92.54	3.102	9.808
8.43	71.07	2.903	9.182	9.03	81.54	3.005	9.503	9.63	92.74	3.103	9.813
8.44	71.23	2.905	9.187	9.04	81.72	3.007	9.508	9.64	92.93	3.105	9.818
8.45	71.40	2.907	9.192	9.05	81.90	3.008	9.513	9.65	93.12	3.106	9.823
8.46	71.57	2.909	9.198	9.06	82.08	3.010	9.518	9.66	93.32	3.108	9.829
8.47	71.74	2.910	9.203	9.07	82.26	3.012	9.524	9.67	93.51	3.110	9.834
8.48	71.91	2.912	9.209	9.08	82.45	3.013	9.529	9.68	93.70	3.111	9.839
8.49	72.08	2.914	9.214	9.09	82.63	3.015	9.534	9.69	93.90	3.113	9.844
8.50	72.25	2.915	9.220	9.10	82.81	3.017	9.539	9.70	94.09	3.114	9.849
8.51	72.42	2.917	9.225	9.11	82.99	3.018	9.545	9.71	94.28	3.116	9.854
8.52	72.59	2.919	9.230	9.12	83.17	3.020	9.550	9.72	94.48	3.118	9.859
8.53	72.76	2.921	9.236	9.13	83.36	3.022	9.555	9.73	94.67	3.119	9.864
8.54	72.93	2.922	9.241	9.14	83.54	3.023	9.560	9.74	94.87	3.121	9.869
8.55	73.10	2.924	9.247	9.15	83.72	3.025	9.566	9.75	95.06	3.122	9.874
8.56	73.27	2.926	9.252	9.16	83.91	3.027	9.571	9.76	95.26	3.124	9.879
8.57	73.44	2.927	9.257	9.17	84.09	3.028	9.576	9.77	95.45	3.126	9.884
8.58	73.62	2.929	9.263	9.18	84.27	3.030	9.581	9.78	95.65	3.127	9.889
8.59	73.79	2.931	9.268	9.19	84.46	3.032	9.586	9.79	95.84	3.129	9.894
8.60	73.96	2.933	9.274	9.20	84.64	3.033	9.592	9.80	96.04	3.130	9.899
8.61	74.13	2.934	9.279	9.21	84.82	3.035	9.597	9.81	96.24	3.132	9.905
8.62	74.30	2.936	9.284	9.22	85.01	3.036	9.602	9.82	96.43	3.134	9.910
8.63	74.48	2.938	9.290	9.23	85.19	3.038	9.607	9.83	96.63	3.135	9.915
8.64	74.65	2.939	9.295	9.24	85.38	3.040	9.612	9.84	96.83	3.137	9.920
8.65	74.82	2.941	9.301	9.25	85.56	3.041	9.618	9.85	97.02	3.138	9.925
8.66	75.00	2.943	9.306	9.26	85.75	3.043	9.623	9.86	97.22	3.140	9.930
8.67	75.17	2.944	9.311	9.27	85.93	3.045	9.628	9.87	97.42	3.142	9.935
8.68	75.34	2.946	9.317	9.28	86.12	3.046	9.633	9.88	97.61	3.143	9.940
8.69	75.52	2.948	9.322	9.29	86.30	3.048	9.638	9.89	97.81	3.145	9.945
8.70	75.69	2.950	9.327	9.30	86.49	3.050	9.644	9.90	98.01	3.146	9.950
8.71	75.86	2.951	9.333	9.31	86.68	3.051	9.649	9.91	98.21	3.148	9.955
8.72	76.04	2.953	9.338	9.32	86.86	3.053	9.654	9.92	98.41	3.150	9.960
8.73	76.21	2.955	9.343	9.33	87.05	3.055	9.659	9.93	98.60	3.151	9.965
8.74	76.39	2.956	9.349	9.34	87.24	3.056	9.664	9.94	98.80	3.153	9.970
8.75	76.56	2.958	9.354	9.35	87.42	3.058	9.670	9.95	99.00	3.154	9.975
8.76	76.74	2.960	9.359	9.36	87.61	3.059	9.675	9.96	99.20	3.156	9.980
8.77	76.91	2.961	9.365	9.37	87.80	3.061	9.680	9.97	99.40	3.158	9.985
8.78	77.09	2.963	9.370	9.38	87.98	3.063	9.685	9.98	99.60	3.159	9.990
8.79	77.26	2.965	9.375	9.39	88.17	3.064	9.690	9.99	99.80	3.161	9.995
8.80	77.44	2.966	9.381	9.40	88.36	3.066	9.695	10.0	100.0	3.162	10.00
N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N	N^2	$\sqrt{N}$	$\sqrt{10N}$

Answers to Odd-Numbered Exercises

(The answers to certain exercises have been intentionally omitted.)

Page 4

7. 7, 16, $\frac{5}{8}$, 2, 8, 4, 10.

Pages 6, 7

1. (*a*) 98, (*b*) 16. **3.** (*a*) -96, (*b*) -18. **5.** (*a*) -76, (*b*) -76. **7.** (*a*) $-5x$, (*b*) $-x$. **9.** (*a*) $-2y$, (*b*) $8y$. **11.** -168. **13.** -24. **15.** $-12x$. **17.** -7. **19.** 8. **21.** -2. **23.** $-42abc$. **25.** $ad+bd+cd$. **27.** 10. **29.** 7. **31.** -3. **33.** 97. **35.** 5, 11. **37.** 2, 3. **39.** 168 ft. below sea level.

Page 8

1. $4x-6$. **3.** $7u-8v-3$. **5.** $20r-27$. **7.** $3s-18t+3$. **9.** -6, 28. **11.** (*a*) $6x-4y+2z+(-9x+4y-5z)$, (*b*) $6x-4y+2z-(9x-4y+5z)$. **13.** (*a*) $9x+y-14z$, (*b*) $5x+7y-4z$. **15.** $3z^3-z^2+5z+10$. **17.** $4r^3+6r^2-10r+5$.

Page 10

1. $39m^9$. **3.** $56a^5b^3c^5d^4$. **5.** $21a^4r^6z^8$. **7.** $6u^6v^4w^2x^5y$. **9.** $6a-12b+3c$. **11.** $4a^3bc+28ab^3c-36abc^3$. **13.** $14rs^3t^3-35r^2s^2t^4+21r^3s^5t$. **15.** $-33a^3b^2c^3+55a^4bc^4+22a^2b^2c^4$. **17.** $8x^2-6x-35$. **19.** $15m^2+34mn-16n^2$. **21.** $3x^3-13x^2-5x+3$. **23.** $x^4-x^2y^2+10xy^3-4y^4$. **25.** $x^3+y^3+z^3+x^2y+xy^2+y^2z+yz^2+z^2x+zx^2$. **27.** $3x^5+x^4-27x^3+43x^2-28x$.

Page 12

1. $4x^5$. **3.** $\dfrac{3xz^2}{4y^2}$. **5.** $\dfrac{7b^6d}{3a^3c^5}$. **7.** $7x^2-11x-6$. **9.** $\dfrac{7xy^2}{2z}+\dfrac{8x^3y^2}{3z^2}+\dfrac{4z}{x}$. **11.** $2x-5$. **13.** $2x^2-x-5-\dfrac{18}{2x-7}$. **15.** $5x-8+\dfrac{20}{3x+2}$. **17.** $4a-3b-\dfrac{8b^2}{2a-5b}$. **19.** $3a^2b-4t^3+\dfrac{17t^6}{a^2b+4t^3}$. **21.** $5v-9+\dfrac{3v+25}{2v^2+7v+9}$. **23.** $7rs+4t-\dfrac{4rst^2+16t^3}{3r^2s^2-rst+4t^2}$. **25.** $m^3n^3-m^2n^2z+2z^3+\dfrac{3z^4}{mn+3z}$. **27.** $5y^4-3y^3+2y^2-8y+7+\dfrac{13}{2y+3}$.

Page 13

1. $6ax+18ay$. **3.** x^2-9. **5.** 9991. **7.** $9x^2+24xy+16y^2$. **9.** $x^2+11x+28$. **11.** $9x^2-12x-5$. **13.** $27x^3+y^3$. **15.** z^4-4z^2-21. **17.** x^3y-xy^3. **19.** $6a^4+19a^2b^2+10b^4$. **21** x^4-16. **23.** x^6-y^6.

Page 14

1. $3y(5x+7y)$. **3.** $(6x-5y)(6x+5y)$. **5.** $(2p+9q)^2$. **7.** $2a(3x-1)^2$.
9. $(ab-3c)(ab+3c)$. **11.** $u(2u+3v)(4u^2-6uv+9v^2)$.
13. $(5z+2)(z-4)$. **15.** $(4a+7b)(5a+3b)$. **17.** $(x-y)^2(x+y)^2$.
19. $(xy-2c^2)(x^2y^2+2xyc^2+4c^4)$. **21.** $(x+y-z)(x+y+z)$.
23. $(3v-u)(3u-v)$. **25.** $(a-b+1)(a^2+ab+b^2+2a+b+1)$.

Page 15

1. $(2a+3b)(3x+5y)$. **3.** $(x+4y)(x-3z)$. **5.** $(x-2)(x+2)(2x+5)$.
7. $(x+2y)(x+2y-7)$. **9.** $(3a-b)(3a+b-5c)$.
11. $(u+3v+3)(u+3v-2)$. **13.** $(x-2y)(x+2y)(x-z)(x+z)$.
15. $(x^2-xy+y^2)(x^2+xy+y^2)$. **17.** $(x-2y-2)(x-2y-3)$.
19. $(y+3z)(y^2-3yz+9z^2-5x)$.

Page 16

1. $72a^2b^6c^4d^3$. **3.** $(x+y)^2(x-y)^2$.
5. $(x-3)(x+2)(x+5)$. **7.** $x^2(x-2)(x+2)(x-3)$; $(x-2)$.
9. $(3x-y-2)(3x-y+2)(3x+y+2)$; $(3x-y+2)$.

Page 18

1. $\frac{7}{3}$. **3.** $\frac{13yz^2}{6x}$. **5.** $\frac{5x^2}{3y}$. **7.** $\frac{x}{3y^2}$. **9.** $\frac{x+1}{2x-3}$. **11.** $\frac{x+b}{x-b}$.
13. $\frac{a+b-c-d}{a-b+c-d}$. **15.** $\frac{2x}{x-y}$. **17.** $\frac{7}{11}$. **19.** $\frac{n}{n+3}$.

Pages 19, 20

1. -1. **3.** $\frac{1}{5x+1}$. **5.** $\frac{4}{99}$. **7.** $\frac{4b}{3a^3}$. **9.** $\frac{19ab^2x^3}{18y^4z^6}$. **11.** $\frac{2a^2}{bc}$.
13. $\frac{y^3}{6x^3}$. **15.** $\frac{3(2x-y)}{2(x+y)(2x+y)}$. **17.** $x-2$. **19.** $\frac{(y-1)(y-2)}{(2y-1)(y+4)}$.
21. $\frac{(x+2y)(x-y)}{(2x+3y)(x-3y)}$. **23.** $\frac{3x+2}{2x-1}$. **25.** $\frac{1}{5}(x-3)(2x-3)$. **27.** $\frac{2r}{s^2}$.

Page 22

1. $\frac{3}{7}$. **3.** $\frac{4x-5}{x}$. **5.** $\frac{5x+11}{10}$. **7.** $\frac{8-3x+10x^2-4x^3}{2x^3}$.
9. $\frac{sy-tx-2s-3t}{xy}$. **11.** $\frac{3x-11}{(x+3)(x-1)}$. **13.** $\frac{11x+17}{(x-1)(x+4)}$.
15. $\frac{3x^2-2x+1}{x+2}$. **17.** $\frac{x^2-18x+5}{(x-1)(x+3)(x-4)}$.
19. $\frac{x^2y+x^2z+y^2z+y^2x+z^2x+z^2y-x^3-y^3-z^3-3xyz}{(x-y)(y-z)(z-x)}$.
21. $\frac{3x^3+2x^2-13x-32}{(x+1)^2(x+3)^2}$.

Page 23

1. $\frac{1}{3}$. 3. $\frac{2-3a}{1+5a}$. 5. $-\frac{x}{y}$. 7. $\frac{b-a}{b+a}$. 9. $a+b$.
11. $\frac{x^2-y^2}{xy}$. 13. $\frac{1}{(x+1)(x+h+1)}$. 15. $\frac{u^2+v^2}{u^2+2uv-v^2}$.

Pages 25, 26

1. Identity. 3. -6. 5. Identity. 7. -5. 9. -4. 11. $\frac{4}{3}$.
13. $-\frac{12}{5}$. 15. -7. 17. $-\frac{3}{7}$. 19. $\frac{4}{3}$. 21. 1.1 23. $1-a$.
25. $\frac{b+a}{b-a}$. 27. $\frac{y-b}{m}$. 29. $\frac{A-P}{nP}$.

Pages 27, 28

1. 411 lbs., 274 lbs. 3. 30 yrs. 5. \$4.80. 7. 7.
9. 1300 bu. 11. 3 qts. 13. 1232 ft.

Pages 29, 30

1. $\frac{5}{14}$. 3. $\frac{2x^3z^5}{5y}$. 5. $\frac{a(x+y)}{bc(x+3y)}$. 7. $\frac{3}{4}$. 17. 20.
19. -51. 21. ± 3. 23. 16 ft., by 28 ft. 25. 10. 27. $\frac{5}{6}$.
29. z^2. 31. 12. 33. $\sqrt{35}$. 35. 9. 37. $\frac{4}{7}$.

Pages 32, 33

1. $S = ke^2$, $S = 6e^2$. 3. $P = kAv^2$, $P = \frac{Av^2}{630}$. 11. 162 lbs.
13. 9.75 in. 15. 2.4. 17. 200 lbs.

Pages 34, 35

1. 81. 3. -16. 5. $\frac{27}{8}$. 7. -0.000027. 9. 16. 11. 1024.
13. x^{12}. 15. z. 17. $81y^4$. 19. h^{2n}.
21. $t^{a^2-b^2}$. 23. $a^{lp}b^{mp}c^{np}$. 25. 20.

Page 36

1. $7, -\sqrt{49} = -7$. 3. $0.5, -\sqrt{0.25} = -0.5$. 5. -1.
7. $-\frac{3x^2}{y^3}$. 9. 19. 11. $\frac{a^6b^{10}}{c^{14}d^2}$.

Pages 38, 39

1. 13. 3. 5. 5. $\frac{1}{32}$. 7. 1. 9. $\frac{3x^2}{y^4}$. 11. $-\frac{a^2}{8}$. 13. $\frac{7}{4a^2}$.
15. $\frac{x-y}{x+y}$. 17. $\frac{1}{x^3+y^3}$. 19. $\frac{xy^2}{x+y^2}$. 21. $3ab^{-1}c^{-2}$.
23. $xy^{-\frac{1}{2}}z^{-\frac{1}{2}}$. 25. $9a\sqrt{a}$. 27. $xz^2\sqrt{2xy}$. 29. $2a^2b^4$. 31. $a^{\frac{17}{24}}$.
33. $x^2 + 3x^{\frac{5}{3}} - 5x$. 35. $y + 7y^{\frac{1}{2}} + 10$. 37. $x^2y^{\frac{1}{3}} - 3xy^{\frac{4}{3}} + x^{\frac{5}{3}}y^{\frac{2}{3}}$.

Page 41

1. $3xy^2\sqrt[3]{6x^2y}$. 3. $\frac{2ab}{3yz^2}\sqrt[4]{\frac{5ab^3c^2}{xy^2z^3}}$. 5. $\frac{x^2y}{z^3}\sqrt[n]{\frac{x^4y}{z^5}}$. 7. $\sqrt[3]{250}$.

9. $\sqrt{\frac{u+v}{u-v}}$. 11. $\frac{\sqrt{ab}}{b}$. 13. $\frac{2x^2y}{5u^2v}\sqrt[3]{75yuv}$. 15. $\frac{\sqrt[3]{x^2-y^2}}{x+y}$.

17. $\sqrt{\frac{3x^3y^5}{5z^7}}$. 19. $\sqrt{2(u-v)^3}$. 21. $\sqrt[3]{xy^n(x+y)^2}$.

23. $2xy^3\sqrt[3]{5xy}$. 25. $\frac{\sqrt{a^2+b^2}}{ab}$. 27. $\frac{x-y}{xy}\sqrt{xy}$.

Pages 42, 43

1. 47.63. 3. 23.81. 5. $(3a-b-6abx)\sqrt{x}$. 7. $(b-2a)\sqrt{a+b}$.

9. $\frac{a+b}{ab}\sqrt{ab}$. 11. $\frac{x+2y-3ab}{xy}\sqrt{3xy}$. 13. $\left(\frac{x}{2}+2x^3\right)\sqrt[4]{2x}$.

15. $\left(\frac{u^{4n+3}}{v^{2n-3}}-\frac{u^3}{v^4}\right)\sqrt[n]{u^2v}$. 17. $\left(x-\frac{2}{x}\right)\sqrt[3]{2y}+\left(\frac{5}{y}-y\right)\sqrt[3]{2x}$.

Page 44

1. $\sqrt[12]{6561}$, $\sqrt[12]{4913}$; $\sqrt[4]{17}$, $\sqrt[3]{9}$.

3. $\sqrt[6]{216}$, $\sqrt[6]{169}$, $\sqrt[6]{143}$; $\sqrt[6]{143}$, $\sqrt[3]{13}$, $\sqrt{6}$.

5. $\sqrt[6n]{x^3y^3}$, $\sqrt[6n]{x^4y^{10}}$. 7. $\sqrt[6]{\frac{6}{55}}$. 9. $\sqrt[6]{\frac{288}{1375}}$. 11. $\sqrt[10]{32u^{17}v^9w^{24}}$.

13. $\sqrt[4]{\frac{18a^5}{b}}$. 15. $\sqrt[6]{600x^{11}y^{19}}$. 17. $\sqrt[mn]{y^{m+n^2}x^{m^2+2mn-n^2}}$.

Page 45

1. $15\sqrt{2}+14\sqrt{5}$. 3. $29\sqrt{30}-74$. 5. $\frac{8y}{z}-\frac{6x}{y}+5\sqrt{\frac{2x}{z}}$.

7. $\frac{52+11\sqrt{15}}{7}$. 9. $\frac{22+3\sqrt{35}}{13}$. 11. $a\sqrt{a}(\sqrt{a+4}+2)$.

13. $\frac{\sqrt{x^2-5}-2}{x-3}$. 15. $2x+2+2\sqrt{x^2+2x-15}$.

17. $2z^2+8+2\sqrt{z^4+10z^2+21}$. 19. $8-4\sqrt{3}-2\sqrt{2}+\sqrt{6}$.

21. $\frac{15b+6a-8\sqrt{6ab}}{3b-2a}$. 23. Yes.

Page 48

1. $V=f(e)=e^3$. 3. $A=f(n)=100+4n$. 5. $h=f(s)=s\sqrt{2}$.

7. 7, -1, 5, 6. 9. 2, $\frac{9}{4}$, $\frac{6\sqrt{6}+1}{6}$, $\sqrt{a}+\frac{1}{a}$.

11. $\frac{5}{13}$, -1, $\frac{3y+y^2}{1+y^2}$, $\frac{3z^2+1}{z^4+1}$. 13. $3x^3+7$, $\frac{3+7y}{y}$, $9x+28$.

15. 4. 17. 4. 19. $-\frac{34}{5}$. 21. (a) $11x+x^2$, (b) $2x^2+22x+121$

Page 50

5. (*a*) 12, (*b*) 35. **7.** (*a*) $(4, -3)$, (*b*) $(3, 6)$.
9. $(6, 6), (-6, 6), (-6, -6), (6, -6)$. **11.** 0.

Pages 58, 59

1. $(4, 3)$. **3.** $(3, -1)$. **5.** $(\frac{2}{3}, \frac{5}{3})$. **7.** $(2, 1)$. **9.** $(\frac{41}{11}, \frac{17}{11})$.
11. $(\frac{25}{11}, -\frac{27}{11})$. **13.** $(2, 3)$. **15.** $(\frac{7}{2}, -1)$. **17.** Dependent.
19. $(5, -2)$. **21.** Inconsistent. **23.** $(a + b, a - b)$. **25.** $\left(\frac{b}{a}, a\right)$.
27. $(1, 2)$. **29.** $(2, 2)$. **31.** $(a, -b)$.

Pages 59, 60

1. $(2, 3, 3)$. **3.** $(-1, 1, 2)$. **5.** $(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3})$. **7.** $(1, -3, 4)$.
9. $\left(\frac{d}{a+b+c}, \frac{d}{a+b+c}, \frac{d}{a+b+c}\right)$. **11.** $(2, -1, 3)$. **13.** $(2, -3, 4, 1)$.

Page 60

1. 5. **3.** -18. **5.** 11. **7.** $5x - 2y$.

Page 62

23. $(2, -3)$. **25.** $(\frac{67}{47}, \frac{3}{47})$. **27.** Inconsistent.

Page 63

1. -83. **3.** 53. **5.** $3x^2 + 5x + 36$.

Pages 65, 66

1. 17, 4. **3.** $\frac{17}{21}$. **5.** Airplane 195 mph; wind 15 mph.
7. 83, 19. **9.** \$3800 at 4%; \$3200 at 5%. **11.** 57.6 ft. by 46.2 ft.
13. $m = \frac{3}{2}, b = -7$. **15.** Mother, \$3900; son, \$2800; daughter, \$1700.

Page 67

1. $2x^2 + 3x - 3 = 0$. **3.** $7x^2 - 14x + 87 = 0$. **5.** $15y^2 - 17y - 52 = 0$.
7. $(a^2 - 4ce)x^2 + (2ab - 4cf - 4de)x + (b^2 - 4df) = 0$.

Page 68

1. $\frac{5}{3}, -\frac{5}{3}$. **3.** 2, 5. **5.** $0, \frac{8}{5}$. **7.** $\frac{4}{3}, \frac{5}{2}$.
9. $-3, 2$. **11.** $\frac{8}{3}, -\frac{1}{2}$. **13.** $b, \frac{1}{a}$. **15.** 0, 1, 3.

Pages 69, 70

1. $9, (x - 3)^2$. **3.** $\frac{4}{25}, (x + \frac{2}{5})^2$. **5.** $-3, 5$. **7.** $7, \frac{5}{3}$. **9.** $\frac{1}{2}, \frac{2}{3}$.
11. $\frac{-5 \pm \sqrt{13}}{4}$; $-2.152, -0.349$. **13.** $\frac{-6 \pm \sqrt{11}}{5}$; $-1.863, -0.537$.
15. $\frac{9 \pm \sqrt{41}}{2}$; 1.299, 7.702. **17.** $\frac{-8 \pm \sqrt{29}}{7}$; $-1.912, -0.374$.

19. $-3 \pm 2i$.
21. $\dfrac{-1 \pm \sqrt{3}i}{2}$.

23. $-3a,\ 7a$.
25. $\dfrac{-1 \pm \sqrt{1-4b^2}}{2b}$.

Page 71

1. $11,\ -3$.
3. $-\frac{1}{2},\ 3$.
5. $-2,\ \frac{5}{2}$.

7. $\dfrac{11 \pm \sqrt{61}}{10}$.
9. $\dfrac{5 \pm \sqrt{13}}{3}$.
11. $\dfrac{-5 \pm \sqrt{37}}{6}$.

13. $\dfrac{-4 \pm \sqrt{5}i}{7}$.
15. $\dfrac{3 \pm \sqrt{11}i}{10}$.
17. $\dfrac{-3 \pm \sqrt{6}i}{3}$.

19. $\dfrac{\sqrt{6} \pm \sqrt{10}i}{4}$.
21. $b-a,\ b+a$.
23. $-1,\ 1-k$.

Page 72

1. $3,\ \frac{9}{4}$.
3. $2,\ \frac{5}{2}$.
5. -2.
7. $1,\ -\frac{3}{2}$.
9. $-2,\ \frac{2}{3}$.
11. $ab,\ -ab$.

Page 73

1. 4.
3. 7.
5. 2, 3.
7. 3.
9. $3,\ \frac{9}{4}$.
11. $1,\ \frac{1}{2}$.
13. $\frac{16}{3}$.
15. -15.
17. $2a^2,\ 3a^2$.

Pages 74, 75

1. $-2,\ 2,\ -3,\ 3$.
3. 49.
5. -7.
7. $-2,\ 1$.
9. $-2,\ 1,\ 5,\ 8$.

11. $\dfrac{5 \pm \sqrt{31}}{2},\ \dfrac{5 \pm \sqrt{33}}{2}$.
13. $-3,\ -\frac{3}{2},\ 1,\ 2$.

15. 3, 5.
17. $-2,\ \frac{1}{2},\ 1,\ \frac{7}{2}$.

Pages 77, 78

1. $(0,\ -4),\ 2,\ -2$.
3. $(\frac{1}{2},\ \frac{25}{4}),\ -2,\ 3$.
5. $(1,\ 0),\ 1,\ 1$.
7. $(-\frac{3}{4},\ -\frac{33}{8}),\ -2.19,\ 0.69$.
9. $(-\frac{1}{2},\ -\frac{21}{4}),\ -2.79,\ 1.79$.
11. $(3,\ 1)$.
15. 625 sq. rods.
17. 36 ft. by 36 ft.

Page 79

1. Real, unequal, rational.
3. Real, unequal, irrational.
5. Real, equal, rational.
7. Imaginary, unequal.
9. Real, unequal, rational.
11. Real, unequal, rational.
13. $\frac{16}{5}$.
15. $5,\ -\frac{5}{11}$.
17. $-5,\ 5$.

Pages 81, 82

1. $-\frac{8}{3},\ \frac{17}{3}$.
3. $-\frac{7}{9},\ -\frac{4}{7}$.
5. $x^2+3x-28=0$.
7. $x^2-4x+3.91=0$.
9. $x^2+6x+4=0$.
11. $x^2+10x+27=0$.
13. $x^2+x+1=0$.
15. $x^2+2px-q=0$.
17. $(3x+2)(5x-1)$.

19. $7\left(x+\dfrac{3-2\sqrt{3}i}{7}\right)\left(x+\dfrac{3+2\sqrt{3}i}{7}\right)$.
21. $(2x-5y+3)(x+3y-1)$.

23. 5.
25. $3x^2+7x-6=0$.
27. $3x^2+2x-2=0$.

Pages 82, 83

1. 11, 14; -14, -11. **3.** 7, 9, 11. **5.** $\frac{5}{2}$, $-\frac{2}{5}$. **7.** $\frac{6}{5}$, $\frac{4}{5}$. **9.** 20 ft.
11. 8 in. by 8 in. or 6 in. by 10 in. **13.** $16\frac{2}{3}\%$. **15.** 11:46 A.M. or 12:46 P.M.

Page 87

11. $(4, -3)$, $(-3, 4)$. **13.** $(5, 1)$, $(7, 5)$.
15. $(-2, 6)(-6, 2)(2, -6)(6, -2)$. **17.** $(-2.6, 8.2)$, $(1.3, 0.4)$.
19. $(0.4, 0.3)$, $(-8.0, 4.5)$.

Page 89

1. $(2, 6)$, $(6, -2)$. **3.** $(5, 17)$, $(-2, 3)$. **5.** $(4, 2)$, $(1, 8)$.
7. $(5, 1)$, $(7, 5)$. **9.** $(1, -2)$, $(1, -2)$.
11. $\left(\dfrac{3+\sqrt{7}i}{4}, \dfrac{1+\sqrt{7}i}{2}\right), \left(\dfrac{3-\sqrt{7}i}{4}, \dfrac{1-\sqrt{7}i}{2}\right)$. **13.** 4, -4.

Pages 89, 90

1. $(3, 4)$, $(-3, 4)$, $(-3, -4)$, $(3, -4)$.
3. $(3, 5)$, $(-3, 5)$, $(-3, -5)$, $(3, -5)$.
5. $(5, 3)$, $(-5, 3)$, $(-5, -3)$, $(5, -3)$.
7. $(\sqrt{3}, \sqrt{5})$, $(-\sqrt{3}, \sqrt{5})$, $(-\sqrt{3}, -\sqrt{5})$, $(\sqrt{3}, -\sqrt{5})$.
9. $(0, 3)$, $(0, 3)$, $(0, -3)$, $(0, -3)$.
11. $(3, 2i)$, $(-3, 2i)$, $(-3, -2i)$, $(3, -2i)$.

Page 91

1. $(1, -3)$, $(-1, 3)$, $(2, 1)$, $(-2, -1)$.
3. $(3, 1)$, $(-3, -1)$, $(1, 4)$, $(-1, -4)$.
5. $(2, 1)$, $(-2, -1)$, $(7, -3)$, $(-7, 3)$.
7. $(2\sqrt{2}, \sqrt{2})$, $(-2\sqrt{2}, -\sqrt{2})$, $(\sqrt{3}, -2\sqrt{3})$, $(-\sqrt{3}, 2\sqrt{3})$.
9. $(2i, -i)$, $(-2i, i)$, $(3i, 2i)$, $(-3i, -2i)$.
11. $(2, -4)$, $(2, -4)$, $(-2, 4)$, $(-2, 4)$.

Pages 92, 93

1. $(6, 6)$, $(6, -6$, $(-3, 9)$, $(-3, -9)$.
3. $(1, 3)$, $(-1, 3)$, $(1, -7)$, $(-1, -7)$.
5. $(2, 1)$, $(1, 2)$, $(\frac{3}{2}, -\frac{1}{2})$, $(-\frac{1}{2}, \frac{3}{2})$.
7. $(1, 2)$, $(2, 1)$, $(-3+\sqrt{2}i, -3-\sqrt{2}i)$, $(-3-\sqrt{2}i, -3+\sqrt{2}i)$.
9. $(1, 2)$, $(2, 1)$. **11.** $(5, 1)$, $(-3, -\frac{5}{3})$. **13.** $(4, 2)$, $(\frac{20}{7}, -10)$.
15. $(1, -\frac{1}{2})$, $(-1, \frac{1}{2})$, $(\frac{2}{3}, -2)$, $(-\frac{2}{3}, 2)$. **17.** $(4, 1)$, $(\frac{5}{3}, \frac{12}{5})$.
19. $(\sqrt{2}, \sqrt{5})$, $(-\sqrt{2}, \sqrt{5})$, $(-\sqrt{2}, -\sqrt{5})$, $(\sqrt{2}, -\sqrt{5})$.
21. $(\frac{1}{2}, \frac{3}{2})$, $(-3, 5)$. **23.** $(2, 13)$, $(3, 24)$.

Pages 93, 94

1. 8 ft., 3 ft. **3.** 12 ft., 5 ft. **5.** 13 in., 13 in., 10 in.
7. $\left(\dfrac{3+\sqrt{5}}{2}, \dfrac{1+\sqrt{5}}{2}\right)$, $\left(\dfrac{3-\sqrt{5}}{2}, \dfrac{1-\sqrt{5}}{2}\right)$. **9.** 7 ft., 3 ft.
11. 12 ft., 5 ft. **13.** \$3600 at 4.5%, \$2800 at 5.5%.

Pages 95, 96

1. $\log_2 32 = 5$. **3.** $\log_{49} 7 = \frac{1}{2}$. **5.** $\log_{81} 3 = 0.25$. **7.** $\log_{25} 5 = \frac{1}{2}$.
9. $\log_{17} 1 = 0$. **11.** $36^{0.5} = 6$. **13.** $2^{-4} = 0.0625$. **15.** $32^{-0.2} = 0.5$.
17. $a^2 = a^2$. **19.** 3. **21.** $\frac{3}{2}$. **23.** -2. **25.** $-\frac{1}{2}$. **27.** -1.
29. $\frac{1}{8}$. **31.** $\frac{1}{7}$. **33.** 1. **35.** 343. **37.** 32. **39.** a^2.

Pages 98, 99

1. $\log_{11} 51 + \log_{11} 896 + \log_{11} 743$.
3. $2 \log_7 43 + \frac{2}{3} \log_7 695 - \frac{1}{3} \log_7 71 - \frac{1}{2} \log_7 563$. **5.** $\log_7 \frac{76^2 \cdot 48^3}{59^5}$.
7. $\log_{10} 16t^2$. **9.** $\log_{10} \frac{4}{3}\pi r^3$. **11.** $\log_{10} \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$.
13. $\frac{2}{3}$. **15.** $\frac{76}{15}$. **17.** $\frac{7}{6}$. **19.** 1.77815. **21.** 0.69897.
23. 2.65321. **25.** 0.31382.

Page 100

1. 3. **3.** 0. **5.** $9 - 10$. **7.** $5 - 10$.
9. 4673.9. **11.** 46739. **13.** 0.46739. **15.** 0.000046739.

Page 102

1. 4.75878. **3.** $7.23968 - 10$. **5.** 3.92034. **7.** 2.86149.
9. $9.53941 - 10$. **11.** 2.18312. **13.** 2.3169. **15.** 0.33447.
17. 48341. **19.** 28.573. **21.** 0.046129. **23.** 0.000047239.

Pages 103, 104

1. 26.105. **3.** 2,840,600. **5.** 387.99. **7.** 0.36998. **9.** 5800.6.
11. 27.148. **13** 0.011763. **15.** 136,480. **17.** 97.796. **19.** 6292.1.
21. -0.18141. **23.** 1.1787. **25.** 4.4256. **27.** 7.1247. **29.** 21.129.
31. 152.60. **33.** 2184.2. **35.** 2903.9. **37.** 1.8618. **39.** 81592.

Page 106

1. 14.206. **3.** 5.3933. **5.** 36.978. **7.** 4.1659.
9. 1.1750. **11.** 1.7076. **13.** 1.9015.

Pages 108, 109

1. 1.6094. **3.** 4.5219. **5.** 1.9213. **7.** -2.3708. **9.** 8.7084.
11. 1752. **13.** 691.0. **15.** 1.282. **17.** 6.5154. **19.** 4.2420. **21.** 3.4510.

Page 111

1. 30°. **3.** 90°. **5.** 15°. **7.** $-390°$. **9.** 114.59°.
11. $\frac{\pi}{3}$. **13.** $-\frac{4\pi}{3}$. **15.** $\frac{11\pi}{6}$. **17.** $\frac{7\pi}{2}$. **19.** 0.087266.

Page 112

1. 53. **3.** 217.46. **5.** 2725.3. **7.** 182. **9.** 0.11988. **11.** 11.753.
13. 17.41°. **15.** 234.41°. **17.** 43.341°. **19.** $\frac{\pi}{3}, \frac{\pi}{4}$. **21.** 298 ft. **23.** 6080 ft.

Page 113

1. 105. **3.** 3.61. **5.** 1520 ft. per sec.

Page 116

1. .29, .96, .31, 3.3, 1.0, 3.4. **3.** .95, .31, 3.1, .32, 3.2, 1.1.
5. − .39, − .92, .42, 2.4, − 1.1, − 2.6.
7. − .56, .83, − .67, − 1.5, 1.2, − 1.8.
9. .87, .50, 1.7, .58, 2.0, 1.2. **11.** − .50, − .87, .58, 1.7, − 1.2, − 2.

Page 117

1. $\frac{5}{13}, \frac{12}{13}, \frac{5}{12}, \frac{12}{5}, \frac{13}{12}, \frac{13}{5}$. **3.** $-\frac{24}{25}, \frac{7}{25}, -\frac{24}{7}, -\frac{7}{24}, \frac{25}{7}, -\frac{25}{24}$.

5. $-\frac{2}{5}, -\frac{\sqrt{21}}{5}, \frac{2\sqrt{21}}{21}, \frac{\sqrt{21}}{2}, -\frac{5\sqrt{21}}{21}, -\frac{5}{2}$.

7. $-\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}, -\frac{1}{2}, -2, \frac{\sqrt{5}}{2}, -\sqrt{5}$.

9. $\frac{1}{5}, -\frac{2\sqrt{6}}{5}, -\frac{\sqrt{6}}{12}, -2\sqrt{6}, -\frac{5\sqrt{6}}{12}, 5$.

Page 121

1. 0.6596. **3.** 0.2846. **5.** 1.6160. **7.** 0.6078. **9.** 0.9525. **11.** 0.8884.
13. 53° 38′. **15.** 42° 40′. **17.** 74° 35′. **19.** 13° 3′. **21.** 83° 34′. **23.** 51° 16′.

Page 122

1. 9.62355 − 10. **3.** 9.82464 − 10. **5.** 9.87430 − 10. **7.** 9.75470 − 10.
9. 9.91172 − 10. **11.** 9.66148 − 10. **13.** 16° 10.4′. **15.** 74° 42.3′.
17. 3° 22.8′. **19.** 33° 13.6′. **21.** 85° 46.2′. **23.** 5° 36.7′.

Page 124

1. $a = 36.20$, $b = 19.25$, $\beta = 28°$. **3.** $\beta = 27° 36'$, $b = 2.867$, $c = 6.187$.
5. $\beta = 55° 43'$, $a = 0.9285$, $c = 1.648$.
7. $\alpha = 49° 54'$, $\beta = 40° 6'$, $a = 6.224$.
9. $\alpha = 30° 18'$, $a = 434.5$, $c = 861.1$.
11. $c = 36.15$, $\alpha = 60° 49'$, $\beta = 29° 11'$.
13. $\alpha = 61° 48'$, $a = 1.298$, $b = 0.6961$.

Page 125

1. 13° 0′. **3.** (*a*) 18° 30′, (*b*) 14° 0′. **5.** 19.4 ft. **7.** 19.6 ft.
9. 146 ft. **11.** 32° 50′. **13.** 93.6 ft., 16.7 ft.
15. 14.1 in. **17.** (*a*) 3.46 in., (*b*) 6.93 in.

Page 126

1. $\beta = 27° 49'$, $a = 173.14$, $c = 195.76$.
3. $\alpha = 36° 11.4'$, $b = 596.10$, $c = 738.62$.
5. $\alpha = 37° 14.2'$, $\beta = 52° 45.7'$, $a = 262.84$.
7. $\beta = 58° 17.5'$, $a = 4426.3$, $b = 7164.5$.

9. $\beta = 35^\circ\ 36.6'$, $b = 3386.8$, $c = 5816.6$.
11. $\alpha = 59^\circ\ 27.9'$, $\beta = 30^\circ\ 32.1'$, $b = 45.176$.
13. $\alpha = 75^\circ\ 33.8'$, $a = 0.70648$, $b = 0.18188$.
15. $\alpha = 17^\circ\ 46.4'$, $a = 2.7743$, $c = 9.0886$.
17. $\alpha = 57^\circ\ 44.0'$, $\beta = 32^\circ\ 16.0'$, $c = 7.2606$.

Page 127

1. 7908.2. **3.** 129,990. **5.** 45,446. **7.** 15,856,000. **9.** 8,008,000.
11. 1729.9. **13.** 0.064247. **15.** 12.006. **17.** 11.898.

Pages 127, 128

1. 275.9, 502.5. **3.** 1361, 1078. **5.** 70.53, 60.38.
7. 28.12, $39^\circ\ 39'$. **9.** 557.1, $21^\circ\ 34'$. **11.** 0.1385, $72^\circ\ 0'$.
13. 123.5 N, 47.95 E. **15.** 206.2 S, 160.5 W.
17. 140.6 mph., N $14^\circ\ 5'$ E. **19.** $AD = 617.1$ yds., $CD = 178.4$ yds.

Page 132

1. -0.8116. **3.** -0.2183. **5.** 0.5176. **7.** -0.6459.
9. -0.4939. **11.** -1.5458. **13.** $9.53020 - 10$. **15.** $9.78089 - 10$ (n).
17. $\sin\theta$. **19.** $-\tan\theta$. **21.** $-\sin\theta$. **23.** $-\sec\theta$.

Pages 137, 138

13. $-\frac{4}{5}, \frac{3}{5}, -\frac{4}{3}, -\frac{3}{4}, \frac{5}{3}, -\frac{5}{4}$. **15.** $-\frac{15}{17}, -\frac{8}{17}, \frac{15}{8}, \frac{8}{15}, -\frac{17}{8}, -\frac{17}{15}$.

17. $-\dfrac{2\sqrt{29}}{29}, -\dfrac{5\sqrt{29}}{29}, \dfrac{2}{5}, \dfrac{5}{2}, -\dfrac{\sqrt{29}}{5}, -\dfrac{\sqrt{29}}{2}$. **19.** $-\frac{3}{8}$. **21.** $\dfrac{3-\sqrt{7}}{4}$.

23. $\pm\sqrt{1-\sin^2 x}, \pm\dfrac{\sin x}{\sqrt{1-\sin^2 x}}, \dfrac{\sqrt{1-\sin^2 x}}{\sin x}, \pm\dfrac{1}{\sqrt{1-\sin^2 x}}, \dfrac{1}{\sin x}$.

25. $\pm\dfrac{\tan x}{\sqrt{1+\tan^2 x}}, \pm\dfrac{1}{\sqrt{1+\tan^2 x}}, \dfrac{1}{\tan x}, \pm\sqrt{1+\tan^2 x}, \pm\dfrac{\sqrt{1+\tan^2 x}}{\tan x}$.

Page 142

1. $60^\circ, 120^\circ, 240^\circ, 300^\circ$. **3.** $0^\circ, 180^\circ$. **5.** $90^\circ, 270^\circ$. **7.** $60^\circ, 300^\circ$.
9. $0^\circ, 180^\circ, 225^\circ, 315^\circ$. **11.** $120^\circ, 240^\circ$. **13.** $45^\circ, 60^\circ, 240^\circ, 315^\circ$.
15. $135^\circ, 315^\circ$. **17.** $0^\circ, 30^\circ, 150^\circ, 180^\circ, 210^\circ, 330^\circ$.
19. $41^\circ\ 49', 138^\circ\ 11', 270^\circ$. **21.** $60^\circ, 180^\circ, 300^\circ$. **23.** $45^\circ, 120^\circ, 225^\circ, 300^\circ$.
25. 0°. **27.** $19^\circ\ 28', 30^\circ, 150^\circ, 160^\circ\ 32'$.
29. $30^\circ, 45^\circ, 135^\circ, 150^\circ, 210^\circ, 225^\circ, 315^\circ, 330^\circ$.

Pages 146, 147

1. $\dfrac{\sqrt{2}+1}{2}, \dfrac{\sqrt{2}+\sqrt{3}}{2}, 1+\sqrt{3}$.

3. $\dfrac{\sqrt{6}-\sqrt{2}}{4}, \dfrac{\sqrt{6}+\sqrt{2}}{4}, 2-\sqrt{3}, 2+\sqrt{3}$.

5. $-\frac{297}{425}, \frac{304}{425}, \frac{87}{425}, \frac{416}{425}$. **7.** $\frac{220}{221}, -\frac{21}{221}, -\frac{140}{221}, -\frac{171}{221}$.

9. $\frac{\sqrt{26}+20\sqrt{13}}{78}$, $\frac{4\sqrt{13}-5\sqrt{26}}{78}$, $\frac{\sqrt{26}-20\sqrt{13}}{78}$, $\frac{4\sqrt{13}+5\sqrt{26}}{78}$.

11. $\frac{\sqrt{2}}{2}(\cos\theta+\sin\theta)$. **13.** $\frac{\tan\theta-1}{\tan\theta+1}$. **15.** $\frac{1}{2}(\sqrt{3}\cos\theta-\sin\theta)$.

25. $\sin\alpha\cos\beta\cos\gamma+\cos\alpha\sin\beta\cos\gamma+\cos\alpha\cos\beta\sin\gamma-\sin\alpha\sin\beta\sin\gamma$.

Pages 147, 148

5. $-\frac{24}{25}, -\frac{7}{25}, \frac{24}{7}, \frac{7}{24}$. **7.** $-\frac{120}{169}, \frac{119}{169}, -\frac{120}{119}, -\frac{119}{120}$. **9.** $-\frac{3}{5}, \frac{4}{5}, -\frac{3}{4}, -\frac{4}{3}$.

Page 149

1. $\frac{1}{2}\sqrt{2-\sqrt{3}}$, $\frac{1}{2}\sqrt{2+\sqrt{3}}$, $2-\sqrt{3}$, $2+\sqrt{3}$.

3. $\frac{1}{2}\sqrt{2+\sqrt{2}}$, $\frac{1}{2}\sqrt{2-\sqrt{2}}$, $\sqrt{2}+1$, $\sqrt{2}-1$.

5. $\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}, \frac{1}{2}, 2$. **7.** $\frac{4\sqrt{41}}{41}, -\frac{5\sqrt{41}}{41}, -\frac{4}{5}, -\frac{5}{4}$.

9. $\frac{\sqrt{15}}{5}, -\frac{\sqrt{10}}{5}, -\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{3}$.

Pages 150, 151

3. $\sin 70^\circ+\sin 40^\circ$. **5.** $\cos 40^\circ-\cos 70^\circ$. **7.** $\frac{1}{2}(\sin 5x+\sin x)$.
9. $\frac{1}{2}(\cos 2x-\cos 8x)$. **11.** $2\sin 35^\circ\cos 15^\circ$. **13.** $2\sin 50^\circ\sin 25^\circ$.
15. $2\cos 3x\sin x$. **17.** $-2\sin 7x\sin 3x$. **19.** $2\sin 28^\circ\sin 6^\circ$. **21.** $\sqrt{3}$.

Page 152

1. 0°, 60°, 180°, 300°. **3.** 60°, 180°, 300°.
5. 0°, 60°, 120°, 180°, 240°, 300°. **7.** 120°, 180°.
9. 0°, 90°, 180°, 210°, 270°, 330°. **11.** 0°, 135°, 180°, 315°.
13. 0°, 30°, 90°, 150°, 180°, 270°.

Page 156

1. $-135^\circ, -45^\circ, 225^\circ, 315^\circ$, etc. **3.** $-240^\circ, -120^\circ, 120^\circ, 240^\circ$, etc.
5. $-225^\circ, -45^\circ, 135^\circ, 315^\circ$, etc.
7. $-292^\circ 10', -247^\circ 50', 67^\circ 50', 112^\circ 10'$, etc.
9. $-353^\circ 34', -173^\circ 34', 6^\circ 26', 186^\circ 26'$, etc.

11. $60^\circ=\frac{\pi}{3}$. **13.** $-60^\circ=-\frac{\pi}{3}$. **15.** $0^\circ=0$. **17.** 37°. **19.** 131° 19′.

21. $\frac{\sqrt{3}}{2}$. **23.** 1. **25.** $\frac{2\sqrt{3}}{3}$. **27.** 45°. **29.** 135°. **31.** -60°.

33. $1-2u^2$. **35.** $\frac{2u}{1-u^2}$. **37.** $\pm\frac{2\sqrt{u^2-1}}{u^2}$.

39. $\pm u\sqrt{1-v^2}\pm v\sqrt{1-u^2}$. **41.** $\frac{u-v}{1+uv}$.

Page 160

1. $a=4.6074$, $b=4.1455$, $\beta=61^\circ 21.0'$.
3. $b=4.7168$, $c=4.4433$, $\beta=78^\circ 10.3'$.

5. $a = 3797.9$, $b = 7951.3$, $\beta = 111° 47.7'$.
7. $a = 4.5089$, $c = 3.9972$, $\gamma = 59° 41.2'$.
9. 5.441 ft., 6.510 ft.
11. 17.06 mi.

Page 163

1. $b_1 = 12.691$, $\beta_1 = 115° 16.7'$, $\gamma_1 = 41° 2.1'$ or $b_2 = 4.1850$, $\beta_2 = 17° 20.9'$, $\gamma_2 = 138° 57.9'$.
3. $a_1 = 4195.0$, $\alpha_1 = 75° 34.3'$, $\gamma_1 = 61° 13.8'$ or $a_2 = 1340.8$, $\alpha_2 = 18° 1.9'$, $\gamma_2 = 118° 46.2'$.
5. $c = 117.61$, $\alpha = 59° 50.0'$, $\gamma = 48° 40.4'$.
7. No solution.
9. $c_1 = 0.54371$, $\beta_1 = 40° 47.4'$, $\gamma_1 = 112° 23.4'$ or $c_2 = 0.14196$, $\beta_2 = 139° 12.6'$, $\gamma_2 = 13° 58.2'$.
11. 77.81 in. or 15.92 in.

Page 164

1. $c = 40.861$, $\alpha = 83° 40.8'$, $\beta = 39° 52.6'$.
3. $b = 414.80$, $\alpha = 21° 48.1'$, $\gamma = 44° 50.1'$.
5. $a = 15.675$, $\beta = 29° 33.9'$, $\gamma = 112° 6.5'$.
7. $a = 10.827$, $\beta = 69° 14.3'$, $\gamma = 93° 54.5'$.
9. 37.58 mi.

Page 165

1. 5.696.
3. 34.90.
5. $\alpha = 49° 41'$, $\beta = 35° 54'$, $\gamma = 94° 25'$.
7. 100.9 mi.

Pages 167, 168

1. $\alpha = 51° 47.0'$, $\beta = 81° 57.8'$, $\gamma = 46° 15.4'$.
3. $\alpha = 39° 10.6'$, $\beta = 24° 42.8'$, $\gamma = 116° 6.6'$.
5. $\alpha = 73° 37.4'$, $\beta = 43° 14.2'$, $\gamma = 63° 8.2'$.
7. $\alpha = 56° 21.0'$, $\beta = 26° 54.0'$, $\gamma = 96° 45.0'$.
9. $\alpha = 43° 55'$, $\beta = 56° 18'$, $\gamma = 79° 47'$.

Page 168

1. 134,750.
3. 114.82.
5. 633.46.

Pages 169, 170

1. 4188 ft., 2074 ft.
3. 4264 ft.
5. 2681 ft.
7. 61.39 ft., 30.08 ft.
9. 29° 40′, 62° 22′.
11. N 35° 34′ E, 126 min.
13. 36.72 ft.
19. 218.8.

Page 172

1. 3.
3. -8.
5. -6.
7. 5.
9. 10.
11. -7.
13. 4, 3.
15. $7, -2$.
17. $-5, -9$.

Pages 173, 174

3. 5.
5. 13.
7. $\sqrt{153}$.
9. $2\sqrt{5}, 2\sqrt{10}, 2\sqrt{17}$.
11. $2\sqrt{37}, \sqrt{61}, \sqrt{113}$.
21. -1.
23. $(-3, 5), (7, 5)$.
25. $x + 2y = 1$.

Page 175

1. (a) $(-3, 2)$, (b) $(2, -4)$.
3. (2, 10), (5, 14), (8, 18).
5. 5:9.
7. (8, 5).

Pages 176, 177

1. $\frac{\sqrt{3}}{3}$. **3.** 0.4877. **5.** 1. **7.** $\sqrt{3}$. **9.** 30°. **11.** 120°. **13.** 27°.
15. 28° 27′. **17.** $\frac{3}{4}$, 36° 52′. **19.** $\frac{7}{4}$, 60° 15′. **21.** $-$ 1.4420, 124° 44′.
29. $\sqrt{3}, -\sqrt{3}, 0$. **31.** $\frac{y-1}{x+2} = 2$.

Pages 179, 180

1. 136. **3.** 63° 26′. **5.** $3x - 2y = 1$. **7.** 45°. **9.** 139° 46′. **11.** $-\frac{7}{2}$.
13. 70° 34′, 41° 49′, 67° 37′. **15.** 65° 13′, 71° 54′, 42° 53′. **17.** 9.

Pages 181, 182

1. $y = 0$. **3.** $x = 3$. **5.** $x^2 + y^2 = 36$. **7.** $2x - y = 4$.
9. $y - 7 = -3(x + 1)$. **11.** $y - 1 = x - 4$. **21.** $y = x$. **23.** $x^2 + y^2 = 6x$.

Pages 184, 185

1. $3x - y - 11 = 0$. **3.** $4x - 3y + 8 = 0$. **5.** $y - 7 = 0$. **7.** $x - y - 4 = 0$.
9. $2x - 3y + 10 = 0$. **11.** $2y = 3x + 8$. **13.** $12x + 15y + 10 = 0$.
15. $2y = 3x + 12$; $3x + y + 3 = 0$; $4x + 5y - 7 = 0$; $y - 3 = 0$.
17. $x - 5 = 0$; $x + 5 = 0$. **19.** 2, 9. **21.** $-\frac{2}{5}$, 3.
23. $3x - 2y = 27$. **25.** $x - 3y - 6 = 0$.

Page 186

1. $3x + 2y - 19 = 0$. **3.** $x + y - 1 = 0$. **5.** $2x + 3y - 9 = 0$.
7. (*a*) $\frac{x}{7} + \frac{y}{2} = 1$, (*b*) $y = -\frac{2}{7}x + 2$.
9. (*a*) $\frac{x}{-5} + \frac{y}{\frac{15}{4}} = 1$, (*b*) $y = \frac{3}{4}x + \frac{15}{4}$.
11. $2x + 3y = 34$, $3x - 2y = 12$. **13.** $4x - 5y + 10 = 0$, $5x + 4y + 33 = 0$.
15. $\frac{x}{\frac{5}{3}} + \frac{y}{\frac{5}{2}} = 1$.
17. $5x - 2y + 11 = 0$, $5x - 2y - 31 = 0$, $x + 8y = 23$, $x + 8y = 65$.
19. $2x - 5y + 14 = 0$, $3x - y = 31$, $4x + 3y = 24$.
21. $5x + 2y = 52$, $x + 3y = 17$, $3x - 4y = 18$; $(\frac{122}{13}, \frac{33}{13})$.

Pages 188, 189

1. 4, 2, $-$ 8. **3.** $-\frac{3}{2}$, 4, 6. **5.** $-\frac{4}{7}, -\frac{5}{2}, -\frac{10}{7}$. **7.** $x + y + 4 = 0$.
9. $3x - 2y + 1 = 0$, $3x - 2y = 7$, $2x + 5y = 50$, $2x + 5y = 11$.
11. $3x - 2y = 16$, $3x + 5y = 2$. **13.** $-b/a$.

Pages 191, 192

1. $\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y - 5 = 0$. **3.** $\frac{1}{2}x + \frac{\sqrt{3}}{2}y + 4 = 0$.
5. $-\frac{1}{2}x + \frac{\sqrt{3}}{2}y - 7 = 0$ **7.** $\frac{1}{2}x + \frac{\sqrt{3}}{2}y - \frac{3}{5} = 0$, 60°, $\frac{3}{5}$.

9. $-\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y = 0$, $135°$, 0. **11.** $-\frac{3}{5}x + \frac{4}{5}y + 3 = 0$, $126°\ 52'$, -3.
13. (*a*) $x - y - 10 = 0$, (*b*) $\sqrt{3}(x-3) + (y+7) = 0$.
15. $\sqrt{3}x - y + 8 = 0$. **17.** $x + y = \pm 7\sqrt{2}$. **19.** $\pm 3x + 4y - 20 = 0$.

Pages 193, 194

1. 5, above. **3.** -2, below. **5.** $\frac{17}{73}\sqrt{73}$, above. **9.** 3. **11.** $\frac{5\sqrt{5}}{2}$.
13. (*a*) $4x + 7y - 5 = 3\sqrt{65}$, (*b*) $4x + 7y - 5 + 7\sqrt{65} = 0$.
15. $4x + 3y + 49 = 0$, $4x + 3y - 31 = 0$.
17. $7x + y + 45 = 0$, $x - 7y - 5 = 0$, the former.

Page 195

1. $\frac{143}{2}$. **3.** 93. **5.** 10. **7.** 13.

Page 197

1. $y = -3x + b$. **3.** $x - y + k = 0$. **5.** $y + 9 = m(x - 2)$.
7. $y = mx + 7$. **9.** $(3x + 5y - 2) + k(2x - 7y + 8) = 0$.
17. (*a*) $y = 4x - 3$, (*b*) $y = -3x - 3$. **19.** $6x - y - 7 = 0$. **21.** $2x + y = 9$.

Pages 200, 201

1. $x^2 + y^2 - 8x - 14y + 40 = 0$. **3.** $x^2 + y^2 + 4x - 16y + 19 = 0$.
5. $x^2 + y^2 - 12x + 12y + 36 = 0$. **7.** $x^2 + y^2 - 24x + 10y = 0$.
9. $x^2 + y^2 + 8x - 6y + 9 = 0$. **11.** $x^2 + y^2 - 8x + 4y - 16 = 0$.
13. $(5, -12)$, 13. **15.** $(4, -5)$, $\sqrt{58}$. **17.** $(2, 3)$, $3\sqrt{-1}$.
19. $(\frac{2}{5}, -\frac{6}{5})$, 3. **21.** $x^2 + y^2 - 10x + 6y + 9 = 0$. **23.** $(5, 3)$, $(-2, 2)$.
25. $x^2 + y^2 = a^2$. **27.** $x^2 + y^2 + Ey = 0$.

Pages 202, 203

1. $x^2 + y^2 - 6x + 8y = 0$. **3.** $x^2 + y^2 - 6x - 2y + 5 = 0$.
5. $x^2 + y^2 + 10x - 2y - 59 = 0$. **7.** $2x^2 + 2y^2 + 6x - 15y - 83 = 0$.
9. $x^2 + y^2 - 17x - 7y + 52 = 0$. **11.** $x^2 + y^2 + 2x - 4y - 40 = 0$.
13. $x^2 + y^2 + 10x - 2y + 10 = 0$. **15.** $x^2 + y^2 + 2x - 8y + 4 = 0$.
17. $x^2 + y^2 - 20x - 38y + 292 = 0$, $x^2 + y^2 + 14x - 4y - 116 = 0$.
19. $x^2 + y^2 - 2x - 10y + 21 = 0$, $x^2 + y^2 - 26x - 2y + 45 = 0$.
21. $x^2 + y^2 - 4x - 4y + 4 = 0$.

Page 204

3. $(3, 1)$, $(-3, 1)$. **5.** $2x^2 + 2y^2 + 15x + y - 23 = 0$.
7. $3x - y - 1 = 0$, $(1, 2)$, $(3, 8)$. **9.** $9x + 4y - 22 = 0$.

Pages 206, 207

1. $4x^2 + 4y^2 - 5x - 21y + 19 = 0$. **3.** $x^2 + y^2 + 12x - 16y + 44 = 0$.
5. $x^2 + y^2 - 5x - 8y + 20 = 0$.

Page 209

19. 10.77, 20. **21.** 4.272, 6.022.

Page 211

1. $(3\sqrt{3}, 3)$. **3.** $(-4, -4\sqrt{3})$. **5.** $(-3, 0)$. **7.** $(4, 60°)$. **9.** $(4, 90°)$.
11. $(2\sqrt{2}, 225°)$. **13.** $4\sqrt{2}$. **15.** $r \sin\theta = -4$.
17. $r(2\cos\theta - 3\sin\theta) = 6$. **19.** $r = 8\cos\theta$.
21. $x^2 + y^2 = 9$. **23.** $y = 2$. **25.** $x = 2$.

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1. $5x - 3y = 9$. **3.** $y = 6$. **5.** $x + \sqrt{3}y + 8 = 0$. **7.** $r(3\cos\theta - 7\sin\theta) = 9$.
9. $r(4\cos\theta - 9\sin\theta) + 11 = 0$. **11.** (*a*) $r\sin\theta + 2 = 0$, (*b*) $r\sin\theta - 3 = 0$, (*c*) $r\sin\theta - 2\sqrt{2} = 0$. **13.** $r\cos\left(\theta - \frac{5\pi}{6}\right) = 5$. **15.** $\sqrt{2}r\cos\left(\theta - \frac{\pi}{4}\right) = 3$.
17. $(4, 60°)$. **19.** 12.

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1. $r = 7$. **3.** $r + 4\cos\theta = 0$. **5.** $r^2 - 12r\cos\left(\theta - \frac{\pi}{6}\right) + 20 = 0$.
7. $r^2 - 2ar\cos\left(\theta - \frac{\pi}{4}\right) = 3a^2$. **9.** $(3, 0°)$, 3. **11.** $\left(4, \frac{\pi}{6}\right)$, 4.
13. $(3, 0°)$, 4. **15.** $\left(2, \frac{\pi}{4}\right)$, 4. **17.** $x^2 + y^2 + 4x = 0$.
19. $x^2 + y^2 = 7x + 7\sqrt{3}y$. **21.** $x^2 + y^2 - 3\sqrt{3}x - 3y + 5 = 0$.
23. $r = 11$. **25.** $r + 6\cos\theta = 0$.
27. $r^2 - 20r\cos\left(\theta - \frac{2\pi}{3}\right) = 44$. **29.** $r^2 - 26r\cos(\theta - 67°\,23') + 169 = 169$.
31. $(6, 60°)$, $(6, 300°)$.

Pages 217, 218

1. $(4, 0)$, $x + 4 = 0$, 16. **3.** $(0, \frac{5}{2})$, $y + \frac{5}{2} = 0$, 10.
5. $(0, \frac{7}{4})$, $y + \frac{7}{4} = 0$, 7. **7.** $(0, \frac{9}{8})$, $y + \frac{9}{8} = 0$, $\frac{9}{2}$.
9. $(0, -\frac{5}{12})$, $y = \frac{5}{12}$, $\frac{5}{3}$. **11.** $y^2 = 28x$. **13.** $x^2 + 16y = 0$.
15. $7x^2 = 4y$. **17.** $y^2 = 20x$. **19.** $(0, 0)$, $(3, 3)$. **21.** $(0, 0)$, $(-4, 6)$.
23. $3\sqrt{13}$. **25.** $4x^2 + 4y^2 - 4px - 3p^2 = 0$.
27. (*a*) 7, (*b*) 10, (*c*) 50, (*d*) 5, 25.
29. $\left(\frac{3p}{2}, \sqrt{3}p\right)$, $\left(\frac{3p}{2}, -\sqrt{3}p\right)$.

Page 222

1. $(\pm 5, 0)$, $(\pm 3, 0)$, $\frac{3}{5}$, 5, 4, $\frac{32}{5}$, $3x = \pm 25$.
3. $(\pm 3, 0)$, $(\pm\sqrt{5}, 0)$, $\frac{\sqrt{5}}{3}$, 3, 2, $\frac{8}{3}$, $\sqrt{5}x = \pm 9$.
5. $(0, \pm 4)$, $(0, \pm\sqrt{7})$, $\frac{\sqrt{7}}{4}$, 4, 3, $\frac{9}{2}$, $\sqrt{7}y = \pm 16$.
7. $(0, \pm 3)$, $\left(0, \pm 3\frac{\sqrt{3}}{2}\right)$, $\frac{\sqrt{3}}{2}$, 3, $\frac{3}{2}$, $\frac{3}{2}$, $y = \pm 2\sqrt{3}$.
9. $(\pm\sqrt{5}, 0)$, $(\pm\sqrt{2}, 0)$, $\frac{\sqrt{10}}{5}$, $\sqrt{5}$, $\sqrt{3}$, $6\frac{\sqrt{5}}{5}$, $\sqrt{2}x = \pm 5$.

11. $3x^2 + 4y^2 = 48$. **13.** $4x^2 + 3y^2 = 108$. **15.** $x^2 + 2y^2 = 18$.
17. $4x^2 + 9y^2 = 180$. **19.** $7x^2 + 4y^2 = 128$.
21. $(1, 2), (1, -2)$. **23.** $x^2 + 4y^2 = a^2$.

Pages 223, 224

1. $3x^2 + 4y^2 = 108$. **3.** $25x^2 + 21y^2 = 2100$.
5. $3x^2 + 4y^2 = 48$. **7.** $9x^2 + 8y^2 = 128$.
9. $9x^2 + 25y^2 = 900$. **11.** $3x^2 + 4y^2 = 48$, $x^2 + 4y^2 = 48$.
13. 15, 7.06, 4.61, 3.38, 3, 3.38, 4.61, 7.06, 15. **15.** 15, 25.
17. 7, 11. **19.** 94.5 and 91.3 million miles.

Page 228

1. $(\pm 12, 0), (\pm 13, 0), \frac{13}{12}, \frac{25}{6}, 12y = \pm 5x, 13x = \pm 144$.

3. $(0, \pm 6), (0, \pm 3\sqrt{5}), \dfrac{\sqrt{5}}{2}, 3, y = \pm 2x, \sqrt{5}y = \pm 12$.

5. $(\pm \frac{2}{3}, 0), \left(\pm \dfrac{\sqrt{97}}{6}, 0\right), \dfrac{\sqrt{97}}{4}, \frac{27}{4}, 4y = \pm 9x, 3\sqrt{97}x = \pm 8$.

7. $(0, \pm \frac{1}{5}), \left(0, \pm \dfrac{\sqrt{34}}{15}\right), \dfrac{\sqrt{34}}{3}, \frac{10}{9}, 5y = \pm 3x, 5\sqrt{34}y = \pm 3$.

9. $9x^2 - 16y^2 = 576$. **11.** $16y^2 - 9x^2 = 144$. **13.** $x^2 - 4y^2 = 9$.
15. $3x^2 - y^2 = 9$. **17.** $9y^2 - 4x^2 = 81$. **19.** $x^2 - y^2 = 8$. **21.** $9x^2 - 16y^2 = 144$.

Pages 229, 230

1. $3x^2 - y^2 = 108$. **3.** $2x^2 - y^2 = 54$. **5.** $9y^2 - 16x^2 = 576$.
7. $144y^2 - 25x^2 = 3600$; $12y = \pm 5x, (\pm 12, 0), (\pm 13, 0), 13x = \pm 144$;
$12y = \pm 5x, (0, \pm 5), (0, \pm 13), 13y = \pm 25$.

9. $9y^2 - x^2 = 4$; $3y = \pm x, (\pm 2, 0), \left(\pm 2\dfrac{\sqrt{10}}{3}, 0\right), \sqrt{10}x = \pm 6$;

$3y = \pm x, (0, \pm \frac{2}{3}), \left(0, \pm 2\dfrac{\sqrt{10}}{3}\right), 3\sqrt{10}y = \pm 2$.

11. $5x^2 - 11y^2 = 55$; $\sqrt{11}y = \pm \sqrt{5}x, (0, \pm \sqrt{5}), (0, \pm 4), 4y = \pm 5$;
$\sqrt{11}y = \pm \sqrt{5}x, (\pm \sqrt{11}, 0), (\pm 4, 0), 4x = \pm 11$.

17. $3\sqrt{13} \pm 3$. **19.** 9, 5.

Page 232

1. $\frac{3}{4}, 4, (8, 0°), (\frac{8}{7}, 180°)$. **3.** $\frac{3}{2}, 12, (-12, 0°), (\frac{12}{5}, 180°)$.
5. $\frac{5}{3}, 6, (\frac{9}{8}, 0°), (-\frac{9}{2}, 180°)$. **7.** $\frac{1}{3}, 10, (\frac{15}{2}, 90°), (\frac{15}{4}, -90°)$
9. $1, 5, (\frac{5}{4}, -90°)$. **11.** $\frac{5}{2}, 7, (1, 90°), (-\frac{7}{3}, -90°)$.

13. $r = \dfrac{6}{1 - \cos\theta}$. **15.** $r = \dfrac{14}{3 + 4\sin\theta}$.

17. $r = \dfrac{18}{5 + 4\cos\theta}$. **18.** $r = \dfrac{10}{2 - \sin\theta}$.

Pages 235, 236

1. $(7, -4)$, $(5, 3)$, $(-3, 4)$, $(0, 1)$, $(2, 0)$. 3. $6x' - 5y' = 0$. 5. $4x'^2 + 25y'^2 = 100$. 7. $2x' + y'^2 = 0$; $4.8, -0.8$. 9. $9x'^2 + 16y'^2 = 144$. 11. $y'^2 - 3x'^2 = 63$. 13. $5x'^2 - 2y' = 0$. 15. $y' = ax'^2$.

Page 237

1. $(-\sqrt{2}, -4\sqrt{2})$, $(-3\sqrt{2}, \sqrt{2})$, $(7\sqrt{2}, 0)$, $(6\sqrt{2}, 6\sqrt{2})$, $(-2\sqrt{2}+2\sqrt{6}, 2\sqrt{2}+2\sqrt{6})$.
3. $(6, 2)$, $(3, 3\sqrt{3})$, $(10, 0)$, $(3-2\sqrt{3}, 3\sqrt{3}+2)$. 5. $x' + 8 = 0$.
7. $x'^2 + 4y'^2 = 4$. 9. $2x'y' = a^2$. 11. $3x'^2 - 2y'^2 = 6$.
13. $12y''^2 - 13x''^2 = 5$. 15. $x'^2 + 4x'y' + y'^2 = 6$, $3x''^2 - y''^2 = 6$.

Page 239

1. $(y+5)^2 = 12(x-1)$, $(1, -5)$, $(4, -5)$, $x + 2 = 0$.
3. $(x+2)^2 = 8(y+1)$, $(-2, -1)$, $(-2, 1)$, $y + 3 = 0$.
5. $\dfrac{(x-3)^2}{25} + \dfrac{(y-2)^2}{9} = 1$, $(3, 2)$, $(8, 2)$, $(-2, 2)$, $(7, 2)$, $(-1, 2)$.
7. $\dfrac{(x+5)^2}{16} - \dfrac{(y-1)^2}{9} = 1$, $(-5, 1)$, $(-1, 1)$, $(-9, 1)$, $(0, 1)$, $(-10, 1)$.
9. $\dfrac{(y+5)^2}{4} - \dfrac{(x-4)^2}{5} = 1$, $(4, -5)$, $(4, -3)$, $(4, -7)$, $(4, -2)$, $(4, -8)$.
11. $\dfrac{(x-\frac{3}{2})^2}{9} + \dfrac{(y+1)^2}{4} = 1$, $(\frac{3}{2}, -1)$, $(\frac{9}{2}, -1)$, $(-\frac{3}{2}, -1)$, $(\frac{3}{2}+\sqrt{5}, -1)$, $(\frac{3}{2}-\sqrt{5}, -1)$.
13. Point ellipse. 15. Imaginary ellipse.

Pages 240, 241

1. $(y+1)^2 = 20(x+3)$. 3. $(x-3)^2 + 12(y+7) = 0$.
5. $9(x-3)^2 + 8(y-1)^2 = 72$. 7. $100(x-5)^2 + 36(y-1)^2 = 225$.
9. $9(y-2)^2 - 16(x-4)^2 = 144$. 11. $4(y-5)^2 - 5(x+3)^2 = 20$.

Pages 242, 243

1. $12x'^2 - 13y'^2 = 48$. 3. $2x'^2 + y'^2 = 5$. 5. $7x'^2 - 19y'^2 = 80$.
7. $19x'^2 - 15y'^2 = 22$. 9. $5y'^2 + 4x' = 0$.

Pages 244, 245

1. $2x''^2 + y''^2 = 4$. 3. $11x''^2 - 2y''^2 = 20$. 5. $5x''^2 + 11y''^2 = 14$.
7. $2\sqrt{29}y''^2 + 3x'' = 0$. 9. $4x''^2 - y''^2 = 0$. 11. $y''^2 - 2x'' = 0$.
13. $x''^2 = 4$. 17. $13x^2 + 48xy + y^2 - 65x + 3y = 0$.
19. $9x^2 + 8xy - 13y^2 - x + 19y - 22 = 0$.
21. $2xy = xy_1 + yx_1$. 23. $x^2 + by = a^2$.

Page 250

1. $4x - 5y = 40$, $5x + 4y = 9$. 3. $8x + 5y = 1$, $5x - 8y = 34$.
5. $5x + 3y = 16$, $3x - 5y = 30$. 7. $7x + 3y + 27 = 0$, $3x - 7y = 5$.

9. $12x - y = 16,\ x + 12y = 98.$ **11.** $11x - 2y = 10,\ 2x + 11y = 70.$
13. $4x - 5y + 12 = 0,\ 5x + 4y = 26.$
15. $(y_1 - k)(y - k) = p(x + x_1 - 2h),\ p(y - y_1) + (y_1 - k)(x - x_1) = 0.$
17. $b^2(x_1 - h)(x - h) - a^2(y_1 - k)(y - k) = a^2b^2,\ b^2(x_1 - h)(y - y_1) + a^2(y_1 - k)(x - x_1) = 0.$

Page 252

1. $y = 3x + 2.$ **3.** $x + 3y = \pm 5.$ **5.** $y = mx - pm - \frac{p}{2}m^2.$

9. $y = mx - \frac{p}{2}m^2.$ **11.** $y - k = m(x - h) + \frac{p}{2m}.$

13. $y - k = m(x - h) \pm \sqrt{a^2m^2 - b^2}.$

Page 256

1. $6x - 7.$ **3.** $21x^2 + 4x.$ **5.** $16x + \frac{2}{x^2} - \frac{6}{x^3}.$ **7.** $10x^4 + 2x.$

9. $\frac{28}{(6x + 5)^2}.$ **11.** $\frac{2x^2 - 6x - 4}{(2x - 3)^2}.$ **13.** $2x^{-\frac{1}{2}} - \frac{50}{3}x^{\frac{2}{3}}.$

15. $\frac{1}{2\sqrt{x}(\sqrt{x} + 1)^2}.$ **17.** $5a_0x^4 + 4a_1x^3 + 3a_2x^2 + 2a_3x + a_4.$

19. 5184 ft. **21.** $\frac{2}{9\pi}$ ft. per sec. **23.** 4 ft. per sec.

Page 257

1. $\frac{3x^2 - 5x}{\sqrt{2x^3 - 5x^2 - 1}}.$ **3.** $\frac{x}{\sqrt{x^2 + 1}} + \frac{x}{\sqrt{(x^2 + 1)^3}}.$ **5.** $\frac{x^3}{\sqrt[4]{(a^4 + x^4)^3}}.$

Pages 257, 258

1. $\sec^2 x.$ **3.** $-\csc^2 x.$ **5.** $-\cos\left(\frac{\pi}{2} - x\right).$ **7.** $k\,e^{kx}.$

9. $-2x\,e^{-x^2}.$ **11.** $\sec x.$ **13.** $2x \cos x^2.$ **15.** $-e^x \sin e^x.$

Page 258

1. $10x - 2,\ 10,\ 0.$ **3.** $4x - \frac{3}{x^2},\ 4 + \frac{6}{x^3},\ -\frac{18}{x^4}.$

5. $-\frac{3}{(x + 1)^2},\ \frac{6}{(x + 1)^3},\ -\frac{18}{(x + 1)^4}.$

Pages 259, 260

1. $(2, -9)$, min. **3.** $(0, 0)$, max.; $(-2, -16)$, $(2, -16)$, min.
5. $(-1, -2)$, max.; $(1, 2)$, min. **7.** $(0, a)$, max.
9. 8 ft. by 8 ft. **11.** 55c.

Pages 261, 262

1. $x^3 + 6x^2 + 7x + C.$ **3.** $\frac{2}{5}x^{\frac{5}{2}} + 2x + C.$ **5.** $\frac{1}{3}(x - 2)^3 + C.$

7. $\frac{x^2}{2} + 4\sqrt{x^3} + 9x + C.$ **9.** $-\frac{1}{x^2 + 1} + C.$

11. $\frac{1}{2}\sin(2x - 5) + C.$ **13.** $-\frac{1}{2}e^{-x^2} + C.$ **15.** $y = 3x^2 - x - 7.$

Page 262

1. 88. **3.** $\frac{18\sqrt{3}+18}{5}$. **5.** $\frac{2}{3}$. **7.** $18+12\sqrt{3}$. **9.** $\frac{2}{5}$.

Page 264

1. 18. **3.** 78. **5.** 57. **7.** 2.

Pages 270, 271

23. $(x^2+y^2)^2+2a^2(y^2-x^2)=0$. **25.** $xy^2+a^2x=a^3$.
27. $x^3+xy^2+ax^2-ay^2=0$.

Page 274

17. (0, 1), (8.3, 1.5).

Page 284

1. $x-2y+10=0$. **3.** $b^2x^2-a^2y^2=a^2b^2$.
5. $x^2-2xy+y^2+2x-6y=0$. **7.** $3xy+x-4=0$. **9.** $a^2y=x^3$.
11. $x^2-2xy+y^2-2ax-2ay+a^2=0$. **13.** $x^3+y^3=3axy$.
15. $dx-by=ad-bc$, d/b. **17.** 264,000 ft., 108 sec.

Page 289

1. $1, 3, 5, \cdots, 21$; $1+3+5+\cdots+21$; $1\cdot 3\cdot 5\cdots 21$.
3. $\frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \cdots, \frac{1}{19}$; $\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\cdots+\frac{1}{19}$; $\frac{1}{5}\cdot\frac{1}{6}\cdot\frac{1}{7}\cdots\frac{1}{19}$.
5. $2, 2^2, 2^3, \cdots, 2^n$; $2+2^2+2^3+\cdots+2^n$; $2\cdot 2^2\cdot 2^3\cdots 2^n$.
7. $3^3, 4^3, 5^3, \cdots, (n+2)^3$; $3^3+4^3+5^3+\cdots+(n+2)^3$; $3^3\cdot 4^3\cdot 5^3\cdots(n+2)^3$.
9. (*a*) $2+2^2+2^3+2^4+2^5+2^6+2^7=254$, (*b*) $2+2^2+2^3+2^4=30$, (*c*) $2+2^2=6$.

Pages 291, 292, 293

1. No. **3.** -5. **5.** No. **7.** 1. **9.** -3. **11.** 27, 99.
13. 41.7, 355.2. **15.** $n=11$, $s=77$. **17.** $l=11$, $s=185$.
19. $a=44$, $s=1218$. **21.** $d=-\frac{5}{11}$, $l=8$. **23.** $n=7$, $l=11$.
25. $\frac{11}{2}$, 9, $\frac{25}{2}$. **27.** $\frac{31}{4}, \frac{21}{2}, \frac{53}{4}, 16, \frac{75}{4}, \frac{43}{2}, \frac{97}{4}, 27, \frac{119}{4}, \frac{65}{2}, \frac{141}{4}$.
29. -68. **31.** n^2. **33.** 156. **35.** 23 ft.
37. 9 yrs., \$7875. **39.** $2ab+a^2c-ac$.

Pages 295, 296

1. 48, 96, 192. **3.** 0.04, 0.004, 0.0004. **5.** 0.08, -0.016, 0.0032.
7. $\frac{4}{625}, \frac{15624}{625}$. **9.** $\frac{243}{2}, \frac{1093}{6}$. **11.** $l=486$, $s=728$.
13. $r=\sqrt{2}$, $s=45+21\sqrt{2}$. **15.** $a=\frac{27}{40}$, $s=\frac{211}{120}$.
17. $s=l\frac{r^n-1}{r^n-r^{n-1}}$. **19.** 140, 350.
21. $21\sqrt{2}, 42, 42\sqrt{2}, 84, 84\sqrt{2}$. **23.** 192. **25.** 4096. **27.** 14.192.
29. $\frac{(1+i)^n-1}{i}$. **31.** $1/r$.

Pages 297, 298

1. 12. **3.** $\frac{125}{9}$. **5.** 2500. **7.** $\frac{17}{33}$.
9. $\frac{110}{37}$. **11.** $\frac{173}{33}$. **13.** 76 ft. **15.** 6 ft.

Page 299

1. 40,320. **3.** 126. **5.** 132. **7.** $n(n+1)$.

Page 301

1. $a^4 + 8a^3 + 24a^2 + 32a + 16$. **3.** $x^{12} - 12x^8 + 54x^4 - 108 + \frac{81}{x^4}$.

5. $x^{10} - 5x^8y^2 + 10x^6y^4 - 10x^4y^6 + 5x^2y^8 - y^{10}$.

7. $x + 4\sqrt[4]{x^3}\sqrt[3]{y^2} + 6\sqrt{x}\sqrt[3]{y^4} + 4\sqrt[4]{x}y^2 + \sqrt[3]{y^8}$.

9. $\frac{a^3}{b^3} - 6\frac{a^2}{b^2} + 15\frac{a}{b} - 20 + 15\frac{b}{a} - 6\frac{b^2}{a^2} + \frac{b^3}{a^3}$.

11. $27a - 54a^{\frac{2}{3}}b^{-\frac{2}{3}} + 36a^{\frac{1}{3}}b^{-\frac{4}{3}} - 8b^{-2}$.

13. $\frac{a^{-8}}{9} + \frac{2\sqrt{6}}{9}a^{-6}b + a^{-4}b^2 + \frac{\sqrt{6}}{3}a^{-2}b^3 + \frac{b^4}{4}$.

15. $a^2b^{-8} + 4a^{\frac{3}{2}}b^{-6}c^{-\frac{1}{2}}d + 6ab^{-4}c^{-1}d^2 + 4a^{\frac{1}{2}}b^{-2}c^{-\frac{3}{2}}d^3 + c^{-2}d^4$. **17.** 1.21550625.

19. $a^2 + 2ab + 2ac + b^2 + 2bc + c^2$. **21.** $x^{18} - 27x^{16}t + 324x^{14}t^2 - 2268x^{12}t^3$.

23. $81u^4 - 216\sqrt{3u^7}v^2 + 756u^3v^4 - 504\sqrt{3u^5}v^6$.

25. $\frac{1}{x^{14}} - \frac{28}{x^{13}y} + \frac{364}{x^{12}y^2} - \frac{2912}{x^{11}y^3}$.

27. $x^{42} + 14x^{39}\sqrt{5y^3} + 455x^{36}y^3 + 1820x^{33}\sqrt{5y^9}$.

29. $a^4 - 8a^{\frac{15}{4}}b^{\frac{1}{2}} + 30a^{\frac{7}{2}}b - 70a^{\frac{13}{4}}b^{\frac{3}{2}}$. **31.** \$201.01. **33.** \$1311.09.

Page 303

1. $210a^4b^6$. **3.** $-1365x^4y^{11}$. **5.** $-8568t^{21}$. **7.** $\frac{140\sqrt{2}}{3}x^3y^2$.

9. $-160x^6y^9$. **11.** $\frac{5}{4}r^3s^4 + \frac{5}{2}r^2s^6$. **13.** $-120u^3w^{14}$. **15.** $504a^{10}b^2$. **17.** $35x^9$.

Page 308

1. 28. **3.** 24, 8. **5.** 48. **7.** 72. **9.** 252. **11.** 480.

Pages 309, 310

1. 30, 24, 6720, 3360, 992. **3.** 10. **5.** 90,000. **7.** 720.
9. 80,640. **11.** 2880. **13.** 1440.

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1. 6. **3.** 6,720. **5.** 1440.

Page 313

1. 56, 924, 1820, 3876, 8436. **3.** 66, 11. **5.** 13. **7.** 153.
9. 8820. **11.** 220, 55. **13.** 60. **15.** 255. **17.** 63.

Pages 315, 316, 317

1. $\frac{7}{16}$. **3.** (a) $\frac{1}{16}$, (b) $\frac{3}{8}$. **5.** $\frac{5}{18}$. **7.** $\frac{2}{9}$. **9.** (a) $\frac{5}{36}$, (b) $\frac{5}{18}$.
11. (a) $\frac{5}{42}$, (b) $\frac{1}{21}$. **13.** (a) $\frac{1}{6}$, (b) $\frac{1}{3}$. **15.** (a) $\frac{5}{9}$, (b) $\frac{16}{81}$.
17. \$3. **19.** \$2.47. **21.** 0.090.

Pages 320, 321

1. $\frac{5}{162}$. **3.** $\frac{5}{27}$. **5.** $\frac{11}{21}$. **7.** \$10. **9.** $\frac{1}{4}$. **11.** $\frac{6561}{15625}$.
13. 4, $\frac{35}{128}$. **15.** $\frac{4553}{15625}$. **17.** (a) 0.58, (b) 0.22.

Pages 323, 324

1. $2i$. **3.** $42i$. **5.** $3xi$. **7.** $8 - 6\sqrt{2}i$. **9.** $-i, 1, i, -i, -i$.
11. $-5 + 3i$. **13.** $3x - 4yi$. **15.** $-3 - 2i$. **17.** $5 - i$.
19. $3 - 6i$. **21.** $-4\sqrt{2} + 6\sqrt{5}i$. **23.** $6\sqrt{15} + 6i$. **25.** $-31 - i$.
27. $x^2 - y^2 + 2xyi$. **29.** $10 + \sqrt{21} + (5\sqrt{7} - 2\sqrt{3})i$. **31.** $-7 + 24i$.
33. $-\frac{7}{5} - \frac{16}{5}i$. **35.** $\frac{39}{17} + \frac{31}{17}i$. **37.** $\dfrac{\sqrt{6} - \sqrt{35}}{10} + \dfrac{\sqrt{15} + \sqrt{14}}{10}i$.
39. $\dfrac{1}{49} + \dfrac{4\sqrt{3}}{49}i$.
41. $(-1, 2)$. **43.** $(2, -1)$.
45. $(2, 1), (-2, 1), (-2, -1), (2, -1)$.
47. $x^2 - 6x + 13$. **49.** $x^2 - 2ax + a^2 + b^2$.
51. $x^2 + 10x + 29 = 0$. **53.** $(x - 2 + 3i)(x - 2 - 3i)$.

Pages 324, 325

9. $4 + 3i, 4 - 3i$. **11.** $0 + 3i, 0 - 3i$.
13. $9 + 0i, 9 - 0i$. **15.** $8 + 3i, 8 - 3i$.

Pages 326, 327

1. $5\sqrt{2}\,(\cos 45° + i \sin 45°)$. **3.** $2(\cos 300° + i \sin 300°)$.
5. $7(\cos 90° + i \sin 90°)$. **7.** $3(\cos 270° + i \sin 270°)$.
9. $5(\cos 180° + i \sin 180°)$. **11.** $\sqrt{29}\,(\cos 338°\,12' + i \sin 338°\,12')$.
13. $2\sqrt{3} + 2i$. **15.** $-3\sqrt{2} - 3\sqrt{2}i$. **17.** $-7 + 0i$. **19.** $0 + 3i$.
21. $8.290 + 5.592i$. **23.** $-4.779 - 3.628i$.

Page 328

1. $12(\cos 150° + i \sin 150°) = -6\sqrt{3} + 6i$.
3. $44(\cos 135° + i \sin 135°) = -22\sqrt{2} + 22\sqrt{2}i$.
5. $48(\cos 120° + i \sin 120°)$. **7.** $65(\cos 111° + i \sin 111°)$.
9. $3(\cos 120° + i \sin 120°) = \dfrac{3}{2} + \dfrac{3\sqrt{3}}{2}i$. **11.** $7(\cos 76° + i \sin 76°)$.

Page 329

1. $2\sqrt{2}\,(\cos 135° + i \sin 135°) = -2 + 2i$.
3. $32(\cos 90° + i \sin 90°) = 0 + 32i$.
5. $64(\cos 240° + i \sin 240°) = -32 - 32\sqrt{3}i)$.

7. $8(\cos 180° + i \sin 180°) = -8 + 0i.$

9. $25(\cos 150° + i \sin 150°) = -\frac{25\sqrt{3}}{2} + \frac{25}{2}i.$

Pages 330, 331

1. $2(\cos 30° + i \sin 30°) = \sqrt{3} + i$; $2(\cos 210° + i \sin 210°) = -\sqrt{3} - i$

3. $\cos 0° + i \sin 0° = 1$; $\cos 120° + i \sin 120° = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$;

$\cos 240° + i \sin 240° - \frac{1}{2} - \frac{\sqrt{3}}{2}i.$

5. $5(\cos 0° + i \sin 0°) = 5$; $5(\cos 90° + i \sin 90°) = 5i$;
$5(\cos 180° + i \sin 180°) = -5$; $5(\cos 270° + i \sin 270°) = -5i.$

7. $3(\cos 45° + i \sin 45°) = \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$; $3(\cos 165° + i \sin 165°)$;
$3(\cos 285° + i \sin 285°).$

9. $2(\cos 12° + i \sin 12°)$; $2(\cos 84° + i \sin 84°)$; $2(\cos 156° + i \sin 156°)$;
$2(\cos 228° + i \sin 228°)$; $2(\cos 300° + i \sin 300°) = 1 - \sqrt{3}i.$

11. $3(\cos 60° + i \sin 60°) = \frac{3}{2} + \frac{3\sqrt{3}}{2}i$; $3(\cos 180° + i \sin 180°) = -3$;

$3(\cos 300° + i \sin 300°) = \frac{3}{2} - \frac{3\sqrt{3}}{2}i.$

13. $\sqrt{10}(\cos 30° + i \sin 30°) = \frac{\sqrt{30}}{2} + \frac{\sqrt{10}}{2}i$; $\sqrt{10}(\cos 120° + i \sin 120°)$

$= -\frac{\sqrt{10}}{2} + \frac{\sqrt{30}}{2}i$; $\sqrt{10}(\cos 210° + i \sin 210°) = -\frac{\sqrt{30}}{2} - \frac{\sqrt{10}}{2}i$;

$\sqrt{10}(\cos 300° + i \sin 300°) = \frac{\sqrt{10}}{2} - \frac{\sqrt{30}}{2}i.$

15. $3(\cos 45° + i \sin 45°) = \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$; $3(\cos 135° + i \sin 135°)$

$= -\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$; $3(\cos 225° + i \sin 225°) = -\frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i$;

$3(\cos 315° + i \sin 315°) = \frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i.$

Page 332

1. $5x^3 + 3x^2 - 2x - 8.$

3. $-9x^4 + 0x^3 + 2x^2 + 4x + 3.$

5. $2x^3 + 4x^2 - 6x + 2.$

7. $x^3 - 6x^2 + 11x - 6.$

9. $-10, -35, -x^3 - 5x^2 - 3x - 1, \frac{x^3}{8} - \frac{5}{4}x^2 + \frac{3}{2}x - 1, 27y^3 - 45y^2 + 9y - 1$

Page 334

1. $x^2 + x + 4$, 21. 3. $2x^2 - 3x - 7$, -9. 5. $2x^2 - x + 4$, -4.
7. $5x^3 - 14x^2 + 14x - 1$, 20. 9. $x^4 + 5x^3 + 3x^2 - x - 5$, -16.
11. $x^2 + (a + 7)x + (a^2 + 7a - 3)$, $a^3 + 7a^2 - 3a + 5$.

Pages 336, 337

1. 5. 3. -13. 5. 14. 7. 6. 9. 30. 11. $-3a^3$.
13. 55. 15. No. 17. No. 19. No. 21. 9.

Page 339

1. 5; 0, 0, 0, $3 - \sqrt{5}$, $3 + \sqrt{5}$. 3. 7; 3, -3, 6, 6, -1, -1, -1.
5. 7; $\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 2i, 2i, -2i, -2i$. 7. 6; 0, 0, 0, 0, 2, -4.
9. $(x - 1)(x - 2)(x - 3)$. 11. $(x - 3)(x + 5)(x + i)(x - i)$.
13. $(x - 1)^2(x + 2 + \sqrt{5})(x + 2 - \sqrt{5})$. 15. $(x + 1)^3(x + i)(x - i)$.
17. $2(x - 2)(x + 2)\left(x + \dfrac{1 + \sqrt{63}i}{2}\right)\left(x + \dfrac{1 - \sqrt{63}i}{2}\right)$.

Page 340

1. $x^3 - 2x^2 - 5x + 6 = 0$. 3. $x^3 - 5x^2 + 2x + 12 = 0$.
5. $x^4 - 6x^3 + 6x^2 + 24x - 40 = 0$. 7. $x^4 - 4x^3 + 8x^2 - 8x + 4 = 0$.
9. $x^6 + 4x^5 - 2x^4 - 12x^3 + 9x^2 = 0$. 11. $x^5 - 44x^3 + 66x^2 + 187x - 210 = 0$.

Page 341

1. $-x^3 - 7x^2 - 7x + 15 = 0$. 3. $x^4 - 5x^2 - 10x - 6 = 0$.
5. $x^4 - 3x^3 - 5x - 3 = 0$. 7. $x^4 - 13x^2 + 36 = 0$. 9. $x^5 - 5x^4 - 2x^3 + 3x = 0$.

Page 342

1. 0 Pos., 2 Neg., 0 zero, 0 Imag.; or 0 Pos., 0 Neg., 0 zero, 2 Imag.; $-2 \pm 2i$.
3. 1 Pos., 0 Neg., 0 zero, 2 Imag.; 3, $\frac{3}{2}(-1 \pm \sqrt{3}i)$.
5. 2 Pos., 2 Neg., 0 zero, 0 Imag.; or 2 Pos., 0 Neg., 0 zero, 2 Imag.; or 0 Pos., 2 Neg., 0 zero, 2 Imag.; or 0 Pos., 0 Neg., 0 zero, 4 Imag.; 2, 3, -2, -3.
7. 1 Pos., 0 Neg., 0 zero, 2 Imag. 9. 1 Pos., 1 Neg., 0 zero, 4 Imag.

Page 343

1. 1, -5. 3. 6, -1. 5. 7, -2. 7. 4, -4.

Page 345

1. $x^3 - 8x^2 - 44x + 240 = 0$. 3. $2x^3 + 3x^2 + 5x - 22 = 0$.
5. $4x^3 + 3x^2 - 275 = 0$. 7. $x^3 + 2x^2 + 3x - 10 = 0$.
9. $x^3 + 10x^2 + 10x - 500 = 0$. 11. $x^4 - 10x^2 + 36x - 56 = 0$.

Page 347

1. -5, 1, 3. 3. $-\frac{2}{3}$, $-2 + \sqrt{6}$, $-2 - \sqrt{6}$. 5. $\frac{3}{2}$, $3 + \sqrt{13}$, $3 - \sqrt{13}$.
7. -5, -2, $-1 + \sqrt{5}$, $-1 - \sqrt{5}$. 9. $-\frac{1}{2}$, $-\frac{1}{2}$, $-3 + \sqrt{5}$, $-3 - \sqrt{5}$.
11. 0, 0, 2, $-\frac{1}{2}$, $-\frac{3}{2}$. 13. 2, -2, 3, -3, i, $-i$.

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11. 0, 2.6, − 2.6. **13.** − 5.8, − 1.3, 1.1. **15.** 3.6. **17.** − 2.8, 2.8.

Page 352

1. 2.332. **3.** 1.811. **5.** − 3.222. **7.** − 1.889, 1.125, 3.764.
9. − 2.751. **11.** 2.705. **13.** − 1.879, 1.532, 0.347.
15. − 4.303, − 0.697, 0.382, 2.618.

Page 354

1. $x^3 - x^2 - 4x + 4 = 0.$ **3.** $x^3 + 2x^2 - 5x + 2 = 0.$
5. $3x^3 - 10x^2 + 8x + 16 = 0.$ **7.** $x^3 - 0.9x^2 + 0.15x - 0.161 = 0.$

Pages 356, 357

17. − 1.069, 1.722, 4.346. **19.** − 1.176. **21.** 1.256, 1.777.
23. 9.447. **25.** 6.445. **27.** 1.632.
29. (− 2, 4), (− 0.946, 0.895), (2.252, 5.072), (4.694, 22.03).

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7. 5, 30. **9.** 2, 3, − 4.

Pages 359, 360

1. $x^2 - 6x + 25 = 0.$ **3.** $x^4 + 12x^3 + 56x^2 + 88x + 68 = 0.$
5. $x^4 - 16x^3 + 98x^2 - 272x + 289 = 0.$ **7.** $2, 3, -1 - i, -1 + i.$
9. $1 + \sqrt{3}i, 1 + \sqrt{3}i, 1 - \sqrt{3}i, 1 - \sqrt{3}i.$ **11.** $(x^2 + 4x + 5)(2x - 3).$
13. $x(x^2 - 2x + 12)^2(2x + 1)(x + 3).$

Page 363

1. $\dfrac{A}{x-4} + \dfrac{B}{x+7} + \dfrac{C}{3x+1}.$ **3.** $x + 2 + \dfrac{A}{x+3} + \dfrac{B}{x+1} + \dfrac{C}{x-1}.$

5. $\dfrac{A}{x+3} + \dfrac{B}{(x+3)^2} + \dfrac{Cx+D}{x^2+1}.$

7. $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{D}{x-1} + \dfrac{E}{(x-1)^2} + \dfrac{Fx+G}{x^2+2} + \dfrac{Hx+I}{(x^2+2)^2}.$

Page 364

1. $\dfrac{5}{2x-3} - \dfrac{2}{x+4}.$ **3.** $\dfrac{7}{x} - \dfrac{4}{2x+5}.$ **5.** $2 + \dfrac{11}{x-5} + \dfrac{4}{x+2}.$

7. $\dfrac{2}{x-1} - \dfrac{5}{x-3} + \dfrac{3}{x-5}.$ **9.** $\dfrac{1}{x+1+\sqrt{5}} + \dfrac{1}{x+1-\sqrt{5}}.$

Page 365

1. $\dfrac{7}{x+3} - \dfrac{12}{(x+3)^2}.$ **3.** $1 + \dfrac{1}{x-3} - \dfrac{4}{(x-3)^2} + \dfrac{7}{(x-3)^3}.$

5. $\dfrac{3}{x+1} - \dfrac{4}{(x+1)^2} + \dfrac{1}{(x+1)^3} - \dfrac{6}{2x+7}.$ **7.** $\dfrac{5}{x^2} - \dfrac{3}{x+1} - \dfrac{2}{(x+1)^2}.$

9. $\dfrac{2}{x-1} - \dfrac{6}{(x-1)^2} + \dfrac{7}{(x-1)^3} - \dfrac{2}{x+1} + \dfrac{9}{(x+1)^3}.$

Page 366

1. $2x - 9 + \frac{4}{x} + \frac{5-6x}{x^2+1}$. 3. $\frac{2}{x-1} - \frac{3}{x+1} + \frac{x-4}{x^2+1}$. 5. $\frac{5x+9}{x^2-x+3} - \frac{3}{x}$.
7. $\frac{7x-1}{2x^2+1} + \frac{4-3x}{x^2+4}$. 9. $\frac{3}{x^2} - \frac{2}{x^3} + \frac{4}{x^2+2x+2}$.

Page 367

1. $\frac{3x-1}{2x^2+x+3} - \frac{8x+5}{(2x^2+x+3)^2}$. 3. $\frac{3x-1}{x^2-2x+5} + \frac{x-9}{(x^2-2x+5)^2} - \frac{3}{x-1}$.
5. $\frac{1}{x^2+1} - \frac{3x+9}{(x^2+1)^2} + \frac{5x+4}{(x^2+1)^3}$. 7. $\frac{3x-7}{(x^2+3)^2} + \frac{1}{x^3}$.

Page 369

3. (*a*) $(x, y, 0)$, $(x, 0, z)$, $(0, y, z)$, (*b*) $(x, 0, 0)$, $(0, y, 0)$, $(0, 0, z)$.
5. $\sqrt{y^2+z^2}$, $\sqrt{x^2+z^2}$, $\sqrt{x^2+y^2}$.
7. (*a*) The *xy*-plane, (*b*) A plane parallel to the *xy*-plane through $(0, 0, 5)$.
9. A line parallel to the *z*-axis through $(5, 3, 0)$.
11. $\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)$, $\left(-\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)$, $\left(-\frac{a}{2}, -\frac{a}{2}, \frac{a}{2}\right)$, $\left(\frac{a}{2}, -\frac{a}{2}, \frac{a}{2}\right)$,
$\left(\frac{a}{2}, \frac{a}{2}, -\frac{a}{2}\right)$, $\left(-\frac{a}{2}, \frac{a}{2}, -\frac{a}{2}\right)$, $\left(-\frac{a}{2}, -\frac{a}{2}, -\frac{a}{2}\right)$, $\left(\frac{a}{2}, -\frac{a}{2}, -\frac{a}{2}\right)$.
13. $(x, y, -z)$, $(x, -y, z)$, $(-x, y, z)$.

Page 370

1. 15. 3. 11. 5. 19. 7. $7\sqrt{2}$. 9. $\sqrt{65}$, $\sqrt{26}$, $\sqrt{91}$. 11. $2\sqrt{6}$.
13. $(x+7)^2 + (y-3)^2 + (z+2)^2 = 25$.
15. A sphere, center $(0, 0, 0)$, radius 6.

Page 373

1. $\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$. 3. $\frac{2}{3}, \frac{11}{15}, \frac{2}{15}$. 5. $\frac{1}{33}\sqrt{33}, \frac{4}{33}\sqrt{33}, -\frac{4}{33}\sqrt{33}$.
7. $\frac{4}{9}, -\frac{7}{9}, \frac{4}{9}$. 9. $-\frac{23}{27}, \frac{2}{27}, \frac{14}{27}$. 11. $\frac{3}{38}\sqrt{38}, \frac{5}{38}\sqrt{38}, \frac{2}{38}\sqrt{38}$.
13. $\frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{2}}{2}$. 15. $-\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}$.
17. 1, 0, 0; 0, 1, 0; 0, 0, 1.

Pages 375, 376

1. $81° 52'$. 3. $162° 15'$. 5. $79° 31'$. 7. $58° 46'$.
9. 7, -4, -1. 13. $83° 44'$.

Pages 377, 378

1. $(3, 3\sqrt{3}, 2)$. 3. $(4\sqrt{3}, 4, -3)$. 5. $(3\sqrt{2}, 135°, 5)$. 7. $(5, 0°, 2)$.
9. $(2\sqrt{6}, 2\sqrt{2}, 4\sqrt{2})$. 11. $(\sqrt{2}, -\sqrt{2}, 2\sqrt{3})$. 13. $(5, 90°, 90°)$.
15. $(3, 315°, 109° 28')$. 17. $x^2 + y^2 = 36$. 19. $x^2 + y^2 = 3y$.
21. $x^2 + y^2 + z^2 = 9$. 23. $y = 2$. 25. $r^2 + z^2 = 25$; $\rho = 5$.
27. $9r^2 + 25z^2 = 225$; $9\rho^2 \sin^2\phi + 25\rho^2 \cos^2\phi = 225$.
29. $r = \rho \sin\phi$, $\theta = \theta$, $z = \rho \cos\phi$.

Pages 381, 382

1. $-\frac{x}{2}-\frac{\sqrt{2}}{2}y+\frac{z}{2}-3=0.$ **3.** $-\frac{\sqrt{2}}{2}y+\frac{\sqrt{2}}{2}z+7=0.$
5. $\frac{2}{3}x-\frac{2}{3}y+\frac{z}{3}\pm 7=0.$ **7.** $\frac{2}{15}x-\frac{10}{15}y-\frac{11}{15}z\pm 4=0.$
9. $\frac{5}{83}\sqrt{83}x+\frac{3}{83}\sqrt{83}y+\frac{7}{83}\sqrt{83}z\pm 6=0.$ **11.** $\frac{2}{7}x-\frac{3}{7}y+\frac{6}{7}z-7=0.$
13. $-\frac{7}{11}x+\frac{6}{11}y+\frac{6}{11}z-11=0.$
15. $\frac{4}{9}x-\frac{7}{9}y+\frac{4}{9}z-2=0;\ \frac{4}{9},-\frac{7}{9},\frac{4}{9};\ 2;\ 4x-7y-18=0, z=0;$ $4x+4z-18=0, y=0;\ 7y-4z+18=0, x=0.$
17. $-\frac{3}{13}x-\frac{12}{13}y+\frac{4}{13}z+3=0;\ -\frac{3}{13},-\frac{12}{13},\frac{4}{13};\ -3;\ 3x+12y-39=0,$ $z=0;\ 3x-4z-39=0, y=0;\ 12y-4z-39=0, x=0.$
19. $\frac{5}{13}x+\frac{12}{13}y-2=0;\ \frac{5}{13},\frac{12}{13},0;\ 2;\ 5x+12y-26=0, z=0;$ $5x-26=0, y=0;\ 12y-26=0, x=0.$ **21.** $\pm 68.$
23. (*a*) $9x-2y+6z+18=0,$ (*b*) $3x+y+4z-1=0.$ **25.** $(-35,-30,16).$

Page 384

1. $69°\ 34'.$ **3.** $70°\ 54'.$ **5.** $6x-2y+3z-19=0, 4x+7y+4z+6=0.$
7. $(-1,3,3).$ **9.** 2. **11.** $6x+2y-3z+40=0, 6x+2y-3z-2=0.$
13. (*a*) $B=0,$ (*b*) $A=0.$

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1. $\frac{x}{10}+\frac{y}{5}+\frac{z}{2}=1.$ **3.** $\frac{x}{-3}+\frac{y}{9}+\frac{z}{6}=1.$
5. $6x+5y-7z=15.$ **7.** $2x-3y+z=6.$ **9.** $4x+5y-16z=4.$
11. $7x-2y-8z=9.$ **13.** $x+2y-2z+6=0.$
15. $4x+11y+5z=10.$ **17.** $6x-3y+z=6.$

Pages 390, 391

1. $\frac{x-2}{6}=\frac{y-1}{-2}=\frac{z+5}{9};\ \frac{6}{11},-\frac{2}{11},\frac{9}{11}.$
3. $\frac{x-4}{2}=\frac{y+1}{1}=\frac{z+7}{2};\ \frac{2}{3},\frac{1}{3},\frac{2}{3}.$
5. $\frac{x-2}{3}=\frac{y-5}{4}=\frac{z+8}{12};\ \frac{3}{13},\frac{4}{13},\frac{12}{13}.$
7. $\frac{x+5}{7}=\frac{y-1}{-4}=\frac{z+3}{4};\ \frac{7}{9},-\frac{4}{9},\frac{4}{9}.$
9. $x+2y=5, x+z=3, 2y-z=2;\ (3,1,0), (5,0,-2), (0,\frac{5}{2},3);\ -\frac{2}{3},\frac{1}{3},\frac{2}{3}.$
11. $x+3y=9, x-z+3=0, 3y+z=12;\ (-3,4,0), (9,0,12), (0,3,3);$ $\frac{3}{19}\sqrt{19}, -\frac{\sqrt{19}}{19}, \frac{3}{19}\sqrt{19}.$ **13.** $83°\ 2'.$
15. $3y+z=12.$ **17.** $\frac{x-1}{2}=\frac{y-3}{-1}=\frac{z-5}{2}.$

Page 394

1. $x^2+y^2+z^2=a^2.$ **3.** $x^2+y^2=az.$ **5.** $x^4=a^2(y^2+z^2).$
7. $y^2+z^2=x^6.$ **9.** $x^2+y^2=z^3+z.$ **11.** $x^2+z^2=e^{2y}.$
13. $(x^2+y^2+z^2+b^2-a^2)^2=4b^2(x^2+y^2).$

Pages 395, 396

1. $x^2 + y^2 + z^2 - 12x + 4y + 18z = 0$.
3. $x^2 + y^2 + z^2 - 10x - 8y + 6z - 14 = 0$.
5. $(-3, 1, 5)$; 4.
7. $(-3, 2, -4)$; 0.
9. $x^2 + y^2 + z^2 - 6x - 6y - 6z + 18 = 0$.
11. $x^2 + y^2 + z^2 + 12x - 2y - 6z - 35 = 0$.
13. $x^2 + y^2 + z^2 + 32x + 16y - 4z = 0$, or $x^2 + y^2 + z^2 - 32x - 16y + 4z = 0$.

Pages 403, 404

1. 40.84 in., 56.02 in., 64.93 in.
3. 68°, 51°, 94°.
5. 2662 mi.

Page 406

1. 3522 mi.
3. 431.6 mi.
5. 65° 47′.

Pages 411, 412

1. $b = 62° 32.6'$, $\alpha = 59° 38.2'$, $\beta = 66° 33.4'$.
3. $b = 115° 22.5'$, $c = 100° 12.4'$, $\alpha = 67° 41.4'$.
5. $c = 97° 51.3'$, $\alpha = 131° 51.6'$, $\beta = 81° 19.7'$.
7. $a = 58° 17.9'$, $b = 25° 59.3'$, $\beta = 29° 48.8'$.
9. $b_1 = 23° 4.4'$, $c_1 = 46° 34.6'$, $\beta_1 = 32° 39.4'$; or
   $b_2 = 156° 55.6'$, $c_2 = 133° 25.4'$, $\beta_2 = 147° 20.6'$.
11. $a = 64° 50.8'$, $\alpha = 69° 4.3'$, $\beta = 122° 49.4'$.
13. $a = 59° 18.6'$, $c = 68° 15.6'$, $\beta = 47° 47.5'$.
15. $a = 19° 30.8'$, $b = 148° 19.0'$, $\alpha = 34° 0.6'$.
17. $a = 58° 48.0'$, $b = 51° 12.3'$, $c = 71° 3.6'$.
19. $a_1 = 145° 24.7'$, $c_1 = 66° 33.2'$, $\alpha_1 = 141° 46.4'$; or
    $a_2 = 34° 35.3'$, $c_2 = 113° 26.8'$, $\alpha_2 = 38° 13.6'$.
21. N 42° 34′ E; 4020 N. M.
23. Lat. 41° 36′, N 81° 14′ W, 584 N. M.
25. H. A. 75° 14′; Dec. 9° 29′.

Pages 412, 413

1. $a = 106° 22.1'$, $\beta = 44° 33.3'$, $\gamma = 74° 1.6'$.
3. $a = 21° 12.0'$, $\alpha = 9° 1.0'$, $\beta = 24° 9.0'$.
5. 3015 N. M.; N 39° 58′ W.
7. Lat. 55° 45′; 3 hr. 21.7 min. P. M.

Pages 413, 414

1. $a = 50° 38.8'$, $\beta = \gamma = 161° 53.5'$.
3. $a = b = 40° 17.3'$, $\gamma = 135° 18.8'$.
5. (*a*) N 70° 34′ W, (*b*) 49° 48′, (*c*) 2398 mi., (*d*) 45 mi.

Pages 420, 421

1. $\alpha = 76° 19.6'$, $\beta = 94° 23.0'$, $\gamma = 53° 43.4'$.
3. $\alpha = 127° 3.4'$, $\beta = 119° 37.4'$, $\gamma = 103° 17.4$.
5. $a = 65° 59.2'$, $b = 69° 56.6'$, $c = 62° 9.0'$.
7. $a = 140° 21.8'$, $b = 134° 37.4'$, $c = 53° 6.0'$.
9. 10,770,000 sq. mi., 18,040,000 sq. mi., 40,560,000 sq. mi., 25,480,000 sq. mi.
11. 25.69 in., 18.89 in., 15.98 in.

Page 422

1. $c = 71° 7.0'$, $\alpha = 55° 50.3'$, $\beta = 22° 18.3'$.
3. $a = 112° 36.4'$, $\beta = 124° 5.3'$, $\gamma = 95° 42.7'$.
5. $b = 74° 2.8'$, $\alpha = 127° 2.2'$, $\gamma = 68° 3.8'$.
7. $a = 96° 39.4'$, $b = 58° 30.8'$, $\gamma = 131° 36.4'$.
9. $b = 68° 56.5'$, $c = 43° 51.7'$, $\alpha = 37° 30.0'$.
11. $a = 148° 31.7'$, $c = 140° 35.7'$, $\beta = 76° 32.6'$.
13. 2490 N. M.; San Diego from Colon, N 50° 13′ W; Colon from San Diego, N 115° 41′ E.
15. 5029 N. M.; Rio from Liverpool, N 143° 21′ W; Liverpool from Rio, N 22° 43′ E.
17. Dec. 1° 9′; 2 hr. 47.5 min.

Pages 423, 424

1. $c = 117° 48.4'$, $\beta = 27° 24.5'$, $\gamma = 144° 44.8'$.
3. $a_1 = 90° 27.6'$, $\alpha_1 = 108° 4.0'$, $\beta_1 = 64° 35.7'$, or $a_2 = 39° 57.6'$, $\alpha_2 = 37° 38.0'$, $\beta_2 = 115° 24.3'$.
5. No solution.
7. $b_1 = 33° 6.4'$, $c_1 = 20° 35.8'$, $\gamma_1 = 37° 24.8'$, or $b_2 = 146° 53.6'$, $c_2 = 145° 25.0'$, $\gamma_2 = 101° 25.8'$.
9. $b = 36° 59.8$, $c = 50° 53.6'$, $\beta = 32° 12.0'$.
11. No solution.
13. Long. 172° 48′ E; S 38° 32′ E; 14.13 knots.
15. N 146° 19′ E; Lat. 54° 0′ N.

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